On $\left(\prod_{i<\kappa} x(i), \leq^*\right)$ and its Bounding Number

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Abstract

For inaccessible cardinals κ, we investigate the relationship between $\left(\prod_{i<\kappa} x(i), \leq^*\right)$ and $(\kappa^\kappa, \leq^*)$ for some $x : \kappa \to \mathrm{Lim} \cap \kappa$. To be precise, we analyse when there is a continuous cofinal embedding

$$\pi : \left(\prod_{i<\kappa} x(i), \leq^*\right) \to (\kappa^\kappa, \leq^*)$$

We show that, under some constraints on x, this question is equivalent to whether Hechler forcing on $\prod_{i<\kappa} x(i)$ adds a κ^κ-dominating function and also to the existence of a certain type of κ-Aronszajn tree. We also prove that there is a $<\kappa$-distributive forcing that adds a κ-Aronszajn tree of this type. It follows that the existence of such an embedding on κ is independent of the theory ZFC+"κ is inaccessible", relative to a weakly compact cardinal. Finally, we discuss a strategy to produce a model where

$$\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) > \mathfrak{b}_\kappa$$

1 Introduction

In this note, we investigate the relationship between the partial orders $(\kappa^\kappa, \leq^*)$ and $\left(\prod_{i<\kappa} x(i), \leq^*\right)$ where x is a reasonable function $x : \kappa \to \kappa$. in the case that κ is an inaccessible cardinal. Here, $\leq^*$ means eventual domination. For the rest, we will fix an inaccessible cardinal κ. This turns out to be connected to generalised versions of Hechler forcing. For the rest of the note, x is assumed to be a function $x : \kappa \to \mathrm{Lim} \cap \kappa$.

Definition 1. (*i*) $\mathbb{H}$ denotes the (unrestricted) generalised Hechler forcing at κ. Conditions are pairs (f, g) where $f \in \kappa^{<\kappa}$ and $g \in \kappa^\kappa$. The order is defined via $(f, g) \leq (f', g')$ iff

(a) $f \supseteq f'$

(b) $g \geq g'$ and

(c) for $i \in \mathrm{dom}(f) \setminus \mathrm{dom}(f')$ $f(i) \geq g'(i)$.

(*ii*) The x-restricted generalised Hechler forcing at κ $\mathbb{H}_x$ is the suborder of $\mathbb{H}$ consisting only of conditions (f, g) with $f \in \prod_{i<\alpha} x(i)$ for some $\alpha < \kappa$ and $g \in \prod_{i<\kappa} x(i)$.

The idea is that the first component is the "working part" giving partial information about a function with domain κ and the second component is a "side condition" making sure that the first component will eventually dominate all ground model functions.

Definition 2. A partial order $\mathbb{P}$ is $<\kappa$-progressively-closed if for any $\lambda < \kappa$ the set of conditions p such that $\mathbb{P} \restriction p$ is λ-closed is dense.

Lemma 3. (*i*) *$\mathbb{H}$ is $<\kappa$-closed and adds a function in κ^κ eventually dominating all ground model functions in κ^κ.*

(*ii*) *If $\mathrm{cof} \circ x$ converges to κ, then $\mathbb{H}_x$ is $<\kappa$-progressively-closed (but not $<\kappa$-closed) and adds a function in $\prod_{i<\kappa} x(i)$ eventually dominating all ground model functions in $\prod_{i<\kappa} x(i)$.*

Also recall the definition of the *bounding* number $\mathfrak{b}(\mathbb{P})$ of a partial order $\mathbb{P}$: It is smallest size of an unbounded subset of $\mathbb{P}$.

The main result here is the following.

Theorem 4. *Assume $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$. Then the following four conditions are equivalent:*

(*i*) *There is a continuous cofinal (w.r.t. both $\leq$ and $\leq^*$) embedding*

$$\pi : \left(\prod_{i<\kappa} x(i); \leq, \leq^*\right) \to \left(\kappa^\kappa; \leq, \leq^*\right)$$

(*ii*) *There is a continuous cofinal embedding*

$$\pi : \left(\prod_{i<\kappa} x(i), \leq^*\right) \to \left(\kappa^\kappa, \leq^*\right)$$

(*iii*) *Forcing with $\mathbb{H}_x$ adds a dominating function for $(\kappa^\kappa, \leq^*)$.*

(*iv*) *There is an x-increasing κ-Aronszajn tree.*

If x only satisfies $\lim_{i<\kappa} \mathrm{cof} \circ x(i) = \kappa$ then the implications from top to bottom still hold.

Remark 5. The assumption on x is satisfied for many natural examples of x, see Lemma 9. We list a few of them.

- $\alpha \mapsto \alpha^+$, or more generally $\alpha \mapsto \alpha^{(+\gamma)}$ for γ a successor ordinal
- $\alpha \mapsto \aleph_{\alpha+1}$, or more generally $\alpha \mapsto \aleph_{\alpha+\gamma}$ for γ a successor ordinal
- $\alpha \mapsto 2^\alpha$
- $\alpha \mapsto \alpha^{\operatorname{cof}|\alpha|}$ if SCH holds
- $\alpha \mapsto$ next inaccessible above α if κ is a limit of inaccessibles, etc.

We still have to explain a few terms. Firstly, an embedding is an order-preserving map. By continuous we mean that the spaces κ^κ and $\prod_{i<\kappa} x(i)$ are endowed with the box-topology w.r.t. the co-bounded filter and each ordinal is considered discrete. I.e. a basic open set N_f is given by all functions extending a partial $f : \alpha \to \kappa$. This is the usual topology on these spaces when they are considered as generalised Baire spaces.

Definition 6. An x-increasing κ-Aronszajn tree is a subtree T of $\bigcup_{\alpha<\kappa} \prod_{i<\alpha} x(i)$ so that

(i) T is a κ-Aronszajn tree and

(ii) whenever $\alpha < \kappa$ and $f, g \in \prod_{i<\alpha} x(i)$ with $f \leq g$ then $f \in T \Rightarrow g \in T$.

Since a weakly compact cardinal has the tree property we get the following corollary:

Corollary 7. *If κ is weakly compact and* $\operatorname{cof} \circ x$ *converges to κ, then $(i)-(iv)$ of Theorem 4 fail.*

It should be stressed, however, that the non-existence of an x-increasing Aronszajn tree is not a trivial property.

Theorem 8. *Suppose* $\operatorname{cof} \circ x$ *converges to κ. There is a $<\kappa$-progressively-closed forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$ there is an x-increasing κ-Aronszajn tree.*

2 The Main Theorem

We start by calculating when $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$. Note that the assumption of Theorem 8 is weaker than condition (ii).

Lemma 9. *The following are equivalent:*

(i) $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$

(*ii*) $\operatorname{cof} \circ x$ *converges to* κ *and* $\operatorname{ran}(\operatorname{cof} \circ x)$ *is non-stationary.*

(*iii*) *There is an increasing unbounded* $y : \kappa \to \kappa$ *with* $y < \operatorname{cof} \circ x$.

Proof. It is straightforward to show that

$$\left(\prod_{i<\kappa} x(i), \leq^*\right) \text{ and } \left(\prod_{i<\kappa} \operatorname{cof}(x(i)), \leq^*\right)$$

are Tukey-equivalent and hence have the same bounding number. Thus $(i) - (iii)$ remain unchanged if x is replaced by $\operatorname{cof} \circ x$. This means we may assume $\operatorname{cof} \circ x = x$ in the proof.
$\underline{(i) \Rightarrow (ii)}$: First assume that $\liminf_{i<\kappa} x(i) < \kappa$. Then x is constant with value β on an unbounded set U. If we set $f_\alpha(i) = \alpha$ for $i \in U$ and $f_\alpha(i) = 0$ for $i \notin U$ then $\{f_\alpha \mid \alpha < \beta\}$ is an unbounded family of $(\prod_{i<\kappa} x(i), \leq^*)$ of size λ, contradiction.
Next, suppose that $S = \operatorname{ran}(x)$ is stationary. Let $C \subseteq \kappa$ be club so that any $i \in C$ is closed under both x and the map

$$\alpha \mapsto \sup\{j < \kappa \mid x(j) \leq \alpha\}$$

Then for $i \in C$ we must have $\forall i \leq \alpha < \kappa \; x(\alpha) \geq i$. Hence for any $i \in C \cap S$ ther is some $h(i) \geq i$ so that $x(h(i)) = i$. Note that $\sup C \cap h(i) = i$ so that h is strictly increasing. We will show that the family $\langle f_j \mid j < \kappa\rangle$ defined by

$$f_j(\alpha) = \begin{cases} j & \text{if } \alpha = h(i) \text{ and } j < i \\ 0 & \text{else} \end{cases}$$

is unbounded in $\left(\prod_{i<\kappa} x(i), \leq^*\right)$.Let g be any function in $\prod_{i<\kappa} x(i)$. Then $g \circ h$ is regressive on $C \cap S$. By Fodor's Lemma, there is some unbounded $U \subseteq C \cap S$ and some j so that $g \circ h$ has value j on U. Hence $g(h(i)) < f_j(\alpha_i)$ for any $i \in U$ so that $f_j \not\leq^* g$. Once again, this contradicts (i).

$\underline{(ii) \Rightarrow (iii)}$: Let C be the club from before and let D be a club disjoint from $\operatorname{ran}(\operatorname{cof} \circ x)$. Then the map y defined by

$$y(i) = \sup(C \cap D \cap i)$$

does the job.

$\underline{(iii) \Rightarrow (i)}$: Let $\{f_j \mid j < \kappa\} \subseteq \prod_{i<\kappa} x(i)$ and let y witness (iii). Then g defined by

$$g(i) = \sup\{f_j(i) \mid j < y(i)\}$$

is in $\prod_{i<\kappa} x(i)$ as the supremum is taken over a set of size $y(i) < \operatorname{cof}(x(i))$. Since y is increasing and unbounded in κ, $f_j \leq^* g$ for any $j < \kappa$. $\square$

Remark 10. Note that ran(cof $\circ x$) can only ever be stationary if κ is Mahlo. Also, if cof $\circ x$ converges to κ, this is equivalent to cof $\circ x = \mathrm{id}$ on a stationary set.

Let us proof the main theorem:

Proof. Fix $x : \kappa \to \mathrm{Lim} \cap \kappa$ with $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$. For convenience, we will assume that x is strictly increasing and takes values in the regular cardinals.

<u>$(i) \Rightarrow (ii)$</u>: This is trivial.

<u>$(ii) \Rightarrow (iii)$</u>: Let $\pi : \left(\prod_{i<\kappa} x(i), \leq^*\right) \to (\kappa^\kappa, \leq^*)$ be a continuous cofinal embedding. After forcing with $\mathbb{H}_x$, let $f \in \prod_{i<\kappa} x(i)$ be the generic function. The continuity of π allows us to make sense of $\pi(f)$ and the cofinality of π will imply that $\pi(f)$ is dominating: We define $\pi(f)$ as

$$\bigcup \{g \mid \exists \alpha < \kappa \; \pi[N_{f \restriction \alpha}] \subseteq N_g\}$$

A simple density argument shows that $\pi(f)$ is indeed a function with domain κ. Now assume $g \in (\kappa^\kappa)^V$. Then we can find $\bar{g} \in \left(\prod_{i<\kappa} x(i)\right)^V$ so that $g \leq^* \pi(\bar{g})$. From another routine density argument it follows that $\pi(\bar{g}) \leq^* \pi(f)$ and hence $g \leq^* \pi(f)$.

<u>$(iii) \Rightarrow (iv)$</u>: We will prove the contraposition $\neg(iv) \Rightarrow \neg(iii)$. The crucial properties of $\mathbb{H}_x$ are that

(a) Any $(f, g), (f', g') \in \mathbb{H}_x$ with $f = f'$ are compatible.

(b) For any $\alpha < \kappa$ there are $<\kappa$-many f with $\mathrm{dom}(f) \subseteq \alpha$ such that $(f, g) \in \mathbb{H}_x$ for some g.

Note that (b) is not true for the unrestricted forcing $\mathbb{H}$. Now let $\dot{h}$ be a $\mathbb{H}_x$-name for a function $h : \kappa \to \kappa$. For $\alpha < \kappa$ let $\mathbb{H}_x^\alpha$ be the set of $(f, g) \in \mathbb{H}_x$ with $\mathrm{dom}(f) \subseteq \alpha$. We have to check that $\dot{h}$ is forced to not be dominating, so that we may as well assume

$$1_{\mathbb{H}_x} \Vdash \dot{h} \notin V \qquad (*)$$

(a) and (b) imply that for any $i, \alpha < \kappa$ there are $<\kappa$-many β such that some $(f, g) \in \mathbb{H}_x^\alpha$ decides $\dot{h}(\check{i})$ to be β. Let us define

$$F_i : \mathbb{H}_x \to \kappa, \; F_i((f, g)) = \min\{\alpha \mid \exists (f', g') \in \mathbb{H}_x^\alpha \text{ deciding } \dot{h}(\check{i}) \wedge (f', g') \leq (f, g)\}$$

and

$$H : \kappa \to \kappa, \; H(i) = \sup\{\beta \mid \exists (f, g) \in \mathbb{H}_x^{\sup F_i[\mathbb{H}_x^i]} \text{ deciding } \dot{h}(\check{i}) \text{ as } \beta\} + 1$$

It will turn out that $H \not\leq^* \dot{h}^G$ for any generic G. But first, we must check that H is well-defined and here we will use our assumption $\neg(iv)$. It is enough to verify the following claim:

Claim 11. $\sup F_i[\mathbb{H}_x^i] < \kappa$ *for any* $i < \kappa$.

Proof. So suppose $\sup F_i[\mathbb{H}_x^i] = \kappa$. There must then be a single first component f with $\mathrm{dom}(f) \leq i$ so that $F_i((f, g))$ gets arbitrarily large by varying g. Let

$$X_\alpha = \{g \in \prod_{i<\kappa} x(i) \mid F_i(f, g) \geq \alpha\}$$

Observe that membership of g in X_α depends only on $g \restriction \alpha$, so (abusing notation) we will say $g \restriction \alpha \in X_\alpha$. Consider the tree T whose α-th level T_α is exactly

$$T_\alpha = X_\alpha \cap \prod_{i<\alpha} x(i)$$

and is ordered by end-extension. Then T is indeed a tree, all levels have size $<\kappa$ and by our choice of f, T has height κ. Moreover, if $f, g \in \prod_{i<\alpha} x(i)$ and $f \leq g$ and $f \in T$ then $g \in T$ as well, since a larger second component only makes it more difficult to extend a condition in $\mathbb{H}_x$. So T satisfies all properties of an x-increasing κ-Aronszajn tree except maybe the non-existence of a cofinal branch. Thus by $\neg(iv)$, T must have cofinal branch. We may think of that branch as a function $g \in \bigcap_{\alpha<\kappa} X_\alpha$. But then $F_i((f, g)) \geq \kappa$, meaning (f, g) has no extension deciding the value of $\dot{h}(\check{i})$. This is clearly a contradiction. ☐

Finally H is not dominated (mod bounded) by $\dot{h}^G$ for any generic G, as the generic will often "pick minimal extensions deciding some value of $\dot{h}$": Let $j < \kappa$ and $p = (f, g) \in \mathbb{H}_x$. Then, by $(*)$ and as $\mathbb{H}_x$ is $<\kappa$-distributive, there must be some $\max\{j, \mathrm{dom}(f)\} < i < \kappa$ so that p does not decide $\dot{h}(\check{i})$. Now let $q \leq p$ be a strengthening with first component of length $F_i(p)$ so that q decides $\dot{h}(\check{i})$ as β. Thus $\beta < H(i)$ and this situation happens densely often.

<u>$(iv) \Rightarrow (i)$</u>: Let T be an x-increasing κ-Aronszajn tree. As a warm up, we describe a continuous cofinal embedding

$$\pi_0 : \left(\prod_{i<\kappa} x(i), \leq\right) \to (\kappa, \leq)$$

The final one will be defined by "squeezing κ-many embeddings similar to π_0 together". This process will turn the $\leq$ into a $\leq^*$ on the left hand side. The map π_0 is simply given by

$$\pi_0(f) = \min\{\alpha < \kappa \mid f \restriction \alpha \notin T\}$$

The minimum is taken over a non-empty set as T has no cofinal branch, so π_0 is welldefined. Also, if $f \leq g \in \prod_{i<\kappa} x(i)$ then $\pi_0(f) \leq \pi_0(g)$. This is a consequence of condition (ii) in the definition of a x-increasing κ-Aronszajn tree. Furthermore, π_0 is cofinal, since T has height κ. It is also easy to see that π_0 is continuous. Now, let's start to construct the final π. By Lemma 9, there is an increasing $y \in \kappa^\kappa$ with $y < x$. We can choose for any $i < \kappa$ a sequence

$$\langle \gamma^i_j \mid j < y(i) \rangle$$

so that any $\gamma^i_j : x(i) \to x(i)$ is strictly increasing and so that

$$\langle \mathrm{ran} \gamma^i_j \mid j < y(i) \rangle$$

is a partition of $x(i)$. Using this, we can turn any $\gamma < x(i)$ into a sequence in $x(i)^{y(i)}$ by mapping γ to

$$\Omega_i(\gamma) := \langle \min\{\xi < x(i) \mid \gamma < \gamma^i_j(\xi)\} \mid j < y(i) \rangle$$

Note that $\gamma \leq \delta < x(i)$ implies $\Omega_i(\gamma) \leq \Omega_i(\delta)$. Putting this together, we turn any $f \in \prod_{i<\kappa} x(i)$ into a map in $\prod_{i<\kappa} x(i)^{y(i)}$:

$$\Omega(f) = \langle \Omega_i(f(i)) \mid i < \kappa \rangle$$

Via

$$\prod_{i<\kappa} x(i)^{y(i)} \cong \prod_{i<\kappa} \prod_{j<y(i)} x(i) \cong \prod_{j<\kappa} \prod_{i<\kappa:\ j<y(i)} x(i)$$

we can also understand $\Omega(f)$ as a sequence

$$\Omega(f) \cong \langle f_j \mid j < \kappa \rangle$$

where

$$\mathrm{dom}(f_j) = \{i < \kappa \mid j < y(i)\}$$

and for $i \in \mathrm{dom}(f_j)$

$$f_j(i) = \Omega(f)(i)(j)$$

Now for any $j < \kappa$ choose some $g_j \in T_{\delta_j}$, where δ_j is the least i with $j < y(i)$, so that g_j has extensions to arbitrarily high levels in T (equivalently $T \restriction g_j$ is an x-increasing κ-Aronszajn tree). Note that $g_j^\frown f_j \in \prod_{i<\kappa} x(i)$. Finally set

$$\pi(f) = \langle \pi_0(g_j^\frown f_j) \mid j < \kappa \rangle$$

Claim 12. *π is a continuous cofinal (w.r.t. both $\leq$ and $\leq^*$ and preserves $\leq$ and $\leq^*$.*

Proof. It follows from the other arguments that π is continuous, so we will not make that explicit. First let's see that

$$\pi(f) =^* \pi(f')$$

if $f =^* f'$ and $f, f' \in \prod_{i<\kappa} x(i)$. The reason is simply that

$$\pi(f)(j) = \pi_0(g_j^\frown f_j)$$

only depends on $f \restriction [\delta_j, \kappa)$ and $\lim_{j<\kappa} \delta_j = \kappa$. Moreover, if only $f \leq^* f'$ then $\pi(f) \leq^* \pi(f')$ since the maps Ω_i are increasing. This also shows that $\pi(f) \leq \pi(f')$ if $f \leq f'$. Finally, let's check for cofinality. Note that it is enough to check cofinality w.r.t. $\leq$. Let $h \in \kappa^\kappa$. For $j < \kappa$, pick $\hat{g}_j$ so that

$$g_j^\frown \hat{g}_j \in T_{h(j)}$$

This is possible by our choice of g_j. We will construct $f \in \prod_{i<\kappa} x(i)$ so that for all $j < \kappa$,

$$\hat{g}_j \leq f_j \restriction \operatorname{dom}(\hat{g}_j)$$

From the properties of π_0, it will then follow that even $\pi(f) \geq h$. Simply put

$$f(i) = \sup\{\sup \gamma_j^i[\hat{g}_j(i)] \mid j < x^-(i) \wedge i < h(j)\} + 1$$

for any $i < \kappa$. $f \in \prod_{i<\kappa} x(i)$ as the inner suprema are $< x(i)$ as they are taken over a set of size $<x(i)$ and the outer supremum is $<x(i)$ as $x^-(i) < x(i)$. Note that $x(i)$ is regular. By chasing definitions, one can indeed verify that $f_j(i) \geq \hat{g}_j(i)$ whenever this makes sense. □

This proves (i).
Finally, we only needed the full assumption on x in $(iv) \Rightarrow (i)$. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ all go through as long as $\operatorname{cof} \circ x$ converges to κ. □

3 Producing x-increasing κ-Aronszajn Trees

Theorem 4 would not be that interesting if the three equivalent conditions there were always false (like they are in case κ is weakly compact). The fourth condition phrases the other ones in terms of existence of familiar set-theoretical objects, namely κ-Aronszajn trees, and so this gives a natural point of attack to force the conditions true. It should also be pointed out here that Cummings-Shelah [CS95] proved that if $\mathbb{Q}$ is a wellfounded poset with $\mathfrak{b}(\mathbb{Q}) \geq \kappa^+$ then there is a forcing extension in which $\mathbb{Q}$ can be cofinally embedded into $(\kappa^\kappa, \leq^*)$. This is however unapplicable to our current situation for multiple reasons. First of all, in general $\left(\prod_{i<\kappa} x(i), \leq^*\right)$ need not satisfy any of the assumptions on $\mathbb{Q}$ in that theorem even for reasonable x. But more importantly, the structure $\left(\prod_{i<\kappa} x(i), \leq^*\right)$ has new objects in the forcing extension, so embedding the ground model structure is different from embedding the new structure. We will now prove Theorem 8.

Proof. Again we will assume for convenience that x is strictly increasing and takes values in the regular cardinals. We modify the forcing, due to Jech, Prikry and Silver, that adds a κ-Souslin tree. Let $\mathbb{P}$ be the forcing whose conditions are subtrees t of $\bigcup_{\alpha<\kappa}\prod_{i<\alpha}x(i)$ with the following properties:

(*i*) $\gamma_t :=$ height(t) is a successor $< \kappa$.

(*ii*) t is a normal tree, i.e. all nodes not on the top level branch and can be extended to the top level.

(*iii*) If $f, g \in \prod_{i<\alpha}x(i)$ and $f \leq g$ then $f \in t \Rightarrow g \in t$.

$\mathbb{P}$ is ordered by $s \leq t$ iff s is an end-extension of t, i.e. $t = s_{\leq\gamma_t}$. We will show that if G is $\mathbb{P}$-generic then $T_G = \bigcup G$ is an x-increasing κ-Aronszajn tree and that $\mathbb{P}$ is $<\kappa$-progressively-closed.

Claim 13. *Any $t \in \mathbb{P}$ can be extended to some s with γ_s arbitrarily high below κ.*

Proof. Let $\gamma_t < \alpha < \kappa$. Put $s = \{g \in \prod_{i<\alpha+1}x(i) \mid g \restriction \gamma_t \in t\}\downarrow$, where $\downarrow$ denotes the closure under initial segments. Then $s \in \mathbb{P}$, $s \leq t$ and $\gamma_s = \alpha + 1$. □

Assume that $\delta < \kappa$ and that $\vec{t} = \langle t_i \mid i < \delta\rangle$ is a strictly decreasing sequence in $\mathbb{P}$. Note that $\vec{t}$ has a lower bound in $\mathbb{P}$ iff in $t_* := \bigcup_{i<\delta} t_i$ every node can be extended to a cofinal branch. A lower bound in this situation is given by $b(\vec{t}) := s$, the downward-closure of all cofinal branches through t_*. However, sometimes we would like to restrict the top level further. If $f \in \prod_{i<\gamma_s}x(i)$ then the downwards-closure of

$$\{g \in s_{\gamma_s-1} \mid f <^* g\}$$

is also a lower bound of $\vec{t}$ which we denote by $b(\vec{t}, f)$. Observe that $f \notin b(\vec{t}, f)$.

Claim 14. *$\mathbb{P}$ is $<\kappa$-progressively-closed.*

Proof. Let $\alpha < \kappa$. Let t_0 be a condition with $\gamma_t > \alpha$. By the claim before the set of such conditions is dense, so it is enough to show that $\mathbb{P}$ is α-closed below t_0. So assume that $\vec{t} = \langle t_i \mid i < \alpha\rangle$ is a decreasing sequence of conditions in $\mathbb{P}$. It is enough to show that every condition in $t_* = \bigcup_{i<\alpha} t_i$ can be extended to a cofinal branch. So let $f_0 \in t_*$ be a node. Since all t_i are normal, we may assume that $f = f_j$ is on the top level of some t_j, in particular dom$(f_0) > \alpha$. For $j \leq i < \alpha$ choose f_j an end-extension of f to the top level of t_i. Then if f_* is defined as the pointwise supremum of the $(f_i)_{j\leq i<\alpha}$ then $f_* \in \prod_{i<\gamma_*}x(i)$ as all the $x(i)$ with $j \leq i < \alpha$ have high cofinality. Moreover, f_* is a cofinal branch through t_* by property (*iii*) of forcing conditions. □

It follows that κ remains inaccessible in $V[G]$ and that T_G is a κ-tree. Since $\mathbb{P}$ is $<\kappa$-progressively-closed, $\mathbb{P}$ does not add new $<\kappa$-sequences of ordinals and hence it is trivial to check that if $f, g \in \prod_{i<\alpha} x(i)$ with $f \leq g$ then $f \in T_G \Rightarrow g \in T_G$.

Claim 15. *In $V[G]$, there is no cofinal branch through T_G.*

Proof. Assume that $t_0 \Vdash$ "$\dot{b}$ is a cofinal branch through $\dot{T}_G$", where $\dot{T}_G$ is the canonical name for T_G. We may assume that $\mathbb{P}$ is σ-closed below t_0 (in fact the whole $\mathbb{P}$ is σ-closed). Construct a strictly descending sequence $\langle t_i \mid i < \omega \rangle$ in $\mathbb{P}$ as well as a sequence $\langle f_i \mid i < \omega \rangle$ such that always

$$t_{i+1} \Vdash \dot{b} \cap (t_i)_{\gamma_{t_i}} = \{\check{f}_i\}$$

Note that the f_i is an end-extension of f_j if $j \leq i < \omega$. Hence $f = \bigcup_{i<\omega} f_i$ is a cofinal branch through $t_* = \bigcup_{i<\omega} t_i$. Then $b(\vec{t}, f)$ is a lower bound of $\vec{t}$, however as $f \notin b(\vec{t}, f)$

$$b(\vec{t}, f) \Vdash \check{f} \notin \dot{T}_G$$

but on the other hand $b(\vec{t}, f)$ extends all t_i so that

$$b(\vec{t}, f) \Vdash \check{f} \in \dot{b}$$

This is clearly a contradiction. □

We have demonstrated that T_G is an x-increasing κ-Aronszajn tree in $V[G]$. □

One could try to adjust the above approach to get an x-increasing κ-*Souslin* tree. That, however, would be a fruitless endeavor.

Lemma 16. *Assume $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$. Then no x-increasing κ-Aronszajn tree is κ-Souslin.*

Proof. Using Lemma 9, find $y : \kappa \to \kappa$ increasing unbounded, $y < \mathrm{cof} \circ x$. For $i < \kappa$, let α_i be the least α with $y(\alpha) \geq i$. Also choose $f_i \in T_{\alpha_i+1}$. By induction on i, define $g_i \in \prod_{j \leq \alpha_i} x(j)$ by

$$g_i(j) = \sup\{f_i(j), g_{i'}(j) \mid i' \leq y(j) \wedge i' < i\} + 1$$

Claim 17. *$A := \{g_i \mid i < \kappa\}$ is an antichain in T.*

Proof. First of all $g_i \in \prod_{j \leq \alpha_i} x(j)$ for all i since $g_i(j)$ is defined as the supremum of a set of size $y(j) < \mathrm{cof}(x(j))$ of ordinals below $x(j)$ $(+1)$ so that $g_i(j) < x(j)$. Since $g_i \geq f_i$ and $f_i \in T$ we have $g_i \in T$ as well. Now suppose $i' < i < \kappa$. Then

$$g_i(\alpha_{i'}) > g_{i'}(\alpha_{i'})$$

by definition of g_i since $i' \leq y(\alpha_{i'})$ and hence g_i and $g_{i'}$ are incompatible in T. □

This shows that A is an antichain of size κ. $\square$

Next up, we consturct x-increasing κ-Aronszajn trees from inifitary combinatorial principles.

Definition 18. [BR17] $\boxtimes^-(\kappa)$ postulates the existence of a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ so that for all $\alpha < \kappa$:

(*i*) $C_\alpha \subseteq \alpha$

(*ii*) If $\alpha \in \mathrm{Lim}$ then C_α is club in α.

(*iii*) If $\beta \in \mathrm{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$.

(*iv*) If $B \subseteq \kappa$ is any cofinal subset then there are stationarily many $\beta < \kappa$ with

$$\sup[(C_\beta \setminus \mathrm{Lim}(C_\beta) \cap B)] = \beta$$

i.e. the "successor points" in C_β meet B cofinally often below β.

Lemma 19. *Assume* $\boxtimes^-(\kappa)$+$\diamondsuit_\kappa$. *Then there is an* x*-increasing* κ*-Aronszajn tree.*

Proof. We roughly follow the construction in [BR17, Proposition 2.3]. Note, however, that a κ-Souslin tree is produced in the reference while the tree here will and must have large antichains in the end.
Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ witness $\boxtimes^-_\kappa$ and let $\langle g_\alpha \mid \alpha < \kappa\rangle$ witness $\diamondsuit_\kappa$ in the sense that any g_α is in α^α and for any $g \in \kappa^\kappa$ the set $\{\alpha < \kappa \mid g \restriction \alpha = g_\alpha\}$ is stationary. Furthermore, fix a wellorder $\trianglelefteq$ of $\bigcup_{\alpha<\kappa}\prod_{i<\alpha} x(i)$.
We construct a normal x-increasing tree T by induction on the levels. We let $T_0 = \{\emptyset\}$ and if T_α is defined then

$$T_{\alpha+1} = \{f \in \prod_{i<\alpha+1} x(i) \mid f \restriction \alpha \in T_\alpha\}$$

It remains to define the limit levels of T. There, our task is to decide which cofinal branches through $T^+_\alpha := \bigcup_{\beta<\alpha} T_\beta$ remain inside the tree. We must make sure that T_α is normal, so for any $t \in T^+_\alpha$ we will construct a cofinal branch b^α_t through T^+_α. Since C_α is unbounded in α, it is enough to do so assuming $t \in T \restriction C_\alpha$. Then we will set

$$T_\alpha = \{b^\alpha_t \mid t \in T \restriction C_\alpha\} \uparrow$$

where $\uparrow$ denotes the upwards closure under $\leq$ to make sure that T will be x-increasing in the end.
Let $\epsilon = \mathrm{dom}(t) \in C_\alpha$ and let $\langle \delta_i \mid i \leq \gamma\rangle$ be the increasing enumeration of $C_\alpha \cup \{\alpha\} \setminus \epsilon$. By induction, construct an increasing sequence $\langle t^\alpha_i \mid i \leq \gamma\rangle$ in T so that

(*i*) $t_0^\alpha = t$

(*ii*) $\forall i < \gamma\ t_i^\alpha \in T_{\delta_i}$

(*iii*) $\forall i < \gamma\ t_{i+1}^\alpha \in Q_{t,i}^\alpha := \{s \in T_{\delta_{i+1}} \mid g_{\delta_{i+1}} <^* s\}$ if $Q_{t,i}^\alpha \neq \emptyset$

(*iv*) $\forall i \in \mathrm{Lim} \cap \gamma + 1\ t_i^\alpha = \bigcup_{j<i} t_j^\alpha$

(*v*) Any t_i^α is $\trianglelefteq$-least with the above properties (relative to $(t_j^\alpha)_{j<i}$).

This construction is straightforward provided that at limit steps below γ we stay inside T. In this case, we set $b_t^\alpha = t_\gamma^\alpha$. We will now check that indeed the construction does not break down. So let $j < \gamma$ be limit.

Claim 20. $t_j^\alpha \in T_{\delta_j}$

Proof. We take a close look at the construction of T_β for $\beta = \delta_j$: Note that $C_\beta = C_\alpha \cap \beta$ so that $t \in T \upharpoonright C_\beta$. We thus have chosen an extension b_t^β of t to level β that we have put in T_β. We have done so by constructing an increasing sequence $\langle t_i^\beta \mid i \leq j\rangle$ w.r.t. the increasing enumeration $\langle \delta_i \mid i \leq j\rangle$ of $C_\beta \cup \{\beta\} \setminus \epsilon$. At each step $i \leq j$ we chose t_i^β as $\trianglelefteq$-minimal w.r.t. a property only depending on δ_i and g_{δ_i}. Hence $t_i^\beta = t_i^\alpha$ for $i \leq j$ and $t_j^\alpha = b_t^\beta \in T_\beta$. □

Now $T = \bigcup_{\alpha<\kappa} T_\alpha$ is a κ-tree. It is clear from the construction that T is x-increasing, so it remains to show that T has no cofinal branch. So assume otherwise that g is a function with $g \upharpoonright \alpha \in T_\alpha$ for all $\alpha < \kappa$. Thus there is a cofinal (even stationary) set

$$B = \{\alpha < \kappa \mid g_\alpha = g \upharpoonright \alpha\}$$

By the properties of a $\boxtimes^-(\kappa)$-sequence, we can find some $\alpha < \kappa$ so that

$$\sup[(C_\alpha \setminus \mathrm{Lim}(C_\alpha)) \cap B] = \alpha$$

Since $g \upharpoonright \alpha \in T_\alpha$, there must be some $t \in T \upharpoonright C_\alpha$ so that $g \upharpoonright \alpha \geq b_t^\alpha$. Let $\langle \delta_i \mid i \leq \gamma\rangle$ and $\langle t_i \mid i \leq \gamma\rangle$ be the sequences from the construction of b_t^α. Now there must be some $i < \gamma$ so that $\delta_{i+1} \in B$ and hence $g_{\delta_{i+1}} = g \upharpoonright \delta_{i+1}$. Note that $Q_{t,i}^\alpha \neq \emptyset$ as T is x-increasing and hence

$$g \upharpoonright \delta_{i+1} = g_{\delta_{i+1}} <^* t_{\delta_{i+1}} \subseteq t_\gamma = b_t^\alpha \leq g \upharpoonright \alpha$$

a contradiction. □

Remark 21. Surprisingly, the above construction works for any $x : \kappa \to \mathrm{Lim} \cap \kappa$.

Corollary 22. *If $V = L$ and $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right) \geq \kappa^+$ then (i) – (iv) of Theorem 4 are equivalent to κ not being weakly compact.*

Proof. Under $V = L$, $\diamondsuit_\kappa$ holds as κ is regular uncountable, while $\boxtimes^-(\kappa)$ is equivalent to κ not being weakly compact (see Proposition 1.9 and Corollary 1.10 in [BR17]). So if κ is not weakly compact then there is an x-increasing κ-Aronszajn tree by Lemma 19 and if κ is weakly compact there are no κ-Aronszajn trees at all. $\square$

4 The Other Direction

One can also ask the inverse question of (*ii*) in Theorem 4, i.e. does the unrestricted Hechler forcing at κ add a $\prod_{i<\kappa} x(i)$-dominating function. It is easy to see that this is impossible if x is unreasonable. We want to point out that also for reasonable x it is consistent, at least relative to a Mahlo cardinal, that $\mathbb{H}$ does *not* add such a function.

Lemma 23. *Let κ be Mahlo and x be the increasing enumeration of regular cardinals. Then $\mathbb{H}$ does not add a $\prod_{i<\kappa} x(i)$-dominating function.*

Proof. The set $S = \{i < \kappa \mid i = x(i)\}$ is stationary in κ and note that any $f \in \prod_{i<\kappa} x(i)$ is regressive on S. Assume that after forcing with $\mathbb{H}$ there is $f \in \prod_{i<\kappa} x(i)$ dominating all such functions in V (mod bounded). Then $f^{-1}(\{\xi\})$ is bounded and thus non-stationary for any $\xi < \kappa$. By Fodor's Lemma this means that S is no longer stationary. However, $\mathbb{H}$ is $<\kappa$-closed and so preserves the stationarity of S, contradiction. $\square$

Remark 24. The above argument also shows that $\mathbb{H}_x$ is an example of a $<\kappa$-progressively-closed forcing that destroys a stationary set. Hence, in general, $<\kappa$-progressive-closure is, in contrast to $<\kappa$-closure, not enough to preserve stationary sets in κ.

5 Con($\mathfrak{b}_\kappa < \mathfrak{b}_x$)?

For $x : \kappa \to \mathrm{Lim} \cap \kappa$ let us abbreviate $\mathfrak{b}\left(\prod_{i<\kappa} x(i), \leq^*\right)$ by $\mathfrak{b}_x$.

Question 25. Is $\mathfrak{b}_\kappa < \mathfrak{b}_x$ consistent?

We will present a construction in an effort to answer the above question. There are two possible outcomes: Either the construction succeeds and results in a model with $\mathfrak{b}_\kappa < \mathfrak{b}_x$ or the construction fails and gives an example where a preservation theorem which holds at ω for finite support iterations of ccc forcings fails at κ for a $<\kappa$-support iteration of κ^+-cc forcings. At the moment it is not clear which direction it will go. We will prove the following:

Theorem 26. *It is consistent relative to an unfoldable cardinal that $2^\kappa = \kappa^+$ and there is $x : \kappa \to \mathrm{Lim} \cap \kappa$ so that if*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa^{++} \rangle$$

is the $<\kappa$*-support iteration of* $\mathbb{H}_x$ *(as defined in the extension) of length* κ^{++} *then:*

(i) $\mathbb{P}$ *is* κ^+*-cc*

(ii) $\mathbb{P}$ *preserves all cofinalities*

(iii) for any $\alpha < \kappa^{++}$*:*

$$\mathbb{P}_\alpha \Vdash \text{“}\mathbb{H}_{\check{x}} \textit{ does not add a } \check{\kappa}^{\check{\kappa}}\textit{-dominating function”}$$

In fact, κ^{++} can be replaced by any ordinal.

Calculating cardinal characteristics of the continuum in forcing extensions by finite or countable support iteration often makes use of preservation theorems that usually are of the form “if $\mathbb{P}$ is an iteration with a specific support and all the iterands are nice then $\mathbb{P}$ does not add a certain object”. For example with the following preservation theorem one can prove that sometimes the ground model functions in ω^ω stay an unbounded family in the extension:

Fact 27. *[JS90] If* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \beta \rangle$ *is a ccc finite support iteration so that for all* $\alpha < \beta$

$$\mathbb{P}_\alpha \textit{ adds no } \omega^\omega\textit{-dominating function}$$

Then $\mathbb{P}$ *adds no* ω^ω*-dominating function.*

One can ask whether the natural generalisation of this fact on κ is still true. We replace ccc by κ^+-cc and finite support by $<\kappa$-support.

Question 28. Suppose $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \beta \rangle$ is a κ^+-cc $<\kappa$-support iteration such that for any $\alpha < \beta$:

$$\mathbb{P}_\alpha \text{ adds no } \kappa^\kappa\text{-dominating function”}$$

Does $\mathbb{P}$ add no κ^κ-dominating function?

Assume we are in the situation of Theorem 26. Let $\mathbb{P}$ be that iteration of length κ^{++}. Then $\mathbb{P}$ is a κ^+-cc iteration with $<\kappa$-support so that at no iterand adds a κ^κ-dominating function. If we extend by $\mathbb{P}$ there are two possible outcomes:

(*i*) $\mathbb{P}$ indeed does not add a κ^κ-dominating function so that in the extension by $\mathbb{P}$, $\mathfrak{b}_\kappa = \kappa^+ < \kappa^{++} = \mathfrak{b}_x$ or

(*ii*) $\mathbb{P}$ adds a κ^κ-dominating function so that in this situation some limit stage of $\mathbb{P}$ is a counterexample to Question 28.

Both outcomes seem to be mathematically interesting.

Question 29. Which one is it?

Proposition 30. *If $\mathfrak{b}_x \geq \kappa^+$ then $\mathbb{H}_x$ is $<\kappa$-strategically closed.*

Proof. By Lemma 9 there is an increasing cofinal function $y : \kappa \to \kappa$ so that $y < \mathrm{cof} \circ x$. Let σ be the strategy of player II that:

- at successor steps depends only on the last move (s, f) of player I and the round i of the game and extends (s, f) to (t, f) with the domain α of t so large that $y(\alpha) \geq i$
- at limit steps takes any lower bound (if possible).

We have to show that if II follows σ then the lower bound at limit steps exists. So let $\beta \in \mathrm{Lim} \cap \kappa$ and $\langle (s_i, f_i) \mid i < \beta \rangle$ a descending sequence in $\mathbb{H}_x$ that is according to σ. Let $s_* = \bigcup_{i<\beta} s_i$. This is clearly a function in $\prod_{i<\alpha} x(i)$ for some $\alpha < \kappa$. By the definition of σ and since y is increasing we must have $y(\alpha) \geq \beta$ and hence $\mathrm{cof}(x(i)) > \beta$ for $\alpha \leq i < \kappa$. Hence we can define $f \in \prod_{i<\kappa} x(i)$ to be 0 below α and

$$f(i) = \sup_{j<\beta} f_j(i)$$

for $\alpha \leq i < \kappa$. Now, (s_*, f_*) is a condition in $\mathbb{H}_x$ and it is clear that it is a lower bound of $\langle (s_i, f_i) \mid i < \beta \rangle$. □

The following two propositions are briefly sketched in the case of (unrestricted) generalised Hechler forcing in [BBTFM18].

Proposition 31. *Assume x satisfies $\mathfrak{b}_x \geq \kappa^+$ and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \beta \rangle$ be the $<\kappa$-support iteration of $\mathbb{H}_x$ for some β. Then the conditions $p \in \mathbb{P}$ so that for all $\alpha \in \mathrm{supp}(p)$, $p \restriction \alpha$ decides the first component of $p(\alpha)$ is dense.*

Proof. We prove this by induction on β. The only non-trivial step is when β is a limit of cofinality $<\kappa$, so we restrict to this case. Let $\gamma = \mathrm{cof}(\beta)$ and find an increasing cofinal sequence $\langle \alpha_i \mid i < \gamma \rangle$. Let $p_0 \in \mathbb{P}$. We construct an descending sequence $\langle p_i \mid i < \gamma \rangle$, writing $p_i(\alpha) = (\dot{s}_i^\alpha, \dot{f}_i^\alpha)$ for $\alpha \in \mathrm{supp}(p_i)$, satisfying for any $i < \gamma$ and $\alpha \in \mathrm{supp}(p_i)$

(*i*) If $\alpha < \alpha_j$ for some $j < i$ then $p_i \restriction \alpha \| \dot{s}_i^\alpha$.

(*ii*) $p(\alpha)$ is forced by $p \restriction \alpha$ to be in the dense open set of conditions in $\dot{\mathbb{Q}}_\alpha$ below which $\dot{\mathbb{Q}}_\alpha$ is $\leq \gamma$-closed.

The construction is straightforward using the induction hypothesis and the fact that each $\dot{\mathbb{Q}}_\alpha$, $\alpha < \beta$, is (forced to be) $<\kappa$-progressively-closed. Finally we can define a condition p_* with support $\bigcup_{i<\gamma} \mathrm{supp} p_i$ so that for $\alpha \in$

$\mathrm{supp}(p_*)$, $p_*(\alpha) = (\check{s}_*, \dot{f}_*)$ with s_* the union of the (eventual) decisions of what $\dot{s}_i^\alpha$ is and $\dot{f}_*$ the name for the function that is 0 on $\mathrm{dom}(s_*)$ and above that the supremum of $\dot{f}_i^\alpha$ for $i < \gamma$ (whenever $\dot{s}_i^\alpha$ resp. $\dot{f}_i^\alpha$ is defined). p_* has the desired property by definition and is below p_0 by construction. □

Proposition 32. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \beta \rangle$ be the $<\kappa$-support iteration of $\mathbb{H}_x$ (as defined in the extensions) of length β. $\mathbb{P}$ is κ^+-cc.*

Corollary 33. *If $\mathfrak{b}_x \geq \kappa^+$ then $<\kappa$-support iterations of $\mathbb{H}_x$ preserve all cofinalities.*

Proof. These iterations preserve cofinalities $\geq \kappa^+$ as they have the κ^+-cc. Using Proposition 30 and Lemma 9, one can show by induction on the length of the iteration that $\mathfrak{b}_x$ stays larger than κ in the extension, so that $\mathbb{H}_x$ stays $<\kappa$-strategically closed so that the whole iteration is $<\kappa$-strategically closed and preserves all cofinalities $\leq \kappa$. □

Proof. By Proposition 31, we can assume that for $p \in \mathbb{P}$ and $i \in \mathrm{supp}(p)$, $p(i)$ is of the form $(\check{s}_i^p, \dot{f}_i^p)$ with $s_i^p \in \prod_{i<\alpha} x(i)$ for some $\alpha < \kappa$ and $\dot{f}_i^p$ a $\mathbb{P}_i$-name for a function in $\prod_{i<\kappa} x(i)$. Suppose $A \subseteq \mathbb{P}$ is of size κ^+. By the Δ-system lemma we may assume that $\{\mathrm{supp}(p) \mid p \in A\}$ is a Δ-system with some root r. Since r is of size $<\kappa$ there are at most $\kappa^{|r|} = \kappa$-many possibilities of

$$\langle s_i^p \mid i \in r \rangle$$

if p ranges over A. Thus there are p, q for which the respective sequences coincide. p and q must be compatible: They are pointwise compatible outside r as at least one of them is trivial there and on r since two conditions in $\mathbb{H}_x$ are compatible if their first coordinates coincide. So A is not an antichain. □

We will produce a model where κ is weakly compact, $2^\kappa = \kappa^+$ and the weak compactness of κ is not destroyed after iterating $\mathbb{H}_x$ with $<\kappa$-support at most κ^{++}-many times. The argument presented here seems to need slightly more than mere weak compactness.

Definition 34. Assume $\kappa \leq \theta$. κ is $\theta + 1$-strongly unfoldable if for any κ-model M there is an elementary embedding $j : M \to N$ with critical point κ and N transitive so that

(i) ${}^{\beth_\theta} N \subseteq N$

(ii) $|N| = \beth_{\theta+1}$

(iii) $j(\kappa) > \theta$

This is not the standard definition of $\theta + 1$-strong unfoldability, but it is equivalent (see [DH06, Lemma 5]). It is easy to see if κ is $\theta + 1$-strongly unfoldable for any $\kappa \leq \theta$ then κ is weakly compact as well.

Lemma 35. *Suppose $\mathfrak{b}_x \geq \kappa^+$, κ is $\kappa+2$-strongly unfoldable and* GCH *holds. Then there is a preparatory forcing after which the weak compactness of κ is indestructible by the $<\kappa$-support iteration of $\mathbb{H}_x$ of length κ^{++}.*

Note that in general this iteration can destroy the weak compactness of κ. For example over L already $\mathbb{H}_x$ does so as it adds a new subset of κ all of which initial segments lie in L.

Proof. Parts of the argument is inspired by the proof of the Main Theorem in [HJ10]. Note, however, that the Main Theorem there only applies to $<\kappa$-closed forcings (that preserve κ^+) while $\mathbb{H}_x$ is not $<\kappa$-closed.
Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa \rangle$ be the Easton-support iteration where $\dot{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for

- the trivial forcing if α is not inaccessible or $x_\alpha := x \restriction \alpha$ fails to be a function with range in α so that $\mathfrak{b}_{x_\alpha} \geq \alpha^+$ and
- for the $<\alpha$-support iteration of $\mathbb{H}_{x_\alpha}$ of length α^{++} otherwise.

We will show that $\mathbb{P}$ is the preparatory forcing we seek. So let G be $\mathbb{P}$-generic over V. In $V[G]$, let $\mathbb{Q}$ be the $<\kappa$-support iteration of $\mathbb{H}_x$ of length κ^{++}. We will show that in $V[G]^{\mathbb{Q}}$, κ is weakly compact. So let g be $\mathbb{Q}$-generic over $V[G]$. Let A be a subset of κ in $V[G*g]$ and let $\dot{A} \in V$ be a name for A. Also let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{Q}$. In V, find a large regular θ and an elementary substructure $X \prec H_\theta$ so that

(*i*) $\mathbb{P}, \dot{A} \in X$ and $\kappa + 1 \subseteq X$

(*ii*) ${}^{<\kappa}X \subseteq X$

(*iii*) $|X| = \kappa$

By Proposition 32, $\mathbb{P} * \dot{\mathbb{Q}}$ is κ^+-cc and thus $G*g$ is generic over X. Let M be the Mostowski collapse of X with $\pi : X \to M$ the collapse map. Since κ^+ is absolute between V and $V[G*g]$, M is of size κ in V. Thus M is a κ-model in V and there is an elementary embedding $j : M \to N$ with critical point κ and N transitive with N closed under κ^+-sequences and of size $2^{2^\kappa} = \kappa^{++}$. Let $\dot{\mathbb{Q}}_0 = \pi(\dot{\mathbb{Q}})$, $\dot{A}_0 = \pi(\dot{A})$ and $g_0 = \pi[g]$ (observe that $G = \pi[G]$ is generic over M and that g_0 is $\dot{\mathbb{Q}}_0^G$ generic over $M[G]$).

Claim 36. *There is a lift $j^{++} : M[G * g_0] \to N[H * h]$ of j.*

Proof. First, we lift j to $j^+ : M[G] \to N[H]$. To do this, we are tasked to find a generic H over N for $j(\mathbb{P})$ with $j[G] \subseteq H$. Note that $j[G] = G$ is $\mathbb{P}$-generic over N. The iterand of $j(\mathbb{P})$ at κ is the $<\kappa$-support iteration of $\mathbb{H}_x$ of length κ^{++} (as computed in N, but the closure of N guarantees that this is the true $\mathbb{Q}$). This means we can factor $j(\mathbb{P})$ as $\mathbb{P} * \dot{Q} * \dot{\mathbb{R}}$ where the latter is now (forced to be) $<\kappa^{++}$-strategically closed in $N[G*g]$ and thus in

$V[G*g]$ as well (standard computations show that ${}^{\kappa^+}N[G*g] \subseteq N[G*g]$). As $N[G*g]$ still has size κ^{++}, one can build a generic H_0 inside $V[G*g]$. We conclude that $H = G*g*H_0$ is $j(\mathbb{P})$-generic over N with $j[G] \subseteq H$ which completes the first lift.
Next up we must further find a generic h for $j^+(\mathbb{Q}_0)$ with $j^+[g_0] \subseteq h$. We will do so via a master condition argument. Note that $\mathbb{Q}_N := j^+(\dot{\mathbb{Q}}_0^G)$ is the $<j(\kappa)$-support iteration of $\mathbb{H}_{j(x)}^{N[H]}$ of length $(j(\kappa)^{++})^{N[H]}$ as computed in $N[H]$. By Proposition 31, we can replace $\mathbb{Q}$ by the dense subset of p for which whenever $\alpha \in \operatorname{supp}(p)$, $p \restriction \alpha$ decides the first component of $p(\alpha)$. Thus we can identify $p(\alpha)$ with the tuple $(s, \dot{f})$ where s is that decision and $\dot{f}$ is (a name for) the second component of $p(\alpha)$. We can do the same thing for $j^+(\mathbb{Q})$. Let $S = j^+[\kappa^{++}] = j[\kappa^{++}]$. The closure of N guarantees $S \in N$ as well as $j^+[g] \in N$. For $i < \beta_0$, let f_i be the generic function added over M by the i-th coordinate of g_0. Now we define p_* to be the condition with support S so that for $j(i) \in S$, $p(j(i)) = (f_i, \dot{g}_i)$ where $\dot{g}_i$ is a $\mathbb{Q}_N \restriction j(i)$-name for the function that is

(*i*) 0 below κ

(*ii*) $\sup\{j(\dot{g})(\check{\alpha}) \mid \exists p \in g_0 \; i \in \operatorname{supp}(p) \wedge \exists s \; p(i) = (s, \dot{g})\}$ for $\kappa \leq \alpha < j(\kappa)$.

Claim 37. *$p_* \in \mathbb{Q}_N$ is below any condition in $j[g_0]$.*

Proof. First of all, $p_* \in N[H]$ as the closure of $N[H]$ in $V[G*g]$ ensures S as well as $j^+ \restriction g_0$ to be elements of $N[H]$. To show that p_* is a condition, it is enough to show that for $i \in S$, $\dot{g}_i$ is forced to be a function in

$$\left(\prod_{k<j(\kappa)} j(x)(k)\right)^{N[H]^{\mathbb{Q}_N \restriction i}}$$

Since there are at most κ-many possibilities for $\dot{g}$ in the definition of $\dot{g}_i(\alpha)$ for a given $\kappa \leq \alpha < j(\kappa)$ (as $M[G]$ has size κ) and since N knows this, it is enough to show that $\operatorname{cof}(j(x)(\alpha)) > \kappa$ in $N[H]^{\mathbb{Q}_N \restriction i}$. As cofinalities in N are preserved by $j(\mathbb{P}) * \mathbb{Q}_N$, it suffices to demonstrate

$$\operatorname{cof}^N(j(x)(\alpha)) > \kappa$$

Consider an increasing cofinal $y : \kappa \to \kappa$, $y \in M$ with $y < \operatorname{cof} \circ x$ guaranteed to exist by Lemma 9. Then $j(y) \in N$ is increasing and cofinal in $j(\kappa)$ with $j(y) < \operatorname{cof}^N \circ j(x)$ and moreover $j(y) \restriction \kappa = y$, so that $y(\kappa) \geq \kappa$. Hence:

$$\operatorname{cof}(j(x)(\alpha))^N > j(y)(\alpha) \geq j(y)(\kappa) \geq \kappa$$

Now let $q \in g_0$, $i \in \operatorname{supp}(q)$ and $q(i) = (s, \dot{g})$. Then s is an initial segment of f_i and $\dot{g}$ appears in the definition of $\dot{g}_i$ at and above κ. Since $p_*(j(i)) = (f_i, \dot{g}_i)$ and as $\dot{g}_i$ dominates $j^+(\dot{g})$ at and above κ we have

$$\Vdash_{\mathbb{Q}_N \restriction j(i)} p_*(j(i)) \leq j^+(s, \dot{g})$$

Since this is true for any i in the support of q, $p_* \leq j^+(q)$. $\square$

This means that p_* is a master condition in our situation, i.e. whenever h is generic for $\mathbb{Q}_N$ over $N[H]$ with $p_* \in h$, then $j^+[g_0] \subseteq h$. Constructing such an h can be done by a standard argument using

(i) ${}^{\kappa^+}N[H] \subseteq N[H]$ in $V[G * g]$

(ii) $|N[H]| = \kappa^{++}$ in $V[G * g]$

(iii) $N[H] \models$ "$\mathbb{Q}_N$ is $<j(\kappa)$-strategically closed"

(iv) $j(\kappa) > \kappa^{++}$

This yields the second and final lift $j^{++} : M[G * g_0] \to N[H * h]$. $\square$

Claim 38. $A \in M[G * g_0]$

Proof. Let $i < \kappa$. In M, Let B be a maximal antichain of conditions that decide whether or not $\check{i} \in \dot{A}_0$. Let $p \in G * g \cap \pi(B)$. Then p decides wheter $\check{i} \in \dot{A}$ depending on whether $i \in A$. As B is of size $\leq \kappa$ in M, $\pi(B)$ is of size $\leq \kappa$ and hence a subset of X. Thus $\pi(p) \in G * g_0$ decides correctly whether $\check{i} \in \dot{A}_0$. This shows $A = \dot{A}_0^{G*g_0} \in M[G * g_0]$. $\square$

Thus, in $V[G * g]$, for any subset of κ there is a κ-model that admits an elementary embedding into a transitive model with critical point κ. We conclude that κ is indeed weakly compact in $V[G * g]$. $\square$

Remark 39. Indeed κ remains $\kappa{+}2$-strongly unfoldable in $V[G{*}g]$, however this is not relevant to the discussion here. It is also possible to make the weak compactness of κ indestructible by longer iterations of $\mathbb{H}_x$ using a similar argument as above if one is willing to assume more strong unfoldability. In fact, one can do this for iterations of arbitrary length if κ is fully strongly compact to begin with. For the longer iterations one must incorporate Laver functions for unfoldability into the above argument.

Proof. (Of Theorem 26) Start with a model V of GCH in which κ is ($\kappa + 2$-)unfoldable. Let $x : \kappa \to \mathrm{Lim} \cap \kappa$ be any function with $\mathfrak{b}_x \geq \kappa^+$, for example the map $\alpha \to \alpha^+$. Apply Lemma 35 to get a model $V[G]$ so that if

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa^{++} \rangle$$

is the $<\kappa$-support iteration of $\mathbb{H}_x$ then κ is weakly compact in $V[G]^{\mathbb{P}}$. Note that $\mathbb{P}$ is κ^+-cc and $<\kappa$-strategically closed by Propositions 32 and 30.

Claim 40. *κ is weakly compact in $V[G]^{\mathbb{P}_\alpha}$ for any $\alpha < \kappa$.*

Proof. Since $\mathbb{P}$ does not add sequences of ordinals of length $<\kappa$, $\mathbb{P}$ factors as $\mathbb{P}_\alpha * \dot{\mathbb{P}}_{>\alpha}$ where the latter is forced to be a $<\kappa$-support iteration of $<\kappa$-strategically-closed forcings. Thus $V[G]^{\mathbb{P}}$ is an extension of $V[G]^{\mathbb{P}_\alpha}$ by $<\kappa$-strategically closed forcing and hence any κ-Aronszajn tree in $V[G]^{\mathbb{P}_\alpha}$ would still have no cofinal branch in $V[G]^{\mathbb{P}_\alpha}$, which is impossible. Thus there are no such trees in $V[G]^{\mathbb{P}_\alpha}$ which means that κ is weakly compact there. $\square$

By Theorem 4, this implies that for all $\alpha < \kappa^{++}$ we have

$$\mathbb{P}_\alpha \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \text{ does not add a } \check{\kappa}^{\check{\kappa}}\text{-dominating function”}$$

The Theorem is proven. $\square$

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