An Iteration Theorem for ω_1-preserving Forcings

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Abstract

We prove an iteration theorem which guarantees for a wide class of nice iterations of ω_1-preserving forcings that ω_1 is not collapse, at the price of needing large cardinals to burn as fuel. More precisely, we show that a nice iteration of ω_1-preserving forcings which force SRP at successor steps and preserves old stationary sets does not collapse ω_1.

1 Introduction

The method of iterated forcing is a powerful yet flexible tool in establishing independence results. Say, the goal is to produce a forcing extension of the universe with a specific property. Frequently, it is the case that it is much easier to find a forcing $\mathbb{P}$, which solves this problem for "a single instance" or "all instances in V", but may add new "unresolved instances" at the same time. One can then hope to iterate $\mathbb{P}$ up to some closure point, usually a sufficiently large regular cardinal κ so that the whole iteration is κ-c.c., so that in end all instances have been dealt with and the full desired property holds. This can only work if the iteration in question preserves the progress of earlier stages up until the end. Theorems which guarantee such a preservation are often called iteration theorems. If the property in question is one about H_{ω_2} then at the very least it is required that ω_1 is preserved or maybe the somewhat stronger property that stationary sets are preserved. We give some examples.

1.1 Iterations of c.c.c. Forcings

The earliest iteration theorem is due to Solovay-Tennenbaum.

Theorem 1.1 (Solovay-Tennenbaum, [ST71])**.** *Suppose $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a finite support iteration of c.c.c. forcings. Then $\mathbb{P}_\gamma$ is c.c.c..*

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A Suslin tree is a tree of height ω_1 with no uncountable chains and antichains. Forcing with a Suslin tree T (with the reverse order) is a c.c.c. forcing and in the forcing extension T is no longer Suslin. Not being a Suslin tree is a $\Sigma_1(\omega_1)$-property and so is upwards absolute to forcing extensions preserving ω_1. As c.c.c. forcings preserve ω_1, iterating forcing with Suslin trees produces models in which there are no Suslin tree and hence in which Suslin's hypothesis holds.

Theorem 1.2 (Solovay-Tennenbaum, [ST71])**.** *There is a c.c.c. forcing $\mathbb{P}$ so that $V^{\mathbb{P}} \models$ "Suslin's hypothesis".*

1.2 Iterations of Proper Forcings

The class of forcings with the countable chain condition is rather small, so not suitable in all cases. Shelah discovered the beautiful notion of proper forcing which is large enough to include both c.c.c. and σ-closed forcing, but nonetheless all such forcings preserve ω_1.

Definition 1.3 (Shelah,[She98])**.** A forcing $\mathbb{P}$ is **proper** if for any large enough regular θ and any countable $X \prec H_\theta$ with $\mathbb{P} \in X$, whenever $p \in \mathbb{P} \cap X$ then there is some $q \leqslant p$ with

$$q \Vdash \check{X} \cap \mathrm{Ord} = \check{X}[\dot{G}] \cap \mathrm{Ord}.$$

Shelah proved a famous iteration theorem for proper forcings. Though, as finite support iterations of non-c.c.c. forcings usually collapse ω_1, the preferred support in this instance is countable support.

Theorem 1.4 (Shelah,[She98])**.** *Suppose $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a countable support iteration of proper forcings. Then $\mathbb{P}_\gamma$ is proper.*

An Aronszajn tree T is a tree of height ω_1 with all countable levels and no cofinal branch. For a tree T of height ω_1 and $A \subseteq \omega_1$, let $T \upharpoonright A$ denote the tree with nodes of a level α of T with $\alpha \in A$ and the tree order inherited from T. Two trees S, T of height ω_1 are club-isomorphic iff there is a club $C \subseteq \omega_1$ so that $S \upharpoonright C \cong T \upharpoonright C$ as partial orders. Given two Aronszajn trees S, T, Abraham-Shelah discovered a proper forcing $\mathbb{P}(T, S)$ which forces S and T to be club-isomorphic. Note that the property "S, T are club-isomorphic" is $\Sigma_1(S, T, \omega_1)$ and thus upwards-absolute to any ω_1-preserving forcing extension.

Theorem 1.5 (Abraham-Shelah, [AS85])**.** *There is a proper forcing $\mathbb{P}$ so that*

$$V^{\mathbb{P}} \models \text{"Any two Aronszajn trees } S, T \text{ are club-isomorphic"}.$$

We remark that Suslin's hypothesis is an immediate consequence of "any two Aronszajn trees are club-isomorphic". There provably is an Aronszajn tree which is *special*, i.e. the union of countably many antichains. Such a tree cannot be club-isomorphic to a Suslin tree.

1.3 Iterations of Semiproper Forcings

Later, Shelah proved another iteration theorem for the even larger class of semiproper forcings.

Definition 1.6 (Shelah, [She98]). A forcing $\mathbb{P}$ is **semiproper** if for any large enough regular θ and any countable $X \prec H_\theta$ with $\mathbb{P} \in X$, whenever $p \in \mathbb{P} \cap X$ then there is some $q \leqslant p$ with

$$q \Vdash \check{X} \cap \omega_1 = \check{X}[\dot{G}] \cap \omega_1.$$

From now on, we will denote $X \subseteq Y \wedge X \cap \omega_1 = Y \cap \omega_1$ by $X \sqsubseteq Y$. So for example above we have $q \Vdash \check{X} \sqsubseteq \check{X}[\dot{G}]$.

Theorem 1.7 (Shelah). *Suppose $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a* RCS*-iteration of semiproper forcings. Then $\mathbb{P}_\gamma$ is semiproper.*

Once again, the notion of support had to be changed. In the argument of Theorem 1.7 it is crucial that if $\alpha < \gamma$ and G_α is $\mathbb{P}_\alpha$-generic over V, then the tail iteration $\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_\beta \mid \xi \leqslant \gamma, \beta < \gamma \rangle$ is still a RCS-iteration. This can fail for countable support iterations as, unlike proper forcings, semiproper forcings can turn regular cardinals into cardinals of countable cofinality. In fact, Theorem 1.7 fails if RCS-support is replaced with countable support.

Suppose $\mathcal{I}$ is an ideal on ω_1. An $\mathcal{I}$-antichain is a set $\mathcal{A} \subseteq \mathcal{P}(\omega_1) - \mathcal{I}$ so that $S \cap T \in \mathcal{I}$ for any $S \neq T \in \mathcal{A}$. The ideal $\mathcal{I}$ is saturated if for all $\mathcal{I}$-antichains $\mathcal{A}$ we have $|\mathcal{A}| \leqslant \omega_1$.

Theorem 1.8 (Shelah, see [Sch11] for a proof). *Assume there is a Woodin cardinal. Then there is a semiproper forcing $\mathbb{P}$ so that*

$$V^{\mathbb{P}} \models \text{“NS}_{\omega_1} \textit{ is saturated”}.$$

If $\mathcal{A}$ is a maximal NS_{ω_1}-antichain then the sealing forcing $\mathcal{S}_\mathcal{A}$ is a natural stationary-set-preserving forcing which turns $\mathcal{A}$ into a maximal NS_{ω_1}-antichain of size $\leqslant\omega_1$ and the statement "$\mathcal{A}$ is a maximal antichain of size $\leqslant\omega_1$" turns out to be $\Sigma_1(\mathcal{A}, \omega_1)$. Now, an instance of the sealing forcing is not semiproper in general, but Shelah shows that when iterating up to a Woodin cardinal and using a sealing forcing only when it is semiproper, it can be arranged that often enough sealing forcings are semiproper that in the end, NS_{ω_1} is saturated.

1.4 Iterations of Stationary-Set-Preserving Forcing

So what are the limits of iteration theorems? We have

$$\text{c.c.c} \Rightarrow \text{proper} \Rightarrow \text{semiproper} \Rightarrow \text{stationary set preserving.}$$

and none of the implications can be reversed. However, while there are always non-c.c.c. proper forcings and non-proper semiproper forcings, consistently the class of semiproper forcings can agree with the class of stationary set preserving forcing, so these two notions are quite close. Nonetheless, there is no analogue of Theorem 1.7 for stationary set preserving forcings. Consistently, a counterexample can be given along the lines of the discussion of Theorem 1.8. In the argument, the Woodin cardinal is used solely to verify that instances of sealing forcing are semiproper often enough, an inaccessible cardinal would suffice otherwise. But a Woodin cardinal is indeed required for the conclusion.

Theorem 1.9 (Steel, Jensen-Steel [JS13]). *Suppose that there is a normal saturated ideal on ω_1. Then there is an inner model with a Woodin cardinal.*

So suppose we work in an model without an inner model with a Woodin cardinal, say $V = L$, and there is an inaccessible cardinal. One could then try to iterate instances of the Sealing forcing along a suitable bookkeeping up to κ. In light of Theorem 1.9, this cannot result in a forcing extension in which NS_{ω_1} is saturated. It follows that the iteration collapses ω_1 at some point, yet instances of the sealing forcing are always stationary set preserving.

A much more serious example is due to Shelah.

Theorem 1.10 (Shelah [She98]). *There is a full support iteration*

$$:= \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$$

of stationary set preserving forcings so that $\mathbb{P}_\omega$ collapses ω_1.

In fact, in the above example it does not matter at all which kind of limit is taken, though we want to mention that countable support, RCS and full support iterations agree on length ω iterations. The first forcing in Shelah's example is semiproper, but all subsequent forcings are not semiproper in the relevant extension. Semiproper forcing is the correct **regularity** property for stationary set preserving forcings in terms of iterations in the sense that

1. all semiproper forcings are stationary set preserving,
2. consistently, all stationary set preserving forcings are semiproper and
3. semiproper forcings can be iterated.

We will define the class of **respectful** forcing which, in a slightly weaker sense, is a regularity property corresponding to the wider class of ω_1-preserving forcings.

1.5 Iterations of ω_1-Preserving Forcings

When iterating ω_1-preserving forcing which kill stationary sets there is another threat to preserving ω_1 in the limit as illustrated in the following folklore example: For $S \subseteq \omega_1$ stationary, the club shooting forcing $\mathrm{CS}(S)$ is the canonical forcing that shoots a club trough S. Conditions are closed countable sets $c \subseteq S$ ordered by $d \leqslant_{\mathrm{CS}(S)} c$ iff $d \cap (\max(c) + 1) = c$. If G is generic for $\mathrm{CS}(S)$ then $\bigcup G$ is a club contained in S, so $\omega_1 - S$ is nonstationary in $V[G]$, but ω_1 is not collapsed, that is $\omega_1^{V[G]} = \omega_1^V$. Now suppose $\langle S_n \mid n < \omega\rangle$ be a partition of ω_1 into stationary sets. Let $\mathbb{P}$ be a length ω iteration of the forcings $\mathrm{CS}(\omega_1 - S_n)$ (it does not matter which limit we take at ω). Then $\mathbb{P}$ must collapse ω_1 as in $V^{\mathbb{P}}$, $\omega_1^V = \bigcup_{n<\omega} S_n$ is a countable union of nonstationary sets and hence must be nonstationary itself. Clearly, this is only possible if $\omega_1^V < \omega_1^{V^{\mathbb{P}}}$.

The issue here does not stem from a lack of regularity of the forcings we used. In fact, for a stationary set $S \subseteq \omega_1$, the club shooting $\mathrm{CS}(S)$ is S-proper. The problem is much more that at each step of the iteration, we come back to V to kill an "old" stationary set. If we avoid the two presented issues of

1. using too many forcings lacking regularity properties and
2. killing old stationary sets

then we can prove an iteration theorem for ω_1-preserving forcings. Without defining respectful forcings, a special case of our main result can be stated as follows.

Theorem 1.11. *Suppose $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ is a nice iteration of ω_1-preserving forcings so that*

(i) if $\alpha + 2 < \gamma$ then $\Vdash_{\mathbb{P}_{\alpha+2}}$ "Strong Reflection Principle" and

(ii) if $\alpha < \gamma$ then $\dot{\mathbb{Q}}_\alpha$ is forced to preserve old stationary sets, i.e.

$$\forall \beta < \alpha \ \Vdash_{\mathbb{P}_{\alpha+1}} \mathrm{NS}_{\omega_1} \cap V[\dot{G}_\beta] = \mathrm{NS}_{\omega_1}^{V[\dot{G}_\alpha]} \cap V[\dot{G}_\beta].$$

Then $\mathbb{P}_\gamma$ preserves ω_1. Moreover, we have for all $\alpha + 1 \leqslant \gamma$

$$\forall \beta \leqslant \alpha \ \Vdash_{\mathbb{P}_\gamma} \mathrm{NS}_{\omega_1} \cap V[\dot{G}_\beta] = \mathrm{NS}_{\omega_1}^{V[\dot{G}_{\beta+1}]} \cap V[\dot{G}_\beta].$$

In fact we will prove something more general which allows, e.g. the preservation of a Suslin tree on the side.

Here, the Strong Reflection Principle is the reflection principle isolated by Todorčević.

Definition 1.12 (Todorčević, [Tod87])**.**

(i) For θ an uncountable cardinal and $\mathcal{S} \subseteq [H_\theta]^\omega$ we define

$$\mathcal{S}^\perp = \{X \in [H_\theta]^\omega \mid \forall Y \in [H_\theta]^\omega (X \sqsubseteq Y \rightarrow Y \notin S)\}.$$

(ii) The Strong Reflection Principle (SRP) holds if: Whenever $\theta \geqslant \omega_2$ is regular, $a \in H_\theta$ and $S \subseteq [H_\theta]^\omega$ then $\mathcal{S} \cup \mathcal{S}^\perp$ contains a continuous increasing ω_1-chain of countable elementary substructures of H_θ containing a, i.e. there is $\langle X_\alpha \mid \alpha < \omega_1\rangle$ so that for all $\alpha < \omega_1$

($\vec{X}.i$) $X_\alpha \prec H_\theta$ is countable,

$(\vec{X}.ii)$ $X_\alpha \in \mathcal{S} \cup \mathcal{S}^\perp$,

$(\vec{X}.iii)$ $a \in X_0$,

$(\vec{X}.iv)$ $X_\alpha \in X_{\alpha+1}$ and

$(\vec{X}.v)$ if $\alpha \in \mathrm{Lim}$ then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

We note that SRP can always be forced assuming large cardinals.

Theorem 1.13 (Shelah)**.** *Suppose there is a supercompact cardinal. Then there is a semiproper forcing* $\mathbb{P}$ *so that* $V^{\mathbb{P}} \models \mathrm{SRP}$.

As a consequence of this, assuming large cardinals, Theorem 1.11 can be understood as a *strategic* iteration theorem. Consider the following two player game IG_γ of length γ.

Player I	$\dot{\mathbb{Q}}_0$		$\dot{\mathbb{Q}}_2$		...	$\dot{\mathbb{Q}}_\omega$		...
Player II		$\dot{\mathbb{Q}}_1$		$\dot{\mathbb{Q}}_3$	...		$\dot{\mathbb{Q}}_{\omega+1}$	...

Player I plays at all even stages, including limit steps. Player I and II cooperate in this way to produce an RCS-iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ of forcings which do not kill old stationary sets. Player II wins iff $\mathbb{P}_\gamma$ preserves ω_1.

Corollary 1.14. *Suppose there is a proper class of supercompact cardinals. Then for any* γ*, Player* II *has a winning strategy for the game* IG_γ.

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2 Notation

First, we fix some notation. We will extensively deal with countable elementary substructures $X \prec H_\theta$ for large regular θ. We will make frequent use of the following notation:

Definition 2.1. Suppose X is any extensional set.

(i) M_X denotes the transitive isomorph of X.

(ii) $\pi_X : M_X \to X$ denotes the inverse collapse.

(iii) $\delta^X := \omega_1 \cap X$.

In almost all cases, we will apply this definition to a countable elementary substructure $X \prec H_\theta$ for some uncountable cardinal θ. In some cases, the X we care about lives in a generic extension of V, even though it is a substructure of H_θ^V. In that case, δ^X will always mean $X \cap \omega_1^V$.

We will also sometimes make use of the following convention in order to "unclutter" arguments.

Convention 2.2. If $X \prec H_\theta$ is an elementary substructure and some object a has been defined before and $a \in X$ then we denote $\pi_X^{-1}(a)$ by $\bar{a}$.

We will make use of this notation only if it is unambiguous.

Definition 2.3. If X, Y are sets then $X \sqsubseteq Y$ holds just in case

(*i*) $X \subseteq Y$ and

(*ii*) $\delta^X = \delta^Y$.

We use the following notions of clubs and stationarity on $[H_\theta]^\omega$:

Definition 2.4. Suppose A is an uncountable set.

(*i*) $[A]^\omega$ is the set of countable subsets of A.

(*ii*) $\mathcal{C} \subseteq [A]^\omega$ is a club in $[A]^\omega$ if

a) for any $X \in [A]^\omega$ there is a $Y \in \mathcal{C}$ with $X \subseteq Y$ and

b) if $\langle Y_n \mid n < \omega \rangle$ is a $\subseteq$-increasing sequence of sets in $\mathcal{C}$ then $\bigcup_{n<\omega} Y_n \in \mathcal{C}$.

(*iii*) $\mathcal{S} \subseteq [A]^\omega$ is stationary in $[A]^\omega$ if $\mathcal{S} \cap \mathcal{C} \neq \varnothing$ for any club $\mathcal{C}$ in $[A]^\omega$.

Next, we explain our notation for forcing iterations.

Definition 2.5. Suppose $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is an iteration and $\beta \leqslant \gamma$. We consider elements of $\mathbb{P}$ as functions of domain (or length) γ.

(*i*) If $p \in \mathbb{P}_\beta$ then $\mathrm{lh}(p) = \beta$.

(*ii*) If G is $\mathbb{P}$-generic then G_β denotes the restriction of G to $\mathbb{P}_\beta$, i.e.

$$G_\beta = \{p \upharpoonright \beta \mid p \in G\}.$$

Moreover, $\dot{G}_\beta$ is the canonical $\mathbb{P}$-name for G_β.

(*iii*) If G_β is $\mathbb{P}_\beta$-generic then $\mathbb{P}_{\beta,\gamma}$ denotes (by slight abuse of notation) the remainder of the iteration, that is

$$\mathbb{P}_{\beta,\gamma} = \{p \in \mathbb{P}_\gamma \mid p \upharpoonright \beta \in G_\beta\}.$$

$\dot{\mathbb{P}}_{\beta,\gamma}$ denotes a name for $\mathbb{P}_{\beta,\gamma}$ in V.

(*iv*) If G is $\mathbb{P}$-generic and $\alpha < \beta$ then $G_{\alpha,\beta}$ denotes the projection of G onto $\mathbb{P}_{\alpha,\beta}$.

There will be a number of instances were we need a structure to satsify a sufficiently large fragment of ZFC. For completeness, we make this precise.

Definition 2.6. *Sufficiently much of* ZFC is the fragment $\mathrm{ZFC}^- +$ "ω_1 exists". Here, ZFC^- is ZFC without the powerset axiom and with the collection scheme instead of the replacement scheme.

3 $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$

We will introduce the combinatorial principle which will parameterize the main iteration theorem. These are generalizations of the principles $\diamondsuit(\omega_1^{<\omega})$ and $\diamondsuit^+(\omega_1^{<\omega})$ isolated by Woodin [Woo10] in his study of $\mathbb{Q}_{\max}$ [Woo10, Section 6.2]. Most results in this Section are essentially due to Woodin and proven in [Woo10, Section 6.2].

Definition 3.1. Suppose $\mathbb{B} \subseteq \omega_1$ is a forcing.

(*i*) We say that f *guesses* $\mathbb{B}$*-filters* if f is a function

$$f\colon \omega_1 \to H_{\omega_1}$$

and for all $\alpha < \omega_1$, $f(\alpha)$ is a $\mathbb{B} \cap \alpha$-filter[1].

(*ii*) Suppose $\theta \geqslant \omega_2$ is regular and $X \prec H_\theta$ is an elementary substructure. We say X is f*-slim*[2] if

(*X.i*) X is countable,

(*X.ii*) $f, \mathbb{B} \in X$ and

(*X.iii*) $f(\delta^X)$ is $\mathbb{B} \cap \delta^X$-generic over M_X.

Definition 3.2. Let $\mathbb{B} \subseteq \omega_1$ be a forcing. $\diamondsuit(\mathbb{B})$ states that there is a function f so that

(*i*) f guesses $\mathbb{B}$-filters and

(*ii*) for any $b \in \mathbb{B}$ and regular $\theta \geqslant \omega_2$

$$\{X \prec H_\theta \mid X \text{ is } f\text{-slim} \wedge b \in f(\delta^X)\}$$

is stationary in $[H_\theta]^\omega$.

$\diamondsuit^+(\mathbb{B})$ is the strengthening of $\diamondsuit(\mathbb{B})$ where (*ii*) is replaced by:

[1] We consider the empty set to be a filter.

[2] We use the adjective "slim" for the following reason: An f-slim $X \prec H_\theta$ cannot be too fat compared to its height below ω_1, i.e. δ^X. If $X \sqsubseteq Y \prec H_\theta$ and Y is f-slim then X is f-slim as well, but the converse can fail.

$(ii)^+$ For any regular $\theta \geqslant \omega_2$

$$\{X \prec H_\theta \mid X \text{ is } f\text{-slim}\}$$

contains a club of $[H_\theta]^\omega$. Moreover, for any $b \in \mathbb{B}$

$$\{\alpha < \omega_1 \mid b \in f(\alpha)\}$$

is stationary.

We say that f *witnesses* $\diamondsuit(\mathbb{B})$, $\diamondsuit^+(\mathbb{B})$ respectively.

Remark 3.3. Observe that if f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{B}$ is separative then $\mathbb{B}$ can be "read off" from f: We have $\mathbb{B} = \bigcup_{\alpha<\omega_1} f(\alpha)$ and for $b, c \in \mathbb{B}$, $b \leqslant_{\mathbb{B}} c$ iff whenever $b \in f(\alpha)$ then $c \in f(\alpha)$ as well. Thus, it is usually not necessary to mention $\mathbb{B}$.

We introduce some convenient shorthand notation.

Definition 3.4. If $\mathbb{B} \subseteq \omega_1$ is a forcing, f guesses $\mathbb{B}$-filters and $b \in \mathbb{B}$ then

$$S_b^f := \{\alpha < \omega_1 \mid b \in f(\alpha)\}.$$

If f is clear from context we will sometimes omit the superscript f.

Note that if f witnesses $\diamondsuit(\mathbb{B})$, then S_b^f is stationary for all $b \in \mathbb{B}$. This is made explicit for $\diamondsuit^+(\mathbb{B})$. This is exactly the technical strengthening over Woodin's definition of $\diamondsuit(\omega_1^{<\omega})$, $\diamondsuit^+(\omega_1^{<\omega})$. Lemma 3.11 shows that this strengthening is natural. Moreover, this implies

$$\diamondsuit(\mathbb{B} \oplus \mathbb{C}) \Rightarrow \diamondsuit(\mathbb{B}) \wedge \diamondsuit(\mathbb{C})$$

whenever $\mathbb{B}, \mathbb{C} \subseteq \omega_1$ are forcings and $\mathbb{B} \oplus \mathbb{C}$ is the disjoint union of $\mathbb{B}$ and $\mathbb{C}$ coded into a subset of ω_1. This becomes relevant in Subsection **??**. Nonetheless, the basic theory of these principles is not changed by a lot.

Definition 3.5. If f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is a forcing, we say that $\mathbb{P}$ *preserves* f if whenever G is $\mathbb{P}$-generic then f witnesses $\diamondsuit(\mathbb{B})$ in $V[G]$.

We remark that if f witnesses $\diamondsuit^+(\mathbb{B})$ then "$\mathbb{P}$ preserves f" still only means that f witnesses $\diamondsuit(\mathbb{B})$ in $V^{\mathbb{P}}$.

Next, we define a variant of stationary sets related to a witness of $\diamondsuit(\mathbb{B})$. Suppose $\theta \geqslant \omega_2$ is regular. Then $S \subseteq \omega_1$ is stationary iff for any club $\mathcal{C} \subseteq [H_\theta]^\omega$, there is some $X \in \mathcal{C}$ with $\delta^X \in S$. f-stationarity results from restricting to f-slim $X \prec H_\theta$ only.

Definition 3.6. Suppose f guesses $\mathbb{B}$-filters.

(i) A subset $S \subseteq \omega_1$ is *f-stationary* iff whenever $\theta \geqslant \omega_2$ is regular and $\mathcal{C} \subseteq [H_\theta]^\omega$ is club then there is some f-slim $X \in \mathcal{C}$ with $\delta^X \in S$.

(*ii*) A forcing $\mathbb{P}$ *preserves f-stationary sets* iff any f-stationary set is still f-stationary in $V^{\mathbb{P}}$.

We make use of f-stationarity only when f witnesses $\diamondsuit(\mathbb{B})$. However, with the above definition it makes sense to talk about f-stationarity in a forcing extension before we know that f has been preserved. Note that all f-stationary sets are stationary, but the converse might fail, see Proposition **??**. We will later see that f-stationary sets are the correct replacement of stationary set in our context. Most prominently this notion will be used in the definition of the MM^{++}-variant $\mathrm{MM}^{++}(f)$ we introduce in Subsection **??**. It will be useful to have an equivalent formulation of f-stationarity at hand.

Proposition 3.7. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent for any set $S \subseteq \omega_1$:*

(*i*) *S is f-stationary.*

(*ii*) *Whenever $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{B}$, the set*

$$\{\alpha \in S \mid \forall \beta < \alpha\ f(\alpha) \cap D_\beta \neq \emptyset\}$$

is stationary.

Proposition 3.8. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent:*

(*i*) *f witnesses $\diamondsuit(\mathbb{B})$.*

(*ii*) *S^f_b is f-stationary for all $b \in \mathbb{B}$.*

(*iii*) *For any $b \in \mathbb{B}$ and sequence $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{B}$,*

$$\{\alpha \in S^f_b \mid \forall \beta < \alpha\ f(\alpha) \cap D_\beta \neq \emptyset\}$$

is stationary.

We mention a handy corollary.

Corollary 3.9. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Any forcing preserving f-stationary sets preserves f.*

Proposition 3.10. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent:*

(*i*) *f witnesses $\diamondsuit^+(\mathbb{B})$.*

(*ii*) *For any $b \in \mathbb{B}$, S^f_b is stationary and all stationary sets are f-stationary.*

(*iii*) *If D is dense in $\mathbb{B}$ then*

$$\{\alpha < \omega_1 \mid\ f(\alpha) \cap D \neq \emptyset\}$$

contains a club and for all $b \in \mathbb{B}$, S^f_b is stationary.

(*iv*) *All countable* $X \prec H_\theta$ *with* $f \in X$ *and* $\theta \geqslant \omega_2$ *regular are* f*-slim and moreover for all* $b \in \mathbb{B}$, S^f_b *is stationary.*

We will now give a natural equivalent formulation of $\diamondsuit^+(\mathbb{B})$.

Witnesses of $\diamondsuit^+(\mathbb{B})$ are simply codes for regular embeddings[3] of $\mathbb{B}$ into $\mathrm{NS}^+_{\omega_1}$.

Lemma 3.11. *The following are equivalent:*

(*i*) $\diamondsuit^+(\mathbb{B})$.

(*ii*) *There is a regular embedding* $\eta\colon \mathbb{B} \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$.

The argument above suggests the following definition.

Definition 3.12. Suppose f witnesses $\diamondsuit(\mathbb{B})$. We define

$$\eta_f\colon \mathbb{B} \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$$

by $b \mapsto [S^f_b]_{\mathrm{NS}_{\omega_1}}$ and call η_f the *embedding associated to* f.

We will now show that $\diamondsuit(\mathbb{B})$ is consistent for any forcing $\mathbb{B} \subseteq \omega_1$, even simultaneously so for all such $\mathbb{B}$. We will deal with the consistency of $\diamondsuit^+(\mathbb{B})$ in the next section.

Proposition 3.13. *Assume* $\diamondsuit$. *Then* $\diamondsuit(\mathbb{B})$ *holds for any poset* $\mathbb{B} \subseteq \omega_1$.

Corollary 3.14. *Suppose* $\mathbb{B} \subseteq \omega_1$ *is a forcing. Then* $\diamondsuit(\mathbb{B})$ *holds in* $V^{\mathrm{Add}(\omega_1,1)}$.

In a number of arguments, we will deal with f-slim $X \prec H_\theta$ that become thicker over time, i.e. at a later stage there will be some f-slim $X \sqsubseteq Y \prec H_\theta$.

Definition 3.15. In the above case of $X \sqsubseteq Y$, we denote the canonical elementary embedding from M_X to M_Y by

$$\mu_{X,Y}\colon M_X \to M_Y.$$

$\mu_{X,Y}$ is given by $\pi_Y^{-1} \circ \pi_X$.

Usually, both X and Y will be f-slim. It is then possible to lift $\mu_{X,Y}$.

Proposition 3.16. *Suppose* f *guesses* $\mathbb{B}$*-filters and* $X, Y \prec H_\theta$ *are both* f*-slim with* $X \sqsubseteq Y$. *Then the lift of* $\mu_{X,Y}$ *to*

$$\mu^+_{X,Y}\colon M_X[f(\delta^X)] \to M_Y[f(\delta^X)]$$

exists.

Proof. As $\delta^X = \delta^Y$, the critical point of $\mu_{X,Y}$ is $>\delta^X$ (if it exists). As $\pi_X^{-1}(\mathbb{B})$ is a forcing of size $\leqslant\omega_1^{M_X} = \delta^X$ and $f(\delta^X)$ is generic over both M_X and M_Y, the lift exists. □

[3]Regular embeddings, also known as complete embeddings, are embeddings between partial orders which preserve maximal antichains.

We consider the above proposition simultaneously as a definition: From now on $\mu^+_{X,Y}$ will refer to this lift if it exists.

Definition 3.17. Suppose f witnesses $\diamondsuit(\mathbb{B})$. NS_f is the ideal of f-nonstationary sets, that is

$$\mathrm{NS}_f = \{N \subseteq \omega_1 \mid N \text{ is not } f\text{-stationary}\}.$$

Lemma 3.18. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. NS_f is a normal uniform ideal.*

3.1 Miyamoto's theory of nice iterations

For all our intents and purposes, it does not matter in applications how the limit our iterations look like as long as we can prove a preservation theorem about it.

We give a brief introduction to Miyamoto's theory of nice iterations. These iterations are an alternative to RCS-itertaions when dealing with the problem described above. In the proof of the iteration theorem for (f-)proper forcings, one constructs a generic condition q by induction as the limit of a sequence $\langle q_n \mid n < \omega\rangle$. In case of ($f$-)semiproper forcings, the length of the iteration may have uncountable cofinality in V but become ω-cofinal along the way. In this case, a sequence $\langle q_n \mid n < \omega\rangle$ with the desired properties cannot be in V. The key insight to avoid this issue is that one should give up linearity of this sequence and instead build a tree of conditions in the argument. Nice supports follow the philosophy of form follows function, i.e. its definitions takes the shape of the kind of arguments it is intended to be involved in. The conditions allowed in a nice limit are represented by essentially the kind of trees that this inductive nonlinear constructions we hinted at above produces.

Miyamoto works with a general notion of iteration. For our purposes, we will simply define nice iterations by induction on the length. Successor steps are defined as usual, that is if $\mathbb{P}_\gamma = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ is a nice iteration of length γ and $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for a forcing then $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma+1, \beta \leqslant \gamma\rangle$ is a nice iteration of length $\gamma + 1$ where $\mathbb{P}_{\gamma+1} \cong \mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$.

Definition 3.19 (Miyamoto, [Miy02])**.** Let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma\rangle$ be a potential nice iteration, that is

($\vec{\mathbb{P}}.i$) $\mathbb{P}_\alpha$ is a nice iteration of length α for all $\alpha < \gamma$,

($\vec{\mathbb{P}}.ii$) $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for all $\alpha + 1 < \gamma$ and

($\vec{\mathbb{P}}.iii$) $\mathbb{P}_\beta \upharpoonright \alpha = \mathbb{P}_\alpha$ for all $\alpha \leqslant \beta < \gamma$.

A *nested antichain* in $\vec{\mathbb{P}}$ is of the form

$$(T, \langle T_n \mid n < \omega\rangle, \langle \mathrm{suc}^n_T \mid n < \omega\rangle)$$

so that for all $n < \omega$ the following hold[4]:

[4] Usually, we identify the nested antichain with T, its first component and write $\mathrm{suc}(a)$ instead of $\mathrm{suc}^n_T(a)$ if n, T are clear from context.

(i) $T = \bigcup_{n<\omega} T_n$.

(ii) $T_0 = \{a_0\}$ for some $a_0 \in \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$.

(iii) $T_n \subseteq \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$ and $\mathrm{suc}^n_T \colon T_n \to \mathcal{P}(T_{n+1})$.

(iv) For $a \in T_n$ and $b \in \mathrm{suc}^n_T(a)$, $\mathrm{lh}(a) \leqslant \mathrm{lh}(b)$ and $b \restriction \mathrm{lh}(a) \leqslant a$.

(v) For $a \in T_n$ and distinct $b, b' \in \mathrm{suc}^n_T(a)$, $b \restriction \mathrm{lh}(a) \perp b' \restriction \mathrm{lh}(a)$.

(vi) For $a \in T_n$, $\{b \restriction \mathrm{lh}(a) \mid b \in \mathrm{suc}^n_T(a)\}$ is a maximal antichain below a in $\mathbb{P}_{\mathrm{lh}(a)}$.

(vii) $T_{n+1} = \bigcup\{\mathrm{suc}^n_T(a) \mid a \in T_n\}$.

Abusing notation, we will usually identify T with

$$(T, \langle T_n \mid n < \omega\rangle, \langle \mathrm{suc}^n_T \mid n < \omega\rangle).$$

If $b \in \mathrm{suc}^n_T(a)$ then we also write $a = \mathrm{pred}^n_T(b)$. If $\beta < \gamma$ then $p \in \mathbb{P}_\beta$ is a *mixture of T up to β* iff for all $\alpha < \beta$, $p \restriction \alpha$ forces

(p.i) $p(\alpha) = a_0(\alpha)$ if $\alpha < \mathrm{lh}(a_0)$ and $a_0 \restriction \alpha \in G_\alpha$,

(p.ii) $p(\alpha) = b(\alpha)$ if there are $a, b \in T$, $n < \omega$ with $b \in \mathrm{suc}^n_T(a)$, $\mathrm{lh}(a) \leqslant \alpha < \mathrm{lh}(b)$ and $b \restriction \alpha \in G_\alpha$,

(p.iii) $p(\alpha) = \mathbb{1}_{\dot{\mathbb{Q}}_\alpha}$ if there is a sequence $\langle a_n \mid n < \omega\rangle$ with $a_{n+1} \in \mathrm{suc}^n_T(a_n)$, $\mathrm{lh}(a_n) \leqslant \alpha$ and $a_n \in G_{\mathrm{lh}(a_n)}$ for all $n < \omega$.

If $\xi \leqslant \gamma$ is a limit, and q is a sequence of length ξ (may or may not be in $\mathbb{P}_\xi$), q is *(T,ξ)-nice* if for all $\beta < \xi$, $q \restriction \beta \in \mathbb{P}_\beta$ is a mixture of T up to β.

We refer to [Miy02] for basic results on nested antichains and mixtures. We go on and define nice limits.

Definition 3.20 (Miyamoto, [Miy02])**.** Suppose $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma\rangle$ is a potential nice iteration of limit length γ. Let $\bar{\mathbb{P}}$ denote the inverse limit along $\vec{\mathbb{P}}$. The *nice limit* of $\vec{\mathbb{P}}$ is defined as

$$\mathrm{nicelim}(\vec{\mathbb{P}}) = \{p \in \bar{\mathbb{P}} \mid \exists T \text{ a nested antichain of } \vec{\mathbb{P}} \text{ and } p \text{ is } (T,\gamma)\text{-nice}\}.$$

$\mathrm{nicelim}(\vec{\mathbb{P}})$ inherits the order from $\bar{\mathbb{P}}$.

Finally, if $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma\rangle$ is a potential nice iteration then

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$$

is a *nice iteration* of length γ where $\mathbb{P}_\gamma = \mathrm{nicelim}(\vec{\mathbb{P}})$.

The fundamental property of nice iterations is:

Fact 3.21 (Miyamoto,[Miy02])**.** *Suppose* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *is a nice iteration and* T *is a nested antichain in* $\mathbb{P}$*. Then there is a mixture of* T*.*

Definition 3.22 (Miyamoto,[Miy02])**.** Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration. If S, T are nested antichains in $\mathbb{P}$ then $S \angle\, T$ iff for any $n < \omega$ and $a \in S_n$ there is $b \in T_{n+1}$ with

$$\mathrm{lh}(b) \leqslant \mathrm{lh}(a) \text{ and } a \upharpoonright \mathrm{lh}(b) \leqslant b.$$

Fact 3.23 (Miyamoto, [Miy02])**.** *Let* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *be a nice iteration of limit length* γ*. Suppose that*

(i) T *is a nested antichain in* $\mathbb{P}$*,*

(ii) p *is a mixture of* T *and* $s \in \mathbb{P}$*,*

(iii) $r \in T_1$*,*

(iv) $s \leqslant r^\frown p \upharpoonright [\mathrm{lh}(r), \gamma)$ *and*

(v) $A \subseteq \gamma$ *is cofinal.*

Then there is a nested antichain S *in* $\mathbb{P}$ *with*

(a) s *is a mixture of* S*,*

(b) *If* $S_0 = \{c\}$ *then* $\mathrm{lh}(r) \leqslant \mathrm{lh}(c) \in A$ *and* $c \upharpoonright \mathrm{lh}(r) \leqslant r$ *and*

(c) $S \angle\, T$*.*

The following describes the tool we use to construct conditions.

Definition 3.24 (Miyamoto, [Miy02])**.** Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration of limit length γ. A *fusion structure* in $\mathbb{P}$ is

$$T, \langle p^{(a,n)},\ T^{(a,n)} \mid n < \omega,\ a \in T_n \rangle$$

where

(i) T is a nested antichain in $\mathbb{P}$

and for all $n < \omega$ and $a \in T_n$

(ii) $T^{(a,n)}$ is a nested antichain in $\mathbb{P}$,

(iii) $p^{(a,n)} \in \mathbb{P}$ is a mixture of $T^{(a,n)}$,

(iv) $a \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(a)$ and if $T_0^{(a,n)} = \{c\}$ then $\mathrm{lh}(a) = \mathrm{lh}(c)$ and

(v) for any $b \in \mathrm{suc}_T^n(a)$, $T^{(b,n+1)} \angle\, T^{(a,n)}$, thus $p^{(b,n+1)} \leqslant p^{(a,n)}$.

If $q \in \mathbb{P}$ is a mixture of T then q is called a *fusion* of the fusion structure.

Fact 3.25 (Miyamoto, [Miy02]). *Let* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *be a nice iteration of limit length* γ*. If* $q \in \mathbb{P}$ *is a fusion of a fusion structure*

$$T, \langle p^{(a,n)},\ T^{(a,n)} \mid n < \omega,\ a \in T_n \rangle$$

and G *is* $\mathbb{P}$*-generic with* $q \in G$ *then the following holds in* $V[G]$*: There is a sequence* $\langle a_n \mid n < \omega \rangle$ *so that for all* $n < \omega$

(*i*) $a_0 \in T_0$,

(*ii*) $a_n \in G_{\mathrm{lh}(a_n)}$,

(*iii*) $a_{n+1} \in \mathrm{suc}^n_T(a_n)$ *and*

(*iv*) $p^{(a_n,n)} \in G$.

We mention one more convenient fact:

Fact 3.26 (Miyamoto, [Miy03]). *Suppose* κ *is an inaccessible cardinal,* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$ *is a nice iteration so that*

(*i*) $|\mathbb{P}_\alpha| < \kappa$ *for all* $\alpha < \kappa$ *and*

(*ii*) $\mathbb{P}$ *preserves* ω_1.

Then $\mathbb{P}$ *is* κ*-c.c..*

Miyamoto proves this for so called simple iterations of semiproper forcings. The proof works just as well for nice iterations of semiproper forcings and finally the proof can be made to work with assuming only $\mathbb{P}$ preserves ω_1 instead of $\mathbb{P}$ being a semiproper iteration.

4 The Iteration Theorem

The full main theorem we are going to prove is the following.

Theorem 4.1. *Suppose* f *witnesses* $\diamondsuit(\mathbb{B})$ *and* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *is a nice iteration of* f*-preserving forcings. Suppose that*

($\mathbb{P}$.*i*) $\Vdash_{\mathbb{P}_{\alpha+2}} \mathrm{SRP}$ *for all* $\alpha + 2 \leqslant \gamma$ *and*

($\mathbb{P}$.*ii*) $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ *preserves* f*-stationary sets from* $\bigcup_{\beta<\alpha} V[\dot{G}_\beta]$.

Then $\mathbb{P}$ *preserves* f.

Note that if $\mathbb{B}$ is the trivial forcing $\{\mathbb{1}\}$ and we take f to be the witness of $\diamondsuit^+(\mathbb{B})$ with $f(\alpha) = \{\mathbb{1}\}$ for all $0 < \alpha < \omega_1$, then we recover the special case mentioned in the introduction.

So what is the basic idea? For the moment, let us assume that f is the trivial witness of $\diamondsuit(\{\mathbb{1}\})$ above for simplicity. As always, we want to imitate the argument of the mother of all iteration theorems, the iteration theorem for proper forcings. Suppose we have a full support iteration

$$\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$$

and for the moment assume only that

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{\mathbb{Q}}_n \text{ preserves } \omega_1\text{”}.$$

We try to motivate some additional reasonable constraints imply $\mathbb{P}$ to be ω_1-preserving. For the moment, we try to consider Shelah's argument as a game: In the beginning there some countable $X \prec H_\theta$ as well as $p_0 \in X \cap \mathbb{P}$. The argument proceeds as follows: In round n, we have already constructed a $(X, \mathbb{P}_n)$-semigeneric condition $q_n \in \mathbb{P} \upharpoonright n$ and have

$$q_n \Vdash \dot{p}_n \upharpoonright n \in \dot{G}_n \cap \check{X}[\dot{G}_n].$$

Next, our adversary hits us with a dense subset $D \subseteq \mathbb{P}$ in X and we must find $\dot{p}_{n+1} \in V^{\mathbb{P}_n}$ and some $(X, \mathbb{P}_{n+1})$-semigeneric q_{n+1} with $q_{n+1} \upharpoonright n = q_n$ and[5]

$$q_{n+1} \Vdash \dot{p}_{n+1} \in \check{D} \wedge p_{n+1} \upharpoonright n+1 \in \dot{G}_{n+1} \cap \check{X}[\dot{G}_{n+1}].$$

Our job is to survive this game for ω-many steps. If we have a winning strategy then we can find a $(X, \mathbb{P})$-semigeneric condition, so in particular $\mathbb{P}$ preserves ω_1.

Destroying stationarity makes it significantly more difficult to survive the above game: Suppose for example that

$$p_0(0) \Vdash \check{S} \in \mathrm{NS}_{\omega_1}$$

for some $S \in X$ with $\delta^X \in S$. Then there is no hope of finding a $(X, \mathbb{P}_1)$-semigeneric condition q with $q \leqslant p_0 \upharpoonright 1$. Hence, we must already be careful with what X we start the game. This leads us to the following definitions.

Definition 4.2. Suppose θ is sufficiently large and regular, $X \prec H_\theta$ is countable. If I is an ideal on ω_1, we say that X *respects* I if for all $A \in I \cap X$ we have $\delta^X \notin A$.

Note that all countable $X \prec H_\theta$ respect NS_{ω_1} and countable $Y \prec H_\theta$ with $f \in Y$ respects NS_f if and only if Y is f-slim.

Definition 4.3. Suppose $\mathbb{P}$ is a forcing and $\dot{I} \in V^{\mathbb{P}}$ is a name for an ideal on ω_1. For p in $\mathbb{P}$, we denote the *partial evaluation of* $\dot{I}$ *by* p by

$$\dot{I}^p := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \dot{I}\}.$$

Back to the discussion, we need to start with an X so that X respects $\dot{I}^{p_0 \upharpoonright 1}$ where $\dot{I}$ is a name for the nonstationary ideal. This gives us a shot at getting past the first round. Luckily, there are enough of these X.

[5]Here, we consider $\dot{p}_n$ also as a $\mathbb{P}_{n+1}$-name.

Definition 4.4. Let A be an uncountable set with $\omega_1 \subseteq A$ and I a normal uniform ideal on ω_1. Then $\mathcal{S} \subseteq [A]^\omega$ is *projective I-positive* if for any $S \in I^+$ the set

$$\{X \in \mathcal{S} \mid \delta^X \in S\}$$

is stationary in $[A]^\omega$.

Proposition 4.5. *Suppose θ is sufficiently large and regular. Let I be a normal uniform ideal on ω_1. Then*

$$\mathcal{S} = \{X \in [H_\theta]^\omega \mid X \prec H_\theta \text{ respects } I\}$$

is projective I-positive.

Proof. Let $\mathcal{C}$ be a club in $[H_\theta]^\omega$ and assume that all elements of $\mathcal{C}$ are elementary substructures of H_θ and contain I as an element. Let

$$\vec{X} := \langle X_\alpha \mid \alpha < \omega_1 \rangle$$

be a continuous increasing chain of elements in $\mathcal{C}$. Let $X := \bigcup_{\alpha<\omega_1} X_\alpha$ and let

$$\vec{A} := \langle A_\alpha \mid \alpha < \omega_1 \rangle$$

be an enumeration of $X \cap I$. Let $C \subseteq \omega_1$ be the set of α so that

$(C.i)$ $\delta^{X_\alpha} = \alpha$ and

$(C.ii)$ $\vec{A} \upharpoonright \alpha$ is an enumeration of $X_\alpha \cap I$

and note that C is club. Let $A = \triangledown_{\alpha<\omega_1} I_\alpha$. As I is normal, $A \in I$. Then $C - A$ is a complement of a set in I and for any $\alpha \in C - A$ we have

$$\delta^{X_\alpha} = \alpha \notin I_\beta$$

for all $\beta < \alpha$. Hence $X_\alpha \in \mathcal{S} \cap \mathcal{C}$. □

Of course, the problem continues. What if we have found a suitable q_1 and now we work in $V[G_1]$ with $q_1 \in G_1$. At the very least, we need that $X[G_1]$ respects $\dot{I}^{p_0 \upharpoonright [1,2)}$, where $\dot{I}$ is now a $\mathbb{P}_{1,2}$-name for the nonstationary ideal. Ensuring this is a matter of being able to pick the right q_1 to begin with. This motivates the following class of forcings.

Definition 4.6. We say that a forcing $\mathbb{P}$ is *respectful* if $\mathbb{P}$ preserves ω_1 and the following is true: Whenever

- θ is sufficiently large and regular,
- $X \prec H_\theta$ is countable with $\mathbb{P} \in X$,
- $\dot{I} \in X$ is a $\mathbb{P}$-name for a normal uniform ideal and
- $p \in \mathbb{P} \cap X$

then exactly one of the following holds:

(Res.i) Either there is some $(X, \mathbb{P})$-semigeneric $q \leqslant p$ so that

$$q \Vdash \text{“}\check{X}[\dot{G}] \text{ respects } \dot{I}\text{”}$$

or

(Res.ii) X does not respect $\dot{I}^p$.

Roughly, this condition states that we can find a $\mathbb{P}$-generic filter G with $p \in G$ so that $X \sqsubseteq X[G]$ respects $\dot{I}^G$ as long as there is no obvious obstruction to it.

Remark 4.7. If $\mathbb{P}$ is respectful and preserves stationary sets then $\mathbb{P}$ is semiproper. However, the converse is not true in general. Similarly, a respectful f-stationary set preserving forcing is f-semiproper, which follows from plugging in a name for NS_f as $\dot{I}$ in the definition of respectfulness.

We require[6] now that

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{\mathbb{Q}}_n \text{ is respectful”}$$

for all $n < \omega$. We then aim to make sure (assuming $\dot{p}_{n+1}$ is already defined) to find q_{n+1} in round n so that in addition to the prior constraints,

$$q_{n+1} \Vdash \text{“}\check{X}[\dot{G}_{n+1}] \text{ respects } \dot{I}\text{”}$$

where $\dot{I}$ is a $\mathbb{P}_{n+1}$ name for the ideal of sets forced to be nonstationary by $\dot{p}_{n+1}(n+1)$. Consider $\dot{I}$ as a $\mathbb{P}_n$-name $\ddot{I}$ for a $\dot{\mathbb{Q}}_n$-name. By respectfulness, this reduces to avoiding an instance of the “bad case” (Res.ii), namely we should make sure that whenever G_n is $\mathbb{P}_n$-generic with $q_n \in G_n$ then

$$X[G_n] \text{ respects } \left(\ddot{I}^{G_n}\right)^{p_{n+1}(n+1)}$$

where $p_{n+1} = \dot{p}_{n+1}^{G_{n+1}}$. he next key insight is that this reduces to

$$\text{“}X[G_n] \text{ respects } J := \{S \subseteq \omega_1 \mid p_{n+1}(n) \Vdash \check{S} \in \mathrm{NS}_{\omega_1}\}\text{”}$$

which we have (almost)[7] already justified inductively, assuming $\dot{\mathbb{Q}}_{n+1}$ only kills new stationary sets: Our final requirement[8] is that

$$\Vdash_{\mathbb{P}_{n+1}} \text{“}\dot{\mathbb{Q}}_{n+1} \text{ preserves stationary sets which are in } V[\dot{G}_n]\text{”}$$

[6]This excludes the first counterexample due to Shelah, but not yet all the counterexamples of the second kind.

[7]We made sure of this if p_{n+1} is replaced by p_n in the definition of J, we ignore this small issue for now.

[8]It is readily seen that this eliminates the counterexamples of the second kind.

for all $n < \omega$. The point is that trivially $\left(\ddot{I}^{G_n}\right)^{p_{n+1}(n)}$ only contains sets in $V[G_n]$, so all such sets will be preserved by $\dot{\mathbb{Q}}_{n+1}$. The sets that are killed are then already killed in the extension by $\dot{\mathbb{Q}}_n^{G_n}$.
Modulo some details we have shown the following.

Theorem 4.8. *Suppose $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ is a full support iteration so that*

($\mathbb{P}.i$) $\Vdash_{\mathbb{P}_n} \dot{\mathbb{Q}}_n$ *is respectful and*

($\mathbb{P}.ii$) $\Vdash_{\mathbb{P}_{n+1}} \dot{\mathbb{Q}}_{n+1}$ *preserves stationary sets which are in* $V[\dot{G}_n]$"

for all $n < \omega$. Then $\mathbb{P}$ does not collapse ω_1.

Two issues arise when generalizing this to longer iterations. The first issue is the old problem that new relevant indices may appear along the iteration in the argument, which we deal with by using nice supports. The second problem is that it seemingly no longer suffices that each iterand individually is respectful. For longer iterations, say of length γ, the argument then requires that

$$\Vdash_\alpha \text{“}\dot{\mathbb{P}}_{\alpha,\beta} \text{ is respectful”}$$

for sufficiently many $\alpha < \beta < \gamma$. This is problematic as we will not prove an iteration theorem of any kind for respectful forcings[9]. This is where we take out the sledgehammer.

Definition 4.9. (‡) holds if and only if all ω_1-preserving forcings are respectful.

Lemma 4.10. SRP *implies* (‡).

Proof. Let $\mathbb{P}$, θ, $\dot{I}$, p be as Definition 4.6. It is easy to see that (Res.i) and (Res.ii) cannot hold simultaneously. It is thus enough to prove that one of them holds. Let λ be regular, $2^{|\mathbb{P}|} < \lambda < \theta$ and $\lambda \in X$ and consider the set

$$\mathcal{S} = \{Y \in [H_\lambda]^\omega \mid Y \prec H_\lambda \wedge \neg(\exists q \leqslant p\ q \text{ is } (Y,\mathbb{P})\text{-semigeneric and } q \Vdash \text{“}\check{Y}[\dot{G}] \text{ respects } \dot{I}\text{”})\}.$$

By SRP, there is a continuous increasing elementary chain

$$\vec{Y} = \langle Y_\alpha \mid \alpha < \omega_1 \rangle$$

so that

($\vec{Y}.i$) $\mathbb{P}, p, \dot{I} \in Y_0$ and

($\vec{Y}.ii$) for all $\alpha < \omega_1$, either $Y_\alpha \in \mathcal{S}$ or there is no $Y_\alpha \sqsubseteq Z \prec H_\theta$ with $Z \in \mathcal{S}$.

[9] Indeed it seems that no useful iteration theorem for respectful forcings is provable in ZFC, see Subsection **??**.

Let $S = \{\alpha < \omega_1 \mid Y_\alpha \in \mathcal{S}\}$.

Claim 4.11. $p \Vdash \check{S} \in \dot{I}$.

Proof. Let G be generic with $p \in G$ and let $I = \dot{I}^G$. Assume toward a contradiction that S is I-positive. Note that $\langle Y_\alpha[G] \mid \alpha < \omega_1\rangle$ is a continuous increasing sequence of elementary substructure of $H_\theta^{V[G]}$. Hence there is a club C of α so that for $\alpha \in C$

$$\delta^{Y_\alpha} = \delta^{Y_\alpha[G]} = \alpha$$

and thus there is a $(Y_\alpha, \mathbb{P})$-semigeneric condition $q \leqslant p$, $q \in G$. Hence by definition of S, for any $\alpha \in S \cap C$, we may find some $N_\alpha \in I \cap Y_\alpha[G]$ so that $\delta^{Y_\alpha} \in N_\alpha$. By normality of I, there is some I-positive $T \subseteq S \cap C$ and some N so that $N = N_\alpha$ for all $\alpha \in T$. But then for $\alpha \in T$, we have

$$\alpha = \delta^Y \in N$$

so that $T \subseteq N$. But $N \in I$, contradiction. □

Thus if $\delta^X \in S$, then S witnesses (Res.*ii*) to hold. Otherwise, $\delta^X \notin S$. Note that $\delta^{Y_{\delta^X}} = \delta^X$ as $\vec{Y} \in X$. We find that $Y_{\delta^X} \sqsubseteq X \cap H_\lambda \prec H_\lambda$. Thus, $X \cap H_\lambda \notin \mathcal{S}$, so that there must be some $q \leqslant p$ that is $(X \cap H_\lambda, \mathbb{P})$-semigeneric and

$$q \Vdash \text{“}(\widetilde{X \cap H_\lambda})[\dot{G}] \text{ respects } \dot{I}\text{”}.$$

This q witnesses that (Res.*i*) holds. □

We will get around this second issue by forcing SRP often along the iteration. Remember that what we really care about is preserving a witness f of $\diamondsuit(\mathbb{B})$ along an iteration of f-preserving forcings, so fix such an f now. It will be quite convenient to introduce some short hand notation.

Definition 4.12. Suppose $\mathbb{P}$ is a forcing and $p \in \mathbb{P}$. Then we let $I_p^{\mathbb{P}}$ denote $\dot{I}^p$ where $\dot{I}$ is a $\mathbb{P}$-name for NS_f. That is

$$I_p^{\mathbb{P}} := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \mathrm{NS}_f\}.$$

Definition 4.13. Suppose f witnesses $\diamondsuit(\mathbb{B})$. An *f-ideal* is an ideal I on ω_1 so that

(*i*) whenever $S \in I^+$ and $\langle D_i \mid i < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{B}$, then
$$\{\alpha \in S \mid \forall \beta < \alpha\ f(\alpha) \cap D_\beta \neq \emptyset\} \in I^+$$

(*ii*) and $S_b^f \in I^+$ for all $b \in \mathbb{B}$.

Recall that NS_f is clearly an f-ideal and it is normal and uniform by Lemma 3.18.

Proposition 4.14. *Suppose $\mathbb{P}$ is a forcing that preserves f and $p \in \mathbb{P}$. Then $I_p^{\mathbb{P}}$ is a normal uniform f-ideal.*

We leave the proof to the reader. The next Lemma gives us a criterion that guarantees the relevant witness f of $\diamondsuit(\mathbb{B})$ to be preserved. We first introduce the notion of a f-semigeneric condition.

Definition 4.15. Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is a forcing, θ is sufficiently large and $X \prec H_\theta$ is a f-slim elementary substructure of H_θ with $\mathbb{P} \in X$. A condition $p \in \mathbb{P}$ is called $(X, \mathbb{P}, f)$-semigeneric if p is $(X, \mathbb{P})$-semigeneric and

$$p \Vdash \check{X}[\dot{G}] \text{ is } \check{f}\text{-slim}.$$

Lemma 4.16. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is a forcing with the following property: For any sufficiently large regular θ and $p \in \mathbb{P}$ there is a normal uniform f-ideal I so that*

$$\{X \in [H_\theta]^\omega \mid X \prec H_\theta \wedge \mathbb{P}, p \in X \wedge \exists q \leqslant p \ q \textit{ is } (X, \mathbb{P}, f)\textit{-semigeneric}\}$$

is projective I-positive. Then $\mathbb{P}$ preserves f.

Proof. Assume $p \in \mathbb{P}$, θ is sufficiently large and regular. Let $b \in \mathbb{B}$,

$$\vec{\dot{D}} = \langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle$$

be a sequence of $\mathbb{P}$-names for dense subsets of $\mathbb{B}$ and $\dot{C}$ a $\mathbb{P}$-name for a club in ω_1. We will find $q \leqslant p$ so that

$$q \Vdash \exists \alpha \in S_b^{\check{f}} \cap \dot{C} \forall \beta < \alpha \ \check{f}(\alpha) \cap \dot{D}_\beta \neq \varnothing.$$

By our assumption, there is some normal uniform f-ideal I so that

$$\{X \in [H_\theta]^\omega \mid X \prec H_\theta \wedge \mathbb{P}, p \in X \wedge \exists q \leqslant p \ q \text{ is } (X, \mathbb{P}, f)\text{-semigeneric}\}$$

is projective I-positive. It follows that we can find some countable $X \prec H_\theta$ so that

$(X.i)$ $\mathbb{P}, p, \vec{\dot{D}}, \dot{C} \in X$ as well as

$(X.ii)$ $b \in f(\delta^X)$

and some $q \leqslant p$ that is $(X, \mathbb{P}, f)$-semigeneric. If G is then any $\mathbb{P}$-generic with $q \in G$, we have

$$X \sqsubseteq X[G] \text{ is } f\text{-slim}$$

and hence $\delta^X \in \dot{C}^G$ as well as

$$\forall \beta < \delta^X \ f(\delta^X) \cap \dot{D}_\beta^G \neq \varnothing.$$

□

We also need to resolve a small issue that we glossed over in the sketch of a proof of Theorem 4.8.

Lemma 4.17. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Further assume that*

- *$\mathbb{P}$ is a respectful, f-preserving forcing and $p \in \mathbb{P}$,*
- *θ is sufficiently large and regular,*
- *$X \prec H_\theta$ is countable, respects $I_p^{\mathbb{P}}$ and $\mathbb{P}, p \in X$ and*
- *$M_X[f(\delta^X)] \models$ "D is dense below $\pi_X^{-1}(p)$ in $\pi_X^{-1}(\mathbb{P})$".*

Then there are Y, q with

(i) *$X \sqsubseteq Y \prec H_\theta$ is countable,*

(ii) *$q \leqslant p$,*

(iii) *Y respects $I_q^{\mathbb{P}}$, in particular Y is f-slim and*

(iv) *$q \in \pi_Y[\mu^+_{X,Y}(D)]$.*

Proof. We may assume that X is an elementary substructure of

$$\mathcal{H} := (H_\theta; \in, \trianglelefteq)$$

where $\trianglelefteq$ is a wellorder of H_θ. As $\mathbb{P}$ is respectful and X respects $I_p^{\mathbb{P}}$, there is a $(X, \mathbb{P})$-semigeneric condition $r \leqslant p$ so that

$$r \Vdash \text{"}\check{X}[\dot{G}] \text{ respects } \mathrm{NS}_{\check{f}}\text{"}$$

i.e. r is $(X, \mathbb{P}, f)$-semigeneric. Let G be $\mathbb{P}$-generic with $r \in G$. Then $X[G]$ is f-slim. Let $Z = X[G] \cap V$, note that $\mu^+_{X,Z}$ exists by Proposition 3.16. Now there is thus some $q \leqslant p$, $q \in G$ with

$$q \in \pi_Z[\mu^+_{X,Z}(D)].$$

Finally, note that q and $Y := \mathrm{Hull}^{\mathcal{H}}(X \cup \{q\})$ have the desired properties. $\square$

Proof of Theorem 4.1. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be an iteration of f-preserving forcings which preserve old f-stationary sets and forces SRP at successor steps. We may assume inductively that $\mathbb{P}_\alpha$ preserves f for all $\alpha < \gamma$. The successor step is trivial, so we may restrict to $\gamma \in \mathrm{Lim}$. Note that we may further assume that (‡) holds in V, otherwise we could work in $V^{\mathbb{P}_1}$. Let $p \in \mathbb{P}$ and let $I := I^{\mathbb{Q}_0}_{p(0)}$. I is a normal uniform f-ideal by Proposition 4.14. Now let θ be sufficiently large and regular, $X \prec H_\theta$ countable with

(X.i) $\mathbb{P}, p, f \in X$ and

(X.ii) X respects I.

By Proposition 4.5 and Lemma 4.16, it suffices to find $q \leqslant p$ that is $(X, \mathbb{P}, f)$-semigeneric. Note that X is f-slim as I is a f-ideal. Let

$$h\colon \omega \to \omega \times \omega$$

be a surjection with $i \leqslant n$ whenever $h(n) = (i, j)$.
Let δ denote δ^X. We will construct a fusion structure

$$T, \langle p^{(a,n)}, T^{(a,n)} \mid a \in T_n,\ n < \omega \rangle$$

in $\mathbb{P}$ as well as names

$$\left\langle \dot{X}^{(a,n)}, \dot{Z}^{(a,n)} \left(\dot{D}_j^{(a,n)}\right)_{j<\omega}, \dot{I}^{(a,n)} \mid a \in T_n, n < \omega \right\rangle$$

so that for any $n < \omega$ and $a \in T_n$

(*F.i*) $T_0 = \{\mathbb{1}\}$, $p^{(\mathbb{1},0)} = p$, $\dot{X}^{(\mathbb{1},0)} = \check{X}$, $\dot{I}^{(\mathbb{1},0)} = \check{I}$,

(*F.ii*) $T^{(\mathbb{1},0)} \in X$ is a nested antichain that p is a mixture of with $T_0^{(\mathbb{1},0)} = \{\mathbb{1}\}$,

(*F.iii*) $a \Vdash_{\mathrm{lh}(a)} \dot{Z}^{(a,n)} = \dot{X}^{(a,n)} \cap V$,

(*F.iv*) $\left(\dot{D}_j^{(a,n)}\right)_{j<\omega}$ is forced by a to be an enumeration of all dense subsets of $\pi^{-1}_{\dot{Z}^{(a,n)}}(\check{\mathbb{P}})$ in

$$M_{\dot{Z}^{(a,n)}}\left[\widetilde{f(\delta)}\right],$$

(*F.v*) $a \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(a)$,

(*F.vi*) $\mathrm{lh}(a)$ is not a limit ordinal,

(*F.vii*) $a \Vdash_{\mathrm{lh}(a)} \check{p}^{(a,n)}, \check{T}^{(a,n)}, \dot{G}_{\mathrm{lh}(a)} \in \dot{X}^{(a,n)}$,

(*F.viii*) $a \Vdash_{\mathrm{lh}(a)} \dot{I}^{(a,n)} = I^{\dot{\mathbb{Q}}_{\mathrm{lh}(a)}}_{\check{p}^{(a,n)}(\mathrm{lh}(a))}$ and

(*F.ix*) $a \Vdash$ "$\check{X} \sqsubseteq \dot{X}^{(a,n)} \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(a)}]}$ is countable and respects $\dot{I}^{(a,n)}$".

Moreover, for any $b \in \mathrm{suc}^n_T(a)$

(*F.x*) $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)}$ "$\check{p}^{(b,n+1)}, \check{T}^{(b,n+1)} \in \dot{X}^{(a,n)}$, in particular $\mathrm{lh}(\check{b}), \mathbb{P}_{\mathrm{lh}(\check{a}),\mathrm{lh}(\check{b})} \in \dot{X}^{(a,n)}$",

(*F.xi*) $b \Vdash_{\mathrm{lh}(b)} \dot{X}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}] \sqsubseteq \dot{X}^{(b,n+1)}$ and

(*F.xii*) if $h(n) = (i, j)$ and $c = \mathrm{pred}^i_T(b)$ then

$$b \Vdash_{\mathrm{lh}(n)} \check{p}^{(b,n+1)} \in \pi_{\dot{X}^{(a,n)}}[\dot{\mu}^+_{c,a}(\dot{D}_j^{(c,i)})].$$

Here, $\mu^+_{c,a}$ denotes[10]

$$\mu^+_{\dot{Z}^{(c,i)},\dot{Z}^{(a,n)}} : M_{\dot{Z}^{(c,i)}}[\check{f}(\check{\delta})] \to M_{\dot{Z}^{(a,n)}}[\check{f}(\check{\delta})].$$

We define all objects by induction on $n < \omega$.

$$T_0 = \{\mathbb{1}\}, p^{(\mathbb{1},0)}, T^{(\mathbb{1},0)}, \dot{X}^{(\mathbb{1},0)}, \dot{Z}^{(\mathbb{1},0)} \left(\dot{D}_j^{(\mathbb{1},0)}\right)_{j<\omega}, \dot{I}^{(\mathbb{1},0)}$$

are given by $(F.i)$-$(F.iv)$ and $(F.viii)$. Suppose we have already defined

$$T_n, \left\langle p^{(a,n)}, T^{(a,n)}, \dot{X}^{(a,n)}, \dot{Z}^{(a,n)}, \left(\dot{D}_j^{(a,n)}\right)_{j<\omega} \mid a \in T_n \right\rangle$$

and we will further construct

$$T_{n+1}, \left\langle p^{(b,n+1)}, T^{(b,n+1)}, \dot{X}^{(b,n+1)}, \dot{Z}^{(b,n+1)}, \left(\dot{D}_j^{(b,n+1)}\right)_{j<\omega} \mid b \in T_{n+1} \right\rangle.$$

Fix $a \in T_n$. Let E be the set of all b so that

$(E.i)$ $b \in \mathbb{P}_{\mathrm{lh}(b)}$ and $\mathrm{lh}(b) < \gamma$,

$(E.ii)$ $\mathrm{lh}(a) \leqslant \mathrm{lh}(b)$ and $b \upharpoonright \mathrm{lh}(a) \leqslant a$,

and there are a nested antichain S in $\mathbb{P}$, $s \in \mathbb{P}$ and names $\dot{X}$, $\dot{I}$ with

$(E.iii)$ $S \angle\, T^{(a,n)}$,

$(E.iv)$ $s \leqslant p^{(a,n)}$ is a mixture of S,

$(E.v)$ if $h(n) = (i,j)$ and $c = \mathrm{pred}^i_T(a)$ then

$$b \Vdash_{\mathrm{lh}(b)} \check{s} \in \pi_{\dot{Z}^{(a,n)}}[\dot{\mu}^+_{c,a}(\dot{D}_j^{(c,i)})],$$

$(E.vi)$ $\mathrm{lh}(b)$ is not a limit ordinal,

$(E.vii)$ $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)} \check{s}, \check{S} \in \dot{X}$,

$(E.viii)$ $b \Vdash_{\mathrm{lh}(b)} \check{s} \upharpoonright \mathrm{lh}(b) \in \dot{G}_{\mathrm{lh}(b)}$,

$(E.ix)$ $b \Vdash_{\mathrm{lh}(b)} \dot{X}^{(a,n)} \sqsubseteq \dot{X}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}] \sqsubseteq \dot{X} \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(\check{b})}]}$,

$(E.x)$ $b \Vdash_{\mathrm{lh}(b)}$ "$\dot{X}$ is countable and respects $\dot{I}$",

$(E.xi)$ $b \Vdash_{\mathrm{lh}(b)} \dot{I} = I^{\dot{\mathbb{Q}}_{\mathrm{lh}(b)}}_{\check{s}(\mathrm{lh}(b))}$ and

$(E.xii)$ if $S_0 = \{c_0\}$ then $\mathrm{lh}(b) = \mathrm{lh}(c_0)$ and $b \leqslant c_0$.

[10] There is some slight abuse of notation here in an effort to improve readability.

Claim 4.18. *$E \upharpoonright \mathrm{lh}(a) := \{b \upharpoonright \mathrm{lh}(a) \mid b \in E\}$ is dense in $\mathbb{P}_{\mathrm{lh}(a)}$.*

Proof. Let $a' \leqslant a$ and let G be $\mathbb{P}_{\mathrm{lh}(a)}$-generic with $a' \in G$. By $(F.v)$, $p^{(a,n)} \upharpoonright \mathrm{lh}(a) \in G$. Work in $V[G]$. Let $h(n) = (i, j)$ and $c = \mathrm{pred}^i_T(a)$. Let

$$X^{(c,i)} = \left(\dot{X}^{(c,i)}\right)^{G_{\mathrm{lh}(c)}} \text{ and } X^{(a,n)} = \left(\dot{X}^{(a,n)}\right)^{G}$$

as well as $Z^{(c,i)} = X^{(c,i)} \cap V$, $Z^{(a,n)} = X^{(a,n)} \cap V$. Find $r \in T_1^{(a,n)}$ with $r \upharpoonright \mathrm{lh}(a) \in G$. As $p^{(a,n)}$ is a mixture of $T^{(a,n)}$, we have

$$r \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(r).$$

Let $\hat{r} = r^\frown p^{(a,n)} \upharpoonright [\mathrm{lh}(r), \gamma)$. Note that $\hat{r} \in X^{(a,n)}$, as

$$p^{(a,n)}, T^{(a,n)}, G \in X^{(a,n)}$$

by $(F.vii)$. Moreover, $\hat{r} \upharpoonright \mathrm{lh}(a) \in G$. Let $\mathbb{Q} := \dot{\mathbb{Q}}^G_{\mathrm{lh}(a)}$ and

$$D := \mu^+_{c,a}((\dot{D}^i_j)^{G_{\mathrm{lh}(c)}}) \in M_{Z^{(a,n)}}[f(\delta)] \subseteq M_{X^{(a,n)}}[f(\delta)].$$

Subclaim 4.19. *There are s, Y with*

(i) $X^{(a,n)} \sqsubseteq Y \prec H^{V[G]}_\theta$,

(ii) $s \leqslant p^{(a,n)}$,

(iii) $s \upharpoonright \mathrm{lh}(a) \in G$,

(iv) $s \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D)]$ *and*

(v) Y *respects* $I^{\mathbb{Q}}_{s(\mathrm{lh}(a))}$.

Proof. Let

$$D_0 := \{t \in D \mid \pi_{X^{(a,n)}}(t) \leqslant p^{(a,n)} \wedge \pi_{X^{(a,n)}}(t) \upharpoonright \mathrm{lh}(a) \in G\}$$

and D_1 be the projection of D_0 onto $\pi^{-1}_{X^{(a,n)}}(\mathbb{Q})$. Observe that

$$M_{X^{(a,n)}}[f(\delta)] \models \text{“}D_1 \text{ is dense below } \pi^{-1}_{X^{(a,n)}}(p^{(a,n)}(\mathrm{lh}(a)) \text{ in } \pi^{-1}_{X^{(a,n)}}(\mathbb{Q})\text{”}.$$

Applying Lemma 4.17 with (making use of the notation there)

- $\mathbb{P} = \dot{\mathbb{Q}}$,
- $p = p^{(a,n)}(\mathrm{lh}(a))$,
- $X = X^{(a,n)}$ and
- $D = D_0$,

we find some countable Y and some s_0 with

(i) $X^{(a,n)} \sqsubseteq Y \prec H_\theta^{V[G]}$,

(ii) $s_0 \leqslant p^{(a,n)}(\mathrm{lh}(a))$,

(iii) $s_0 \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D_1)]$ and

(iv) Y respects $I^{\mathbb{Q}}_{s_0}$.

By definition of D_1, there is $s \leqslant p^{(a,n)}$ with

(s.i) $s \upharpoonright \mathrm{lh}(a) \in G$,

(s.ii) $s \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D)]$ and

(s.iii) $s(\mathrm{lh}(a)) = s_0$.

Y, s have the desired properties.

□

We can now apply Fact 3.23 in Y and get a nested antichain $S \in X^{(a,n)}$ with

(S.i) s is a mixture of S,

(S.ii) if $S_0 = \{d\}$ then $\mathrm{lh}(r) \leqslant \mathrm{lh}(d)$, $d \upharpoonright \mathrm{lh}(r) \leqslant r$ and $\mathrm{lh}(d)$ is not a limit ordinal and

(S.iii) $S \angle\, T^{(a,n)}$.

Let $\dot{X}$ be a name for $Y[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(d)}]$ and $\dot{I}$ a name for $I^{\dot{\mathbb{Q}}_{\mathrm{lh}(d)}}_{s(\mathrm{lh}(d))}$.

Subclaim 4.20. *In $V[G]$, we have*

$$\dot{I}^{s\upharpoonright \mathrm{lh}(d)} = I^{\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)+1}}_{s\upharpoonright \mathrm{lh}(d)+1} = I^{\mathbb{Q}}_{s(\mathrm{lh}(a))}.$$

Proof. The first equality is simply by definition of $\dot{I}$. The second equality follows as we preserve old f-stationary sets along the iteration and since $\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)+1}$ preserves f by our inductive hypothesis. □

It follows that

$$Y \text{ respects } \dot{I}^{s\upharpoonright \mathrm{lh}(d)}.$$

As $\mathrm{lh}(a)$ is not a limit ordinal, (‡) holds in $V[G]$, so that $\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)}$ is respectful by Lemma 4.10. Thus there is $b \in \mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)}$, $b \leqslant s \upharpoonright \mathrm{lh}(d)$ so that

$$b \Vdash_{\mathrm{lh}(b)} \text{“}\check{Y} \sqsubseteq \check{Y}[\dot{G}] \text{ respects } \dot{I}\text{”}.$$

Since $b \upharpoonright \mathrm{lh}(a) \in G$, we may assume further that $b \upharpoonright \mathrm{lh}(a) \leqslant a'$. $s, S, \dot{X}, \dot{I}$ witness $b \in E$. □

To define T_{n+1}, fix a maximal antichain $A \subseteq E \upharpoonright \mathrm{lh}(a)$, and for any $e \in A$ choose $b_e \in E$ with $b_e \upharpoonright \mathrm{lh}(a) = e$. We set $\mathrm{suc}^n_T(a) = \{b_e \mid e \in A\}$. For any $b \in \mathrm{suc}^n_T(a)$, let $S, s, \dot{X}, \dot{I}$ witness $b \in E$. We then let

- $p^{(b,n+1)} = s,\ T^{(b,n+1)} = S,\ \dot{X}^{(b,n+1)} = \dot{X},\ \dot{I}^{(b,n+1)} = \dot{I}$,
- $\dot{Z}^{(b,n+1)}$ be a name for $\dot{X} \cap V$ and
- $\left(\dot{D}^{(b,n+1)}_j\right)_{j<\omega}$ be a sequence of names that are forced by b to enumerate all dense subsets of $\pi^{-1}_{\dot{Z}^{(b,n+1)}}(\mathbb{P})$ in $M_{\dot{Z}^{(b,n+1)}}\left[\widetilde{f(\delta)}\right]$.

This finishes the construction.

By Fact 3.23, there is a mixture q of T. Let G be $\mathbb{P}$-generic with $q \in T$. By Fact 3.25, in $V[G]$ there is a sequence $\langle a_n \mid n < \omega\rangle$ so that for all $n < \omega$

($\vec{a}.i$) $a_0 = q_0$,

($\vec{a}.ii$) $a_{n+1} \in \mathrm{suc}^n_T(a_n)$ and

($\vec{a}.iii$) $p^{(a_n,n)} \in G$.

For $n < \omega$, let $\alpha_n = \mathrm{lh}(a_n) < \gamma$. For $n < \omega$ we let

$$X_n := \left(\dot{X}^{(a_n,n)}\right)^{G_{\alpha_n}}$$

and also

$$X_\omega = \bigcup_{n<\omega} X_n[G_{\alpha_n,\gamma}].$$

Further, for $n \leqslant \omega$ let

$$Z_n := X_n \cap V \text{ and } \pi_n := \pi_{Z_n}.$$

We remark that

$$X_n[G_{\alpha_n,\gamma}] \sqsubseteq X_m[G_{\alpha_m,\gamma}] \prec H^{V[G]}_\theta$$

follows inductively from ($F.vii$) and ($F.ix$) for $n \leqslant m < \omega$ so that $X_\omega \prec H^{V[G]}_\theta$. We aim to prove that

$$X \sqsubseteq X_\omega \text{ is } f\text{-slim.}$$

In fact, we will show

($Z_\omega.i$) $X \sqsubseteq Z_\omega$,

($Z_\omega.ii$) Z_ω is f-slim and

($Z_\omega.iii$) $\pi^{-1}_\omega[G]$ is generic over $M_\omega[f(\delta)]$,

which implies the above.

Claim 4.21. $Z_\omega = \bigcup_{n<\omega} Z_n$.

Proof. " $\supseteq$ " is trivial, so we show " $\subseteq$ ". Let $x \in Z_\omega$ and find $i < \omega$ with $x \in X_i[G_{\alpha_i,\gamma}]$. Note that there is $\dot{x} \in Z_i$ a $\mathbb{P}$-name for a set in V with $x = \dot{x}^G$. Let $D \in M_i$ be the dense set of conditions in $\pi_n^{-1}(\mathbb{P})$ deciding $\pi_i^{-1}(\dot{x})$. There must be some $j < \omega$ so that

$$\left(\dot{D}_j^{(a_i,i)}\right)^G = D.$$

Now find n with $h(n) = (i, j)$. We then have

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{a_i,a_{n+1}}(D)]$$

by $(F.xii)$. We have that $p^{(a_{n+1},n+1)}$ decides $\dot{x}$ to be some $z \in X_n$, and as $p^{(a_{n+1},n+1)} \in G$,

$$x = \dot{x}^G = z \in X_n \cap V = Z_n.$$

□

As $X \sqsubseteq X_n$ is f-slim by $(F.ix)$ for $n < \omega$, $(Z_\omega.i)$ and $(Z_\omega.ii)$ follow at once. It remains to show $(Z_\omega.iii)$.
As Z_ω is f-slim and by Claim 4.21, we have that

$$\langle M_\omega[f(\delta)], \mu^+_{n,\omega} \mid n < \omega\rangle = \varinjlim \langle M_n[f(\delta)], \mu^+_{n,m} \mid n \leqslant m < \omega\rangle$$

for some $(\mu^+_{n,\omega})_{n<\omega}$. Let $E \in M_\omega[f(\delta)]$ be dense in $\pi_\omega^{-1}(\mathbb{P})$. Then for some $i, j < \omega$, $E = \mu^+_{i,\omega}(D)$ for

$$D := \left(\dot{D}_j^{(a_i,i)}\right)^G.$$

Find n with $h(n) = (i, j)$. By $(F.xii)$,

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{i,n}(D)] \subseteq \pi_\omega[\mu^+_{i,\omega}(D)] = \pi_\omega[E].$$

As $p^{(a_{n+1},n+1)} \in G$, we have $E \cap \pi_\omega^{-1}[G] \neq \emptyset$, which is what we had to show. □

5 f-Proper and f-Semiproper Forcings

Suppose f witnesses $\diamondsuit(\mathbb{B})$. We already used the term $(X, \mathbb{P}, f)$-semigeneric which suggests there should be a notion of f-semiproperness. Indeed there is and it behaves roughly like semiproperness. In fact, there are several other classes associated to f which mirror well-known forcing classes.

Definition 5.1. A forcing $\mathbb{P}$ is f-complete if for any sufficiently large regular θ, for any f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$ and any $g \subseteq \bar{\mathbb{P}}$ generic over $M_X[f(\delta^X)]$, there is a some $p \in \mathbb{P}$ with

$$p \Vdash \dot{G} \cap \check{X} = \pi_X[\check{g}].$$

Definition 5.2. A forcing $\mathbb{P}$ is f-proper if for any sufficiently large regular θ, any f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$ and any $p \in X \cap \mathbb{P}$, there is a $(X, \mathbb{P}, f)$-generic condition $q \leqslant p$, that is a condition q with

$$q \Vdash \dot{G} \cap \check{X} \text{ is generic over } \check{X} \wedge \check{X}[\dot{G}] \text{ is } \check{f}\text{-slim''}.$$

Definition 5.3. A forcing $\mathbb{P}$ is f-semiproper if for any sufficiently large regular θ, any f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$ and any $p \in X \cap \mathbb{P}$, there is a $(X, \mathbb{P}, f)$-semigeneric condition $q \leqslant p$.

The following graphic collects all provable relations between the relevant forcing classes.

Classical		$\diamondsuit$-Forcing
complete($\approx$ σ-closed)	$\Longrightarrow$	f-complete
$\Downarrow$		$\Downarrow$
proper		f-proper
$\Downarrow$		$\Downarrow$
semiproper		f-semiproper
$\Downarrow$		$\Downarrow$
stationary set preserving		f-stationary set preserving
$\Downarrow$		$\Downarrow$
ω_1-preserving	$\Longleftarrow$	f-preserving

We also get the expected iteration theorems.

Theorem 5.4. *Any countable support iteration of f-complete (resp. f-proper) forcings is f-complete (resp. f-proper).*

Theorem 5.5. *Any nice iteration of f-semiproper forcings is f-semiproper.*

The proof is much easier than that of Theorem 4.1, so we omit it.

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