Set-Theoretic Geology

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0 Abstract

Set-Theoretic Geology is the study of grounds, the base models of forcing extensions, and the generic multiverse and was initially founded by Hamkins and Reitz in an effort to find regular structure under the generic "dust" added by forcing. Although their hope was not quite fulfilled, this investigation left open many interesting questions about the nature of forcing, until the recent results of Usuba about the strong downwards directed grounds hypothesis. For example the mantle, the intersection of all grounds, turned out to be a model of ZFC and the largest forcing invariant definable class.
The first chapter of this thesis deals with basic theory of forcing and provides a foundation for the rest of this thesis.
After that, we dive right into the theory of Set-Theoretic Geology. The main theorem, with which we will start, could be described as the fundamental theorem of this topic, the uniform definability of grounds. Moreover, we will examine the implications of Usuba's breakthrough. This serves as a good motivation for chapter 3, where we will proof these results.
In the following chapter, we will discuss the interplay between the generic multiverse, its mantle and large cardinals, including Usuba's results on extendible and hyper-huge cardinals. In addition to this, we will investigate how large cardinals at and below the level of a supercompact relate to the mantle. The supercompacts will come out as the most flexible large cardinals, they can both be found in the mantle with no sign of them in the entire generic multiverse and lose their supercompactness (even weak compactness) by passing to the mantle. Also, we will find a connection between the generic multiverse and the mantle regarding smaller large cardinals, that will make the first situation impossible for them.
Chapter 6 serves as an addendum and deals with findings crucial for our analysis in chapters 2 and 3. But since their nature is not inherently geologic and have been known for longer than this topic exists, we skip the proofs in these chapters. Most prominently, one can find a discussion of the inner model criterion, i.e. a first order sentence that checks whether or not a given class is an inner model ZFC, and a proof of Bukovský's Theorem in there.
We finish with a conclusion.

Contents

1 Preliminaries on Forcing

In this section, we discuss basic theory on forcing which serves as a basis for constructions later on in this thesis. The results in this chapter can be found in most standard introductions to forcing, such as [Jec03] and [Kun83].

1.1 Δ-System Arguments

Definition 1.1.1. Given two ordinals $\kappa, \lambda > 0$, the forcing $Add(\kappa, \lambda)$ consists of functions p of size $< \kappa$ with $dom(p) \subseteq \kappa \times \lambda$ and $ran(p) \subseteq 2$, ordered by reverse inclusion.

It is standard to show that if G is $Add(\kappa, \lambda)$-generic over V then $f = \bigcup G : \kappa \times \lambda \to 2$ is a function with $f \upharpoonright (\kappa \times \{\alpha\}) \notin V$ for all $\alpha < \lambda$. In particular $Add(\kappa, \lambda)$ adds λ many new subsets of κ.

Lemma 1.1.2. *(Δ-system lemma) Suppose $\kappa < \lambda$ are cardinals such that λ is regular and $\alpha^{<\kappa} < \lambda$ for all $\alpha < \lambda$. If X is a set of size λ such that $|x| < \kappa$ for every $x \in X$, then there is some $Y \subseteq X$ of size λ and a set r with $x \cap y = r$ for all $x \neq y \in Y$. In this situation, Y is called a Δ-system with root r.*

Proof. Since $|\bigcup X| \leqslant \sum_{x \in X} |x| \leqslant \lambda \cdot \kappa = \lambda$, we can assume that $X \subseteq \mathcal{P}(\lambda)$ and so we will identify $x \in X$ with its increasing enumeration $\langle x(\alpha) | \alpha < otp(x) \rangle$. For notational simplicity, we will allow us to switch between viewing $x \in X$ as a function and a subset of λ. By regularity of λ, we can furthermore impose without loss of generality that all $x \in X$ have the same ordertype γ. If $\alpha < \lambda$ then $\alpha^{\gamma} < \lambda$ and hence $X \nsubseteq \mathcal{P}(\alpha)$. This implies that $\bigcup X$ is unbounded in λ and as λ is regular and $\gamma < \kappa < \lambda$, there must be some minimal $\beta < \gamma$ such that $\{x(\beta) | x \in X\}$ is unbounded in λ. Let $\rho = sup \bigcup \{ran(x \upharpoonright \beta) | x \in X\}$. We must have $\rho < \lambda$ as a consequence of λ being regular. We construct a sequence $\langle x_\alpha | \alpha < \lambda \rangle$ by induction on α. If x_δ is already constructed for all $\delta < \alpha < \lambda$ then $\bigcup_{\delta < \alpha} x_\delta$ is bounded in λ. Hence there must be some $x_\alpha \in X$ with $x_\alpha(\beta) > sup \bigcup_{\delta < \alpha} x_\delta$. This ensures that $x_\alpha \cap x_\delta = (x_\alpha \upharpoonright \beta) \cap (x_\delta \upharpoonright \beta) \subseteq \rho$ for all $\delta < \alpha$ (where we identify $x \upharpoonright \beta$ with its range).
Now $\{x_\alpha \cap \rho | \alpha < \lambda\} \subseteq \mathcal{P}(\rho)$ and thus has size at most $2^\rho < \lambda$. Hence there must be some $r \subseteq \rho$ and $A \subseteq \lambda$ of size λ such that $x_\alpha \cap \rho = r$ for all $\alpha \in A$. Let $Y = \{x_\alpha | \alpha \in A\}$. Then for $x \neq y$ both in Y, we have $x \cap \rho = r = y \cap \rho$ and $x \cap y \subseteq \rho$ and hence $x \cap y = r$. □

Lemma 1.1.3. *Let κ be a regular cardinal and θ any ordinal.*

(i) If $\kappa^{<\kappa} = \kappa$ then $Add(\kappa, \theta)$ is κ^+-cc.

(ii) If $2^\kappa = \kappa^+$ then $Add(\kappa, \theta)$ is κ^{++}-cc.

Proof. Suppose $A \subseteq Add(\kappa, \theta)$ and let $X = \{dom(p) | p \in A\}$.

(i) Assume A is of size κ^+. Apply the Δ-system lemma (1.1.2) to X with $\lambda = \kappa^+$. This yields $B \subseteq A$ of size κ^+ and some $r \subseteq \kappa \times \theta$ of size less than κ such that $dom(p) \cap dom(q) = r$ for all $p \neq q \in B$. But as $\kappa^{<\kappa} = \kappa$, $2^{|r|} \leqslant \kappa < \kappa^+$ which implies that there must be some $C \subseteq B$ of size κ^+ with $p \upharpoonright r = q \upharpoonright r$ for all $p, q \in C$. But in this case, p and q must be compatible witnessed by $p \cup q$.

(ii) Here, assume that A is of size κ^{++} and apply the Δ-system lemma to X with $\lambda = \kappa^{++}$. To do this we have to check that $(\kappa^+)^{<\kappa} < \kappa^{++}$. For any $\theta < \kappa$, this simplifies to $(\kappa^+)^\theta = (2^\kappa)^\theta = 2^\kappa = \kappa^+$. Since $\kappa^{<\kappa} \leqslant \kappa^\kappa = \kappa^+$, we can coclude as above that there must be some subset of A of cardinality κ^{++} that consists of pairwise compatible conditions.

□

1.2 Counting Nice Names

Definition 1.2.1. A nice $\mathbb{P}$-name for a subset of an ordinal α is a $\mathbb{P}$-name of the form $\dot{x} = \bigcup_{\beta<\alpha} \{\check{\beta}\} \times A_\beta$ where A_β is an antichain of $\mathbb{P}$ for each β.

Lemma 1.2.2. *(The counting nice names argument) Let $\mathbb{P}$ be a forcing and G be $\mathbb{P}$-generic.*

(i) *If $x \in \mathcal{P}(\alpha)^{V[G]}$ then there is a nice name $\dot{x}$ with $x = \dot{x}^G$.*

(ii) *If λ, κ are cardinals such that $\mathbb{P}$ has the κ-cc then*

$$(2^\lambda)^{V[G]} \leqslant ((|\mathbb{P}|^{<\kappa})^\lambda)^V$$

Furthermore if $\lambda \geqslant (|\mathbb{P}|^{<\kappa})$ then $(2^\lambda)^{V[G]} = (2^\lambda)^V$.

Proof. (i) Find a name $\dot{z}$ for x. For $\beta < \alpha$, let $D_\beta = \{p \in \mathbb{P} | p \Vdash \check{\beta} \in \dot{z}\}$ and find an antichain $A_\beta \subseteq D_\beta$ that is maximal in D_β. Let $\dot{x} = \bigcup_{\beta<\alpha}\{\check{\beta}\} \times A_\beta$. We have to show $\dot{x}^G = x$.
"$\subseteq$" : If $\beta \in \dot{x}^G$ then $A_\beta \cap G \neq \emptyset$. Find p in this intersection. Then $p \Vdash \check{\beta} \in \dot{z}$, so we have $\beta \in x$.
"$\supseteq$" : Let $\beta \in x$. Find $p \in G$ with $p \Vdash \check{\beta} \in \dot{z}$, i.e. $p \in D_\beta$. Let $E = \{q \leqslant p | \exists a \in A_\beta \; q \leqslant a\}$. If $q \leqslant p$ then still $q \in D_\beta$. As A_β was chosen maximal in D_β, there is $a \in A_\beta$ and $r \leqslant a, q$. Then $r \in E$ and $r \leqslant q$ which shows that E is dense below p. This implies $G \cap A_\beta \neq \emptyset$ and so $\beta \in \dot{x}^G$.

(ii) The first part shows that the size of $\mathcal{P}(\lambda)^{V[G]}$ is bounded by the size of the set of all nice $\mathbb{P}$-names for subsets of λ in V. A nice name is basically a function that maps each $\beta < \alpha$ to an antichain in $\mathbb{P}$. As $\mathbb{P}$ has the κ-cc, we can understand such a nice name as a function $f : \lambda \to \mathcal{P}_\kappa(\mathbb{P})$, where $\mathcal{P}_\kappa(\mathbb{P})$ is the set of all subsets of $\mathbb{P}$ of size $< \kappa$. As $|\mathcal{P}_\kappa(\mathbb{P})| = |\mathbb{P}|^{<\kappa}$, there are $(|\mathbb{P}|^{<\kappa})^\lambda$ many of these functions in V. If now $\lambda \geqslant (|\mathbb{P}|^{<\kappa})$ then first of all $(2^\lambda)^V \geqslant \kappa$ is still a cardinal in $V[G]$. We calculate:

$$(2^\lambda)^V \leqslant (2^\lambda)^{V[G]} \leqslant ((|\mathbb{P}|^{<\kappa})^\lambda)^V \leqslant (\lambda^\lambda)^V = (2^\lambda)^V$$

□

Remark 1.2.3. In any case, $\mathbb{P}$ always has the $|\mathbb{P}|^+$-cc and thus the above lemma shows that if G is $\mathbb{P}$-generic, then the continuum function of V and $V[G]$ coincide from $2^{|\mathbb{P}|}$ onwards.

Similarly as for subsets of λ, there are nice names for functions with domain λ and range in V. In fact, we will only need that there are small names.

Proposition 1.2.4. *Suppose that κ is an infinite cardinal, $\mathbb{P}$ a forcing of size κ, G a $\mathbb{P}$-generic filter and $f : \lambda \to Ord$ a function in $V[G]$. Then there is a $\mathbb{P}$-name for f in V of size $\kappa \cdot \lambda$.*

Proof. Find a $\mathbb{P}$-name $\dot{f}$ for f so that:

$$\mathbb{1}_\mathbb{P} \Vdash \text{“}\dot{f} : \check{\lambda} \to Ord \text{ is a function”}$$

For $\alpha < \lambda$, let D_α be the (dense) set of conditions that decide $\dot{f}(\check{\alpha})$ and for $p \in D_\alpha$ let $d(p)$ be the corresponding decision. Then

$$\dot{g} = \left\{ \left(op\left(\check{\alpha}, \widecheck{d(p)}\right), p\right) | \alpha < \lambda, p \in D_\alpha \right\}$$

is another $\mathbb{P}$-name for f, where $op(\dot{x}, \dot{y})$ is the canonical $\mathbb{P}$-name for the corresponding ordered pair. Furthermore, $\dot{g}$ has size $\kappa \cdot \lambda$. □

1.3 Degrees of Closure

In this section we will introduce both a strengthening and a weakening of the usual $\leqslant \lambda$-closure conditions which prescribes that any decreasing sequence $\langle p_\alpha | \alpha < \gamma \rangle$ for $\gamma \leqslant \lambda$ has a lower bound.

Definition 1.3.1. Let $\mathbb{P}$ be a forcing.

(*i*) A subset X of $\mathbb{P}$ is directed if for any $p, q \in X$ there is $r \in X$ with $r \leqslant p, q$.

(*ii*) $\mathbb{P}$ is $\leqslant \lambda$-directed closed if any directed $X \subseteq \mathbb{P}$ of size $\leqslant \lambda$ has a lower bound in $\mathbb{P}$.

Directed closure is certainly a stronger condition than mere closure as any decreasing sequence is directed. This concept is important with regard to Laver indestructibility, which comes up in chapter 4. In the same context, we will will force that a certain combinatorial property holds in the generic extension. This forcing will in general not have the desired closure property, but will be sufficiently *strategically* closed.

Definition 1.3.2. Let $\mathbb{P}$ be a forcing and α an ordinal.

(*i*) For α an ordinal, $G(\mathbb{P}, \alpha)$ is the following two player game of perfect information of length α: The goal is to construct a decreasing sequence $\langle p_\beta | \beta < \alpha \rangle$ in $\mathbb{P}$. The game starts with player II playing $p_0 = \mathbb{1}_{\mathbb{P}}$. If $\langle p_\beta | \beta < \gamma \rangle$ has already been played for some $\gamma < \alpha$ then it is player I to play if γ is odd and player II to play if γ is even. In any case, a legal move is a p_γ which extends every p_β for $\beta < \gamma$. If there is no such legal move, player I wins the game (notice that this can only happen at a limit stage γ, where it is player II to play). Otherwise, if the game reaches stage α, player II wins.

(*ii*) A strategy in the game $G(\mathbb{P}, \alpha)$ for a player is a complete plan of action for every possible configuration where this player is asked to make a move. More formally, a strategy for player II is a function which maps every decreasing sequence $\langle p_\beta | \beta < \gamma \rangle$ to a legal move p_γ (if there is one) for γ even, similarly for player I with γ odd. If σ_I, σ_{II} are strategies for players I and II respectively, then there is a unique outcome $O(\sigma_I, \sigma_{II}) = \langle p_\beta | \beta < \gamma \rangle$ which is the result of always playing according to the strategies. If player I has won, then $\gamma < \alpha$ and the sequence can not be extended further. Otherwise, $\gamma = \alpha$ and player II has won. A strategy is a winning strategy if it wins against every possible strategy of the opposing player.

(*iii*) The forcing $\mathbb{P}$ is said to be $\leqslant \lambda$-strategically closed if player II has a winning strategy in the game $G(\mathbb{P}, \lambda + 1)$. Notice that the last move player II plays in this game extends a decreasing sequence of length λ. $\mathbb{P}$ is $< \lambda$-strategically closed if player II has a winning strategy in the game $G(\mathbb{P}, \lambda)$. Note that this is in general a stronger assumption than being $< \alpha$-strategically closed for all $\alpha < \lambda$.

Remark 1.3.3. We have defined a strategy as a function that prescribes a legal move at any possible state of the game. However, when we explicitly define strategies, we will usually only prescribe an action at positions that are important for the argument and neglect positions that are irrelevant.

Any $\leqslant \lambda$-closed forcing is $\leqslant \lambda$-strategically closed since any strategy for player II is in fact a winning strategy in the game $G(\mathbb{P}, \lambda+1)$. On the other hand, every $\leqslant \lambda$-strategically closed forcing does not add new sequences of ordinals of length $\leqslant \lambda$. This latter concept is known as $\leqslant \lambda$-distributivity. To be precise, a forcing is $\leqslant \lambda$-distributive if the intersection of λ-many dense open subsets is again dense. This directly implies that $\mathbb{P}$ does not add new ordinal sequences of length $\leqslant \lambda$ and the inverse implication is true for all separative forcings. Since all forcings considered in this thesis will be separative, we will use these two properties interchangeably.

Proposition 1.3.4. *If $\mathbb{P}$ is $\leqslant \lambda$-strategically closed then it is $\leqslant \lambda$-distributive. Consequently, if $\mathbb{P}$ is $< \lambda$-strategically closed then it is $< \lambda$-distributive.*

Proof. Assume $\langle D_\alpha | \alpha < \lambda \rangle$ is a sequence of dense open subsets of $\mathbb{P}$. Let σ_I be the strategy for player I which demands him to extend $p_{2\alpha}$ to a condition $p_{2\alpha+1} \in D_\alpha$ at stage $2\alpha + 1$. Let σ_{II} be a winning strategy for player II. These strategies build a sequence $O(\sigma_I, \sigma_{II}) = \langle p_\alpha | \alpha \leqslant \lambda \rangle$ and by the choice of σ_I, $p_\lambda \in \bigcap_{\alpha<\lambda} D_\alpha$. □

Lemma 1.3.5. *[Cum10] If λ is a cardinal and $\mathbb{P}, \mathbb{Q}$ are forcings such that $\mathbb{P}$ is λ-cc and $\mathbb{Q}$ is $< \lambda$-strategically closed, then:*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\check{\mathbb{Q}} \textit{ is } < \check{\lambda}\textit{-distributive”}$$

Proof. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$ generic over V. Assume $\dot{f}$ is a $\mathbb{P} \times \mathbb{Q}$-name with $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} \Vdash$ “$\dot{f} : \check{\gamma} \to Ord$ is a function” for some $\gamma < \lambda$. Define subsets of $\mathbb{Q}$ for $\alpha < \gamma$:

$$D_\alpha = \{q \in \mathbb{Q} | \exists A \subseteq \mathbb{P} \text{ max. AC such that } \forall p \in A \ (p, q) \parallel \dot{f}(\check{\alpha})\}$$

The D_α are dense: Let $q \in \mathbb{Q}$. Define a decreasing sequence $\langle q_\alpha | \alpha < 2\delta \rangle$ in $\mathbb{Q}$ and a maximal AC $\langle p_\alpha | \alpha < \delta \rangle$ in $\mathbb{P}$. Find $q_1 \leqslant q$ and $p_0 \in \mathbb{P}$ such that $(p_0, q_1) \parallel \dot{f}(\check{\alpha})$. q_1 shall be the first move of player I. Let player II play according to a winning strategy in the game $G(\mathbb{Q}, \lambda)$. If $q_{2\beta+1}, p_\beta$ are defined for all $\beta < \xi$ then first of all $\xi < \lambda$ as $\{p_\beta | \beta < \xi\}$ is an antichain. If this is a maximal antichain, stop the procedure and let $\delta = \xi$. Else, find $\tilde{p}$ incompatible with all p_β and let $q_{2\xi}$ be the next move of player II that extends all q_β for $\beta < 2\xi$. Now let $(p_\xi, q_{2\xi+1}) \leqslant (\tilde{p}, q_{2\xi})$ with $(p_\xi, q_{2\xi+1}) \parallel \dot{f}(\check{\alpha})$.
By the λ-cc of $\mathbb{P}$, δ must be less than λ. We let player II play one last move to find some $q_{2\delta}$ that lies below all q_β for $\beta < 2\delta$. Now $\{p_\beta | \beta < \delta\}$ is a maximal antichain and $(p_\beta, q_{2\delta}) \parallel \dot{f}(\check{\alpha})$ holds for all $\beta < \delta$. Thus $q_{2\delta} \in D_\alpha$ and $q_{2\delta} \leqslant q$. This implies that D_α is dense and it is certainly open as if $q \in D_\alpha$ witnessed by $A \subseteq \mathbb{P}$ then this A works for any $q' \leqslant q$.
Finally, find $q \in H \cap \bigcap_{\alpha<\gamma} D_\alpha$ using the $< \lambda$-distributivity of $\mathbb{Q}$ (which is a consequence of $< \lambda$-strategical closure). Let A_α be a maximal antichain of $\mathbb{P}$ that witnesses $q \in D_\alpha$. We see that $\dot{f}^{G\times H}(\alpha)$ is the unique x such that the unique $p \in G \cap AS_\alpha$ decides $\dot{f}(\check{\alpha})$ as x. Hence $\dot{f}^{G\times H}$ is definable in $V[G]$.
□

1.4 Elementary Embeddings and Extenders

We mostly follow [Kan09] and [Cum10] here.

Definition 1.4.1. Given two $\in$-models $\langle M, \in\rangle$ and $\langle N, \in\rangle$, a map $j : M \to N$ is an elementary embedding if for every $\in$-formula $\varphi(x_0, \dots, x_{n-1})$ and parameters $a_0, \dots, a_{n-1}$ in M, the following holds:

$$\langle M, \in\rangle \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow \langle N, \in\rangle \models \varphi(j(a_0), \dots, j(a_{n-1}))$$

Usually we just write M for $\langle M, \in\rangle$ and N for $\langle N, \in\rangle$. Moreover, the critical point $crit(j)$ of j is the least ordinal moved by j, if there is any.

Lemma 1.4.2. *[Cum10] Suppose M is an inner model and $j : V \to M$ is an elementary embedding, G is $\mathbb{P}$-generic over V and H is $j(\mathbb{P})$-generic over M. Suppose $j[G] \subseteq H$. Then j lifts to an elementary embedding*

$$j^+ : V[G] \to M[H]$$

with $j^+ \restriction V = j$.

Proof. We have to extend j to evaluations of $\mathbb{P}$-names. Thus we define $j^+(\dot{x}^G) = j(\dot{x})^H$. Notice that since $\dot{x}$ is a $\mathbb{P}$-name, $j(\dot{x})$ is a $j(\mathbb{P})$-name.

Claim 1.4.3. *j^+ is welldefined.*

Proof. Suppose $\dot{x}^G = \dot{y}^G$. Find $p \in G$ with $p \Vdash \dot{x} = \dot{y}$. By elementarity, $M \models j(p) \Vdash j(\dot{x}) = j(\dot{y})$. By our assumption, $j(p) \in H$. Hence $j(\dot{x})^H = j(\dot{y})^H$. □

j^+ really extends j as

$$j^+(x) = j(\check{x}^G) = j(\check{x})^H = \widecheck{j(x)}^H = j(x)$$

and so it is only left to show that $j^+ : V[G] \to M[H]$ is elementary. Let φ be an $\in$-formula. For simplicity, we assume that φ only has one free variable. Let $a = \dot{a}^G \in V[G]$. We have:

$$\begin{aligned} V[G] \models \varphi(a) &\Leftrightarrow \exists p \in G\ V \models p \Vdash_{\mathbb{P}} \varphi(\dot{a}) \\ &\Rightarrow \exists q \in H\ M \models q \Vdash_{j(\mathbb{P})} \varphi(\dot{a}) \Leftrightarrow M[H] \models \varphi(a) \end{aligned}$$

The middle implication follows by elementarity as $j[G] \subseteq H$. Lastly, observe that just this one direction is enough for elementarity. □

Next, we look into a way to approximate elementary embeddings by much better controllable ones, that is embeddings which are induced by ultrafilters. This has the advantage that we know exactly how the target models look like. They will be quite thin after some point, which allows us to lift these embeddings more easily.

Definition 1.4.4. Suppose M is an inner model and $j : V \to M$ a nontrivial elementary embedding with critical point κ. Let $\beta > \kappa$ and ζ minimal with $\beta \leqslant j(\zeta)$. β will be the variable that controls the degree of approximation. In most cases, we will have $\beta = j(\kappa)$ and thus $\zeta = \kappa$. For $a \in \beta^{<\omega}$, let

$$E_a = \{X \subseteq \zeta^{|a|} | a \in j(X)\}$$

and $\mathcal{E} = \{E_a | a \in \beta^{<\omega}\}$. Then $\mathcal{E}$ is the (κ, β)-extender derived from j.

The standard arguments show that the E_a are all ultrafilters and thus induce an elementary embedding $j_a : V \to Ult(V, E_a)$. One can check that for every $a \in \beta^{<\omega}$, j factors as $k_a \circ j_a$ where $k_a : Ult(V, E_a) \to M$ is an elementary embedding defined via $k_a([f]_{E_a}) = j(f)(a)$. In particular, $Ult(V, E_a)$ is wellfounded and so we will identify them with their transitive collapse. Now suppose $a \subseteq b \in \beta^{<\omega}$, say $b = \{\alpha_0, \dots, \alpha_{n-1}\}$ and $a = \{\alpha_{i_0}, \dots, \alpha_{i_{m-1}}\}$ in increasing order respectively. Let $\pi_{ab} : \beta^n \to \beta^m$ be given by

$$\pi_{ab}(\{\beta_0, \dots \beta_n\}) = \{\beta_{i_0}, \dots, \beta_{i_{m-1}}\}$$

where the sets are again represented in increasing order. Then the map

$$j_{ab} : Ult(V, E_a) \to Ult(V, E_b),\ j_{ab}([f]_{E_a}) = [f \circ \pi_{ab}]_{E_b}$$

is an elementary embedding between the corresponding ultrapowers and these maps cohere in the sense that $j_{ab} \circ j_a = j_b$.

Definition 1.4.5. In this situation, we will write $(M_{\mathcal{E}}, \langle j_{a\mathcal{E}} | a \in \beta^{<\omega}\rangle)$ for the direct limit of the directed system

$$(\langle Ult(V, E_a) | a \in \beta^{<\omega}\rangle, \langle j_{ab} | a \subseteq b \in \beta^{<\omega}\rangle)$$

and $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ for the elementary embedding given by $j_{a\mathcal{E}} \circ j_a$ (for any $a \in \beta^{<\omega}$).

Since each $Ult(V, E_a)$ embeds into M via k_a, and since these embeddings cohere with the j_{ab} via $j_{ab} \circ k_a = k_b$, the universal property of the direct limit yields an elementary embedding $k : M_{\mathcal{E}} \to M$ with $k \circ j_a = k_a$ for all $a \in \beta^{<\omega}$ and thus $k \circ j_{\mathcal{E}} = j$. In particular, $M_{\mathcal{E}}$ is wellfounded and thus we will always assume that $M_{\mathcal{E}}$ is transitive (and hence an inner model).

Fact 1.4.6. *[Kan09, Lemma 26.1] The following hold:*

(i) $M_{\mathcal{E}} = \{j_{\mathcal{E}}(f)(a) | a \in \beta^{<\omega}, f : \kappa^{|a|} \to V\}$

(ii) If $|M_\gamma|^M \leqslant \beta$ *then* $M_\gamma = (M_{\mathcal{E}})_\gamma$ *and* $k_{\mathcal{E}}(x) = x$ *for all* $x \in (M_{\mathcal{E}})_\gamma$.

(iii) $crit(j_{\mathcal{E}}) = \kappa$ *and* $j_{\mathcal{E}}(\kappa) \geqslant \beta$.

Remark 1.4.7. With the above fact, we can conclude that if β is inaccessible in M (as will be the case in any application in this thesis), the maps $j_{a\mathcal{E}}$ are given by:

$$j_{a\mathcal{E}}([f]_{E_a}) = j_{\mathcal{E}}(f)(a)$$

The reasoning being that $k(\alpha) = \alpha$ for all $\alpha < \beta$ is a consequence of (ii) by our assumption on β and hence $k(a) = a$ for all $a \in \beta^{<\omega}$. We compute:

$$\begin{aligned} k(j_{a\mathcal{E}}([f]_{E_a})) =& k \circ j_{a\mathcal{E}}([f]_{E_a}) = k_a([f]_{E_a}) \\ =& j(f)(a) = k(j_{\mathcal{E}}(f))(k(a)) = k(j_{\mathcal{E}}(f)(a)) \end{aligned}$$

And so the assertion follows from the injectivity of k.

Without deriving such an extender from some embedding j, it is possible to axiomatize how a system of ultrafilters $(E_a)_{a\in\beta^{<\omega}}$ shall behave in order to make the above construction work nonetheless. This yields a first order definition of an (κ, β)-extender $\mathcal{E}$ such that the (κ, β) extender derived from the resulting embedding $j_{\mathcal{E}}$ is again $\mathcal{E}$. This implies that a large cardinal axiom which we will define later is first order definable. We state another fact that we will use later on.

Fact 1.4.8. *[Cum10, Proposition 9.4 1.] If $\mathcal{E}$ is a derived (κ, β)-extender (again with $\beta \leqslant j(\kappa)$) and we lift $j_{\mathcal{E}}$ to $j^+_{\mathcal{E}} : V[G] \to M[H]$ as in Lemma 1.4.2 and $\mathcal{E}^+$ is the (κ, β)-extender derived from $j^+_{\mathcal{E}}$, then $j_{\mathcal{E}^+} = j^+_{\mathcal{E}}$ and $M_{\mathcal{E}^+} = M_{\mathcal{E}}[H]$.*

Lemma 1.4.9. *Suppose $j_{\mathcal{E}} : V \to M_{\mathcal{E}}$ is a (κ, β)-extender embedding derived from j with $\beta \leqslant j(\kappa)$. Suppose that $\mathbb{P}$ is $\leqslant \kappa$-distributive. If G is $\mathbb{P}$-generic over V then the upwards closure H of $j_{\mathcal{E}}[G]$ in $j_{\mathcal{E}}(\mathbb{P})$ is $j_{\mathcal{E}}(\mathbb{P})$-generic over $M_{\mathcal{E}}$. Thus $j_{\mathcal{E}}$ lifts to an embedding $j^+_{\mathcal{E}} : V[G] \to M_{\mathcal{E}}[H]$.*

Proof. Suppose that $D \subseteq j_{\mathcal{E}}(\mathbb{P})$, $D \in M_{\mathcal{E}}$ is dense open. We have to show that $H \cap D \neq \varnothing$. Since $j_{\mathcal{E}}$ is a derived (κ, β)-extender embedding, we can find a map $f : \kappa^{<\omega} \to V$ and a finite sequence $a \in \beta^{<\omega}$ such that $j_{\mathcal{E}}(f)(a) = D$. Now $j_{\mathcal{E}}$ factors as $j_{a\mathcal{E}} \circ j_a$. Let $g = f \restriction \kappa^{|a|}$. Then $j_{a\mathcal{E}}([g]_a) = j_{\mathcal{E}}(f)(a)$ and hence $[g]_a$ is a dense open subset of $j_a(\mathbb{P})$ in $Ult(V, E_a)$ by elementarity of $j_{a\mathcal{E}}$. By Łoś's Theorem:

$$\{b \in \kappa^{|a|} | f(b) = g(b) \text{ is a dense open subset of } \mathbb{P}\} \in E_a$$

Thus we may assume without loss of generality that $ran(f)$ only contains dense open subsets of $\mathbb{P}$. Since $\mathbb{P}$ is $\leqslant \kappa$-distributive in V, $\bigcap ran(f)$ is still dense open. Since G is $\mathbb{P}$-generic over V, there is some $p \in G \cap \bigcap ran(f)$. Hence by elementarity, $j_{\mathcal{E}}(p) \in j_{\mathcal{E}}[G] \cap \bigcap ran(j_{\mathcal{E}}(f))$ and in particular, $j_{\mathcal{E}}(p) \in j_{\mathcal{E}}[G] \cap j_{\mathcal{E}}(f)(a) \subseteq H \cap D$. $\square$

Proposition 1.4.10. *Suppose $\mathcal{E}$ is a derived (κ, β)-extender (say from j). Again let ζ be minimal with $j(\zeta) \geqslant \beta$. If $\lambda \geqslant \zeta$,$j_{\mathcal{E}}[\lambda] \in M_{\mathcal{E}}$ and ${}^{\lambda}(\beta^{<\omega}) \subseteq M_{\mathcal{E}}$ then ${}^{\lambda}M_{\mathcal{E}} \subseteq M_{\mathcal{E}}$.*

Proof. It is enough to show that every set $x \subseteq M$ of size λ is in M. Write $x = \{j_{\mathcal{E}}(f_\gamma)(a_\gamma) | \gamma < \lambda\}$. Let $h : \zeta^{<\omega} \to V$ such that

$$h(a) : \lambda \to V, h(a)(\gamma) = f_\gamma(a)$$

for all $a \in \zeta^{<\omega}$. For $n < \omega$, let $h_n = h \restriction \zeta^n$ and similarly for g. For a given a, $[h_{|a|}]_{E_a}$ is a function with domain $j_a[\lambda] \subseteq dom([h_n]_{E_a})$, by Łós's Theorem. Furthermore, if $|a| = |a_\gamma|$, then $[h_{|a|}]_{E_a}(j_a(\gamma)) = [f_\gamma]_{E_a}$ since

$$\{c \in \beta^{|a|} | h(c)(\gamma) = f_\gamma(c)\} = \zeta^{<\omega} \in E_a$$

and thus

$$\begin{aligned} j_{a\mathcal{E}}([h_n]_{E_a})(j_{\mathcal{E}}(\gamma)) &= j_{a\mathcal{E}}([h_n]_{E_a})(j_{a\mathcal{E}}(j_a(\gamma))) \\ =& j_{a\mathcal{E}}([h_n]_{E_a}(j_a(\gamma))) = j_{a\mathcal{E}}([f_\gamma]_{E_a}) = j_{\mathcal{E}}(f_\gamma)(a) \end{aligned}$$

This shows:

$$j_{\mathcal{E}}(h)(a_\gamma)(j_{\mathcal{E}}(\gamma)) = j_{a_\gamma \mathcal{E}}([h_{|a_\gamma|}]_{E_{a_\gamma}}(j_{a_\gamma}(\gamma))) = j_{\mathcal{E}}(f_\gamma)(a_\gamma)$$

Let $A = \{(a_\gamma, j(\gamma)) | \gamma < \lambda\}$. After currying, we may write (with abuse of notation):

$$x = j_{\mathcal{E}}(h)[A]$$

Hence it is enough to show $A \in M_{\mathcal{E}}$ and since $j_{\mathcal{E}}[\lambda] \in M_{\mathcal{E}}$ we may only show $(a_\gamma)_{\gamma<\lambda} \in M_{\mathcal{E}}$, but this is given by one of our assumptions. □

1.5 Miscellaneous

In this section we present a few result that resisted to fit into a prior category.

Proposition 1.5.1. *[Gol93] If κ is regular and $x \in H_\kappa$ then there is an ordinal $\lambda < \kappa$ and a sequence $\langle x_\alpha | \alpha \leqslant \lambda \rangle$ of sets in H_κ with the following properties:*

(i) $\forall \alpha \leqslant \lambda \; x_\alpha \subseteq \{x_\beta | \beta < \alpha\}$

(ii) $x_\lambda = x$

Proof. Suppose the claim holds for all $y \in x$. Enumerate x as $x = \{y^\delta | \delta < \gamma\}$ and find a witnessing sequences $\langle y^\delta_\alpha | \alpha \geqslant \lambda^\delta \rangle$ for every y^δ. Let $\lambda = \Sigma_{\delta<\gamma} \lambda^\delta$ be the ordinal sum and let $\langle x_\alpha | \alpha < \lambda \rangle$ be the concatenation of the sequences $\langle y^\delta_\alpha | \alpha \geqslant \lambda^\delta \rangle$ for $\delta < \gamma$. Since κ is regular and $\gamma < \kappa$, $\lambda < \kappa$. Finally, set $x_\lambda = x$. □

Lemma 1.5.2. *[Gol93] If κ is regular, $\mathbb{P}$ is κ-cc, $\mathbb{P} \subseteq H_\kappa$, $\dot{x}$ is a $\mathbb{P}$-name and $\mathbb{1}_\mathbb{P} \Vdash \dot{x} \in H_{\check{\kappa}}$, then there is a $\mathbb{P}$-name $\dot{y} \in H_\kappa$ with $\mathbb{1}_\mathbb{P} \Vdash \dot{x} = \dot{y}$.*

Proof. First of all, $\mathbb{P}$ does not collapse κ by the κ-cc. Find $\mathbb{P}$-names $\dot{S}$ and $\dot{\lambda}$ such that for every generic G, $\dot{S}^G$ is a sequence of length $\dot{\lambda}^G$ that witness the statement of Proposition 1.5.1 for x. Since $\mathbb{P}$ is κ-cc and κ regular, we can cover $\dot{\lambda}$ in V by a set of size $< \kappa$ and thus find some $\lambda < \kappa$ with $\mathbb{1}_\mathbb{P} \Vdash \dot{\lambda} \leqslant \check{\lambda}$. Since we can always extend the sequence $\dot{S}^G$ in $V[G]$ to a sequence of length λ with the same properties, we can assume that $\mathbb{1}_\mathbb{P} \Vdash \dot{\lambda} = \check{\lambda}$. Now, for each $\alpha \leqslant \lambda$ find a $\mathbb{P}$-name $\dot{x}_\alpha$ such that $\mathbb{1}_\mathbb{P}$ forces $\dot{x}_\alpha$ to be the α-th point in the sequence $\dot{S}$. For every $\beta < \alpha \leqslant \lambda$, find a maximal antichain $A_{\beta,\alpha}$ in

$$D_{\beta,\alpha} = \{p \in \mathbb{P} | p \Vdash \dot{x}_\beta \in \dot{x}_\alpha\}$$

and by induction define

$$\dot{y}_\alpha = \{(\dot{y}_\beta, p) | \beta < \alpha \wedge p \in A_{\beta,\alpha}\}$$

It is left to show by induction that $\dot{y}_\alpha \in H_\kappa$ and $\mathbb{1}_\mathbb{P} \Vdash \dot{x}_\alpha = \dot{y}_\alpha$. So assume this is true for all $\beta < \alpha$. Since every $\dot{y}_\beta$ and $p \in A_{\beta,\alpha}$ is in H_κ, κ is regular and $|A_{\beta,\alpha}| < \kappa$, we can conclude that $\dot{y}_\alpha \in H_\kappa$. Now suppose that G is $\mathbb{P}$-generic over V. We calculate:

$$\begin{aligned} \dot{y}_\alpha^G &= \{\dot{y}_\beta^G | \beta < \alpha \wedge G \cap A_{\beta,\alpha} \neq \varnothing\} \\ &= \{\dot{x}_\beta^G | \beta < \alpha \wedge G \cap A_{\beta,\alpha} \neq \varnothing\} \\ &= \{\dot{x}_\beta^G | \beta < \alpha \wedge \dot{x}_\beta^G \in \dot{x}_\alpha^G\} = \dot{x}_\alpha^G \end{aligned}$$

Here, the first equality holds by induction and the second by choice of $A_{\beta,\alpha}$. Finally, $\dot{y} = \dot{y}_\lambda$ is as desired. □

Using the above result, one can prove the following by an induction over the complexity of formulas. The essence of the next statement is that if p forces a formula to hold in H_κ, then H_κ knows about this.

Fact 1.5.3. *[LS16, Lemma 1.2.3] Suppose κ is regular and $\mathbb{P} \in H_\kappa$. Then for $p \in \mathbb{P}$ and any formula $\varphi(x_0, \ldots x_{n-1})$ and $\mathbb{P}$-names $\dot{x}_0, \ldots, \dot{x}_{n-1}$ with $p \Vdash \dot{x}_i \in H_{\check{\kappa}}$ there are $\mathbb{P}$-names $\dot{y}_0, \ldots, \dot{y}_{n_1} \in H_\kappa$ with $p \Vdash \dot{x}_i = \dot{y}_i$ for $i < n$ so that*

$$p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})^{H_{\check{\kappa}}} \Leftrightarrow H_\kappa \models p \Vdash \varphi(\dot{y}_0, \ldots, \dot{y}_{n-1})$$

Lemma 1.5.4. *Suppose that κ is a cardinal of uncountable cofinality and $\mathbb{P}$ is a forcing of size $< cof(\kappa)$. Then all stationary subsets of κ in V remain stationary in $V[G]$.*

Proof. Suppose $C \in V[G]$ is a club in κ.

Claim 1.5.5. *There is a club $D \subseteq C$ in V.*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for C so that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{C} \subseteq \check{\kappa}$ is a club". Let D be the set of all $\alpha < \kappa$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \check{\alpha} \in \dot{C}$. It is clear that D is closed. Given $\alpha < \kappa$ we have to find some element of D above α. Construct an increasing sequence $\langle \alpha_n \in \kappa | n < \omega \rangle$ by induction. Let $\alpha_0 = \alpha$. Given α_n, we can find for every $p \in \mathbb{P}$ some $q_p^n \leqslant p$ and $\beta_p^n > \alpha_n$ so that $q_p^n \Vdash \breve{\beta_p^n} \in \dot{C}$. Let $\alpha_{n+1} = sup\{\beta_p^n | p \in \mathbb{P}\}$. Since $\mathbb{P}$ has size $< cof(\kappa)$, $\alpha_{n+1} < \kappa$.
Let $\alpha_\star = sup_{n<\omega}\ \alpha_n < \kappa$. By construction, the set

$$D_n = \{q \in \mathbb{P} | q \Vdash \exists \beta\ \check{\alpha}_n < \beta \leqslant \check{\alpha}_{n+1} \wedge \beta \in \dot{C}\}$$

is dense for every $n < \omega$. Since $\dot{C}$ is forced to be closed, $\mathbb{1}_{\mathbb{P}} \Vdash \check{\alpha}_\star \in \dot{C}$. □

If D is as above and $S \subseteq \kappa$ stationary in V then $C \cap S \supseteq D \cap S \neq \emptyset$. □

Lemma 1.5.6. *Suppose $\mathbb{P}$ is κ-cc and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a κ-cc forcing for κ regular. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc.*

Proof. It is enough to show that if $\dot{\alpha}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for an ordinal then there is a set X of size $< \kappa$ with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \dot{\alpha} \in \check{X}$. We can naturally identify $\dot{\alpha}$ with a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name $\ddot{\alpha}$, so that whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic then $\dot{\alpha}^{G*\dot{H}} = (\ddot{\alpha}^G)^H$. If G is $\mathbb{P}$-generic then as $\mathbb{Q} = \dot{\mathbb{Q}}^G$ is κ-cc, there is a cardinal $\lambda < \kappa$ and a function $f : \lambda \to Ord$ so that $\mathbb{1}_{\mathbb{Q}} \Vdash \ddot{\alpha}^G \in ran\ \check{f}$. Going back to V, this shows that there are $\mathbb{P}$-names $\dot{\lambda}, \dot{f}$ that are forced by $\mathbb{1}_{\mathbb{P}}$ to have the above properties in the extension. Now as $\mathbb{P}$ is κ-cc, there is a set Z of cardinals $< \kappa$ of size $< \kappa$ that covers $\dot{\lambda}$. Let $\theta = sup\ Z$ which is less than κ by regularity of κ. Again by the κ-cc of $\mathbb{P}$, we can find a set X_β of size $< \kappa$ so that $\mathbb{1}_{\mathbb{P}} \Vdash \check{\beta} \in dom\ \dot{f} \to \dot{f}(\check{\beta}) \in \check{X}_\beta$ for every $\beta < \theta$. Now $X = \bigcup_{\beta<\theta} X_\beta$ is as desired. □

2 Set-Theoretic Geology

Set-Theoretic Geology is motivated by a change of perspective regarding the tool forcing. Usually, forcing is used to construct a model that satisfies some specific properties. One starts with a model V and produces a larger model $V[G]$ that contains some generic object G. That means that traditionally, the perspective is directed *upwards* in this setting. Here, we direct our attention *downwards*. We do not ask "Where are we going?", but rather "Where are we coming from?". This view is already cemented in the most basic definition in this context. We shift from looking at extensions to focusing on *grounds*

Definition 2.0.1. If $V = W[G]$ is a forcing extension of some inner model W of ZFC, then W is called a ground of V.

We will mainly follow [FHR15] and [Rei06] in this chapter.

2.1 Definability of Grounds

As we want to analyze the structure of grounds, it is necessary for models of ZFC to be able to talk about their grounds in the first place. The following could thus be called the fundamental theorem of Set-Theoretic Geology.

Theorem 2.1.1. *(Definability of Grounds Theorem) The grounds of V are uniformly definable. This means that there is a first order formula $\phi(x, r)$ with the following properties: For $r \in V$ set $W_r = \{x | \phi(x, r)\}$.*

(*i*) *For any r, W_r is a ground of V with $r \in W$.*

(*ii*) *If W is a ground of V then for some $r \in V$, $W = W_r$.*

Remark 2.1.2. For the second condition to make sense, we implicitly suppose that V is a countable set in some large background model. A different approach would be to formulate this statement in the second order set theory GBC.

To show that grounds are definable, it is essential to be able to uniquely characterize a ground W and its initial segments via simple properties. In the end, we want to be able to say that "W_α is the unique subset of V_α in which the bounded subsets of δ are exactly r and which has certain properties (that depend on δ)" for arbitrarily large α and some (single) δ. This allows us to define W over its initial segments. Next we introduce these properties which were first formulated by Hamkins in [Ham03] and have proven very useful in Set-Theoretic Geology.

Definition 2.1.3. Assume $M \subseteq N$ are transitive classes and δ is a regular cardinal in N.

(*i*) $M \subseteq N$ has the δ-cover property if for every $x \subseteq M$, $x \in N$ of size $< \delta$ in N, there is a cover $y \in M$ of x, $x \subseteq y$, of size $< \delta$ in M. We will call such y a δ-cover of x.

(*ii*) Given $x \subseteq M$, $x \in N$ and $a \in M$ of size $< \delta$ (in M), we call $x \cap a$ a δ-approximation of x in M. $M \subseteq N$ has the δ-approximation property if every such x for which all δ-approximations of x in M are members of M, are itself in M, i.e. $x \in M$.

Remark 2.1.4. The δ-cover and approximation properties are respectively equivalent to the δ-cover and approximation properties restricted only to sets of ordinals, given that both classes are transitive models of (a suitable fragment of) ZFC. This is because if $x \subseteq M$, then $x \subseteq M_\alpha$ for $\alpha = rk(x)^N$. Then one can find a bijection $f : M_\alpha \to \kappa$, $f \in M$ for some cardinal κ and apply the respective properties to $f[x] \subseteq \kappa$. Reversing this construction yields $x \in M$.

The unique characterization of certain subsets of V_α only works in case we have a sufficiently large fragment of ZFC present. This fragment was isolated by Jonas Reitz in his dissertation [Rei06].

Definition 2.1.5. Let $S_\delta = \{\in, \dot{\delta}\}$ be the first order language consisting of the usual binary relation symbol $\in$ and a constant symbol $\dot{\delta}$. ZFC_δ is the S_δ-theory consisting of the following axioms:

(*i*) the axioms of Zermelo set theory (extensionality, set existence, pairing, separation, union, power set, infinity)

(*ii*) the well-ordering theorem

(*iii*) the statement "every set is coded as a set of ordinals"

(*iv*) "$\dot{\delta}$ is a regular cardinal"

(*v*) the δ-replacement scheme: For a $\{\in\}$-formula $\phi(x, y, z_0, \dots, z_{n-1})$ this scheme contains the formula

$$\forall z_0, \dots, z_{n-1} \forall x \in \dot{\delta} \exists! y \ \phi(x, y, \vec{z}) \to \exists a \forall b (b \in a \leftrightarrow \exists c \in \dot{\delta} \ \phi(c, b, \vec{z}))$$

Remark 2.1.6. (*i*) Usually, when working with a S_δ-structure M, we will write δ for $\dot{\delta}^M$. If we have previously defined a regular cardinal δ, we will always assume that any S_δ-structure appearing afterwards has $\dot{\delta}^M = \delta$.

(*ii*) Since not all versions of the axiom of choice are equivalent under just Zermelo set theory, we chose the well-ordering theorem as a representative and for convenience.

(*iii*) Notice that the δ-replacement scheme is just the standard replacement scheme restricted to functions with domain δ.

The word "code" is used quite freely in set theory, so let us make sure we specify exactly what is meant here.

Definition 2.1.7. Given a class M and $x \in M$, we say that x is coded as a set of ordinals in M if there is an ordinal α, $R \subseteq \alpha \times \alpha$ such that $\langle \alpha, R\rangle \cong \langle tc(\{x\}), \in\rangle$.

Remark 2.1.8. (*i*) The above definition might seem strange as R is not a set of ordinals. However, one can usually think of R as a set of ordinals if one takes its pointwise image under an injection $\alpha \times \alpha \to Ord$. One can even choose this injection uniformly, for example as (the inverse of) the Gödel pairing function. Nonetheless, the above definition manages to refrain from making further assumptions on M.

(*ii*) Under ZFC, every set is coded as a set of ordinals (de facto this is equivalent to the axiom of choice under ZF). Given a set x, find a bijection $f : \alpha \to tc(\{x\})$ for some ordinal α. We let

$$R = \{(\beta, \gamma) \in \alpha \times \alpha | f(\beta) \in f(\gamma)\}$$

so that $f : \langle \alpha, R\rangle \to \langle tc(\{x\}), \in\rangle$ is an isomorphism.

In the next argument, we want to be able to apply Mostowski's theorem in the context of ZFC_δ, however we cannot do so in general as one needs full replacement to do so. Observe however that ZFC_δ proves Mostowski's theorem for structures $\langle A, E\rangle$ of size $\leqslant \delta$. This is in fact all we need.

Lemma 2.1.9. *Assume U is a transitive model of ZFC_δ and $M, N \subseteq U$ are transitive substructures that also satisfy ZFC_δ. Suppose the following:*

(*i*) *δ^+ is constant across M, N and U.*

(*ii*) *$M, N \subseteq U$ both satisfy the δ-cover and approximation properties*

(*iii*) $({}^{<\delta}2)^M = ({}^{<\delta}2)^N$

Then already $M = N$.

Proof. First of all, the δ-cover property of $M, N \subseteq U$ assures that the statement "x has size $< \delta$" is absolute between these models. This is because any bijection $f : \alpha \to x$ for some $\alpha < \delta$ in U can be covered by $A \in M$, $A \subseteq \alpha \times x$ of size $\kappa < \delta$ in M. From A, M is able to construct a surjection $g : \alpha \times \lambda \to x$ for some $\lambda < \delta$, so x has size $< \delta$ in M. The same works for N. Furthermore, as δ^+ is evaluated equally in all three models, the statement "x has size δ" is also absolute between all three of them. Statements about

sizes will always be regarded from the perspective of U.
The proof will proceed through several steps.

Step 1: $\mathcal{P}(\delta)^M = \mathcal{P}(\delta)^N$.
Let $x \in \mathcal{P}(\delta)^M$. Clearly $x \in U$. By (iii), every δ-approximation of x in N is in N. As $N \subseteq U$ has the δ-approximation property, $x \in N$ and hence $\mathcal{P}(\delta)^M \subseteq \mathcal{P}(\delta)^N$. Equality follows from the symmetry of the argument.

Step 2: M and N have the same sets of ordinals of size $< \delta$.
Assume $x \in M$ is a set of ordinals of size $< \delta$. I claim that x is contained in some $y \in M \cap N$ of size δ. That means that there is some uniform cover that is contained both in M and N. The construction will take place in U and will make use of the well-ordering theorem by implicitly choosing desired sets via a well-order on $\mathcal{P}(sup\ x)^U$. We define y^K_α for $\alpha < \delta$ and $K = M, N$ by induction. Let $y^K_0 \in K$ be a δ-cover of x. If y^K_γ is defined for all $\gamma < \alpha$ and $K = M, N$, then let $x_\alpha = (\bigcup_{\gamma<\alpha} y^M_\gamma) \cup (\bigcup_{\gamma<\alpha} y^N_\gamma)$. First of all, $x_\alpha \in U$ as U models Zermelo set theory and the δ-replacement scheme. Furthermore, x_α has size $< \delta$ as δ is regular. Thus we can define $y^K_\alpha \in K$ to be a δ-cover of x_α for $K = M, N$.
Let $y = \bigcup_{\alpha<\delta} y^M_\alpha = \bigcup_{\alpha<\delta} y^N_\alpha$. We use the δ-approximation property of $M, N \subseteq U$ to show that $y \in M \cap N$. Let K be either M or N. Let $a \in K$ of size $< \delta$. By regularity of δ, there is some $\alpha < \delta$ such that $y \cap a \subseteq \bigcup_{\gamma<\alpha} y^K_\gamma$. This implies that $y \cap a = y^K_{\alpha+1} \cap a \in K$. Since every δ-approximation of y is in K, $y \in K$.
Without loss of generality, y only contains ordinals. Now find a well-order $\tilde{<} \in M$ on a subset of δ of ordertype $otp(y)$. The Gödel Pairing function restricted to δ, $G : \delta \times \delta \to \delta$, is contained in both M and N. By Step 1, $G[\tilde{<}] \in N$ and thus $\tilde{<} \in N$. $\tilde{<}$ induces a homomorphism $f : \langle y, \in\rangle \to \langle \delta, \tilde{<}\rangle$. To be precise, f is the concatenation of the Mostowski collapse of $\langle y, \in\rangle$ and the inverse collapse of $\langle \delta, \tilde{<}\rangle$, which both exist in N as these structures have size δ. In N, we can now reconstruct x from $y \in N$, $f \in N$ and $f[x] \in \mathcal{P}(\delta)^M \subseteq N$. That $f[x]$ is a set in M follows from the δ-replacement scheme as x is a set of size $< \delta$. We conclude $x \in N$. By the symmetry of this argument, M also contains all sets of ordinals of size $< \delta$ of N.

Step 3: M and N contain the same sets of size $< \delta$.
If $A \in M$, then A is coded as a set of ordinals in M, i.e. there is α an ordinal $R \subseteq \alpha \times \alpha$ with $\langle tc(\{A\}), \in\rangle \cong \langle \alpha, R\rangle$. Via Gödel pairing, we can understand R as a set of ordinals. By Step 2, $R \in N$. Applying the Mostowski collapse to the structure $\langle \alpha, R\rangle$ in N yields a structure $\langle B, \in\rangle$ that is transitive from the perspective of N and is isomorphic to $\langle \alpha, R\rangle$. As N is transitive, B is really transitive. As M is transitive, $tc(\{A\})$ is transitive as well. This implies $B = tc(\{A\})$. Using the transitivity of N once again we get $A \in N$. As usual the other direction follows from symmetry.

Step 4: $M = N$.

Let $A \in M$ be any set. Step 2 shows that N contains all δ-approximations of A in N. By the δ-approximation property of $N \subseteq U$, $x \in N$ and so $N \subseteq M$. Again by symmetry, the reverse inclusion follows.

□

The following definition is thus justified.

Definition 2.1.10. Suppose N is a model of ZFC_δ and $M \subseteq N$ is a submodel of ZFC_δ so that δ^+ is the same in M, N and so that the δ-cover and approximation properties hold. Then we call M the (unique) r-substructure of N where $r = ({}^{<\delta}2)^M$.

Fortunately, the necessity of satisfying ZFC_δ in the above lemma isn't too restrictive. The next proposition shows that $V_\alpha \models ZFC_\delta$ for class many α for any given interpretation of $\dot{\delta}$ as a regular cardinal.

Proposition 2.1.11. *If δ is a regular cardinal and κ is a $\beth$-fixed point of cofinality $> \delta$ then $V_\kappa \models ZFC_\delta$.*

Proof. We must show that $(i) - (v)$ of Definition 2.1.5 are satisfied in V_κ.

$(i) + (ii)$ As κ is an infinite limit ordinal, V_κ satisfies Zermelo set theory and well-ordering theorem.

(iii) By induction one sees that $|V_{\omega+\alpha}| = \beth_\alpha$ for all ordinals α. Since κ is a $\beth$-fixed point, and of course $\omega + \kappa = \kappa$, we have $|V_\kappa| = \kappa$. If $x \in V_\kappa$, then $x \in V_\beta$ for some $\beta < \kappa$. As V_β is transitive, $tc(\{x\}) \subseteq V_\beta$. In particular $\alpha = |tc(\{x\})| \leqslant |V_\beta| \leqslant \beth_\beta < \kappa$ and thus $\alpha \in V_\kappa$. With this we can see that after the construction of Remark 2.1.8, the code R for x is in V_κ. Thus every set in V_κ is coded as a set of ordinals in V_κ.

(iv) δ is a regular cardinal in V_κ as it is in V.

(v) Here, we have to show that V_κ is a model of the δ-replacement scheme. So let $\phi(x, y, z_0, \ldots, z_{n-1})$ be a $\in$-formula that is functional on δ in V_κ for given parameters $\vec{z} \in V_\kappa$. This induces a function $f : \delta \to V_\kappa$. As $cof(\kappa) > \delta$, the function $rk \circ f : \delta \to \kappa$ must be bounded by some β. But then $ran(f) \in V_{\beta+1} \subseteq V_\kappa$.

□

Remark 2.1.12. Actually, we will need a little more. The above theorem states that the structure $\langle V_\kappa, \in, \delta\rangle$ satisfies the system ZFC_δ from the perspective of the meta-theory. What we actually need is that $V \models$ "$V_\kappa \models \ulcorner ZFC_\delta \urcorner$". The brackets $\ulcorner, \urcorner$ indicate that this is the theory ZFC_δ as formalized in V, opposed to the ZFC_δ of the meta-theory. This includes the formalized single axioms of ZFC_δ, as well as the formalized axiom schemes of separation $\ulcorner Sep \urcorner$ and δ-replacement $\ulcorner Rep_\delta \urcorner$. The single axioms ϕ (for example the extensionality axiom) are unproblematic as $V_\kappa \models \phi$ is equivalent to $V \models$ "$V_\kappa \models \ulcorner\phi\urcorner$". However, the formalized schemes contain every formalized instance of it for every formalized formula.This means we have to show for example $V \models$ "$\forall k \in Fml\ V_\kappa \models \ulcorner Sep\urcorner(k)$" where Fml is the set of formalized formulas and $\ulcorner Sep\urcorner(k)$ is the formalized separation scheme with instance $k \in Fml$. Fml includes all "standard formulas" of the form $\ulcorner\phi\urcorner$, but might contain nonstandard formulas that cannot be represented in this from. Showing $V \models$ "$V_\kappa \models \ulcorner ZFC_\delta\urcorner$" is however virtually the same as the above proof. For formalized separation this amounts to showing that $\{a \in x | V_\kappa \models k(a, x, \vec{y})\} \in V_\kappa$ for a formalized formula k and that is clear. In the same way for formalized δ-replacement, any functional $k \in Fml$ generates a function f as in the proof above.

The next definition is only for convenience for now, but becomes more essential later on.

Definition 2.1.13. A ground W is called a δ-ground if there is a forcing $\mathbb{P}$ of size $< \delta$ in W such that V is an extension of W via $\mathbb{P}$.

Since we want to use Lemma 2.1.9 to define grounds, we need to show that grounds satisfy the δ-cover and approximation properties for some large enough δ.

Proposition 2.1.14. *[HJ10] Suppose δ is regular and $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration such that $\mathbb{P}$ has size $\leqslant \delta$, is nontrivial and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is $\leqslant \delta$-strategically closed. Then for any generic $G * H$, $V \subseteq V[G * H]$ satisfies the δ^+-cover and approximation properties.*

Proof. $V \subseteq V[G]$ has the δ^+-cover property: Assume $x \subseteq V$, $x \in V[G]$ is of size $\leqslant \delta$. Let $\dot{x}$ be a $\mathbb{P}$-name for x and $\dot{f}$ a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{f} : \check{\theta} \to \dot{x}$ is surjective" for some $\theta \leqslant \delta$. Let

$$y = \{a | \exists p \in \mathbb{P},\ \alpha < \theta\ p \Vdash \dot{f}(\check{\alpha}) = \check{a}\}$$

be the set of possible elements of $\dot{x}$. For every $p \in \mathbb{P}$ and $\alpha < \theta$ there is at most one a with $p \Vdash \dot{f}(\check{\alpha}) = \check{a}$. Hence y has size at most $|\mathbb{P}| \cdot |\theta| \leqslant \delta$ and $\mathbb{1}_{\mathbb{P}} \Vdash \dot{x} \subseteq \check{y}$.
Since H does not add any new subsets of V of size $< \delta$, $V \subseteq V[G * H]$ satisfies the δ^+-cover property.
$W \subseteq V$ has the δ^+-approximation property: First of all, enumerate $\mathbb{P}$ as $\{p_\alpha | \alpha \leqslant \delta\}$ (not necessarily injective). Suppose $x \in V[G * H]$ is not in V. Thus we can find a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{x}$ for x such that $1_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \dot{x} \notin V$, i.e. $1_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \dot{x} \neq \check{z}$ for every $z \in V$.

Claim 2.1.15. *For any $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ there is a set $a = a(p, \dot{q})$ and a $\mathbb{P}$-name $\dot{q}' = q(p, \dot{q}) \in dom(\dot{\mathbb{Q}})$ such that:*

(*i*) $1_{\mathbb{P}} \Vdash \dot{q}' \leqslant q$

(*ii*) *There are* $p^0, p^1 \leqslant p$ *with* $(p^0, \dot{q}') \Vdash \check{a} \in \dot{x}$ *and* $(p^1, \dot{q}') \Vdash \check{a} \notin \dot{x}$.

Proof. By our assumption on $\dot{x}$, there must be some a such that $(p, \dot{q}) \nVdash$ "$\check{a} \in \dot{x}$". This means we can find $(p^i, \dot{q}^i) \leqslant (p, \dot{q})$ with $(p^0, \dot{q}^0) \Vdash \check{a} \in \dot{x}$ and $(p^1, \dot{q}^1) \Vdash \check{a} \notin \dot{x}$. Since $\mathbb{P}$ is nontrivial, we may choose p^0, p^1 incompatible with one another. We can extend these two conditions to a maximal antichain A of $\mathbb{P}$. We can now build a $\mathbb{P}$-name $\dot{q}'$ so that $(\dot{q}')^{\bar{G}} = (\dot{q}^i)^{\bar{G}}$ whenever p^i is the unique element of $\bar{G} \cap A$ and $(\dot{q}')^{\bar{G}} = \dot{q}^{\bar{G}}$ if $p^i \notin \bar{G} \cap A$ for $i < 2$, for any $\mathbb{P}$-generic $\bar{G}$. Without loss of generality, $\dot{q}' \in dom(\dot{\mathbb{Q}})$. Then $\dot{q}'$ is as desired. □

Now let $\dot{\sigma}_{II}$ be a $\mathbb{P}$-name for the winning strategy witnessing the $\leqslant \delta$-strategic closure of $\dot{\mathbb{Q}}$ in the extension by $\mathbb{P}$.

Claim 2.1.16. *There is a sequence* $\langle \dot{q}_\alpha | \alpha \leqslant \delta \rangle$ *in* $dom(\dot{\mathbb{Q}})$ *such that for all* $\alpha \leqslant \delta$:

(*i*) $\mathbb{1}_{\mathbb{P}} \Vdash \dot{q}_\alpha \leqslant \dot{q}_\beta$ *for* $\beta \leqslant \alpha$

(*ii*) *If* $\alpha = 2\beta + 1$, *there is a set* a_β *and* $p^i_\beta \leqslant p_\beta$ *for* $i < 2$ *such that* $(p^0_\beta, \dot{q}_\alpha) \Vdash \check{a}_\beta \in \dot{x}$ *and* $(p^1_\beta, \dot{q}_\alpha) \Vdash \check{a}_\beta \notin \dot{x}$.

Proof. Assume $\alpha = 2\beta \leqslant \delta$. By induction, we can assume that $\mathbb{1}_{\mathbb{P}}$ forces that the sequence constructed up to α is the outcome of a play in $G(\dot{\mathbb{Q}}, \check{\alpha})$ where player I played according to some strategy $\dot{\sigma}^\alpha_I$ and II according to the strategy $\dot{\sigma}_{II}$ restricted to this shorter game. Let $\dot{q}_\alpha \in dom(\dot{\mathbb{Q}})$ be a name for the next play according to $\dot{\sigma}_{II}$. If $\alpha < \delta$, we let $\dot{\sigma}^{\alpha+2}_I$ be a $\mathbb{P}$-name for the strategy extending $\dot{\sigma}^\alpha_I$ by playing $q(p_\beta, \dot{q}_\alpha)$ at stage $\alpha+1$. Set $a_\beta = a(p_\beta, q_\alpha)$. By the first claim, we have extended the sequence as desired. □

In the above argument, we may as well incorporate into the strategy of player I to make sure that his first move is below a given $\dot{q} \in dom(\dot{\mathbb{Q}})$. This shows that the endpoints of sequences with the above properties are forced to be dense in $\dot{\mathbb{Q}}$, so that we can assume $\dot{q}^G_\delta \in H$. Let $A = \{a_\beta | \beta < \delta\}$. Suppose $x_0 = x \cap A \in V$. Then there is $(p, \dot{q}) \leqslant (\mathbb{1}_{\mathbb{P}}, \dot{q}_\delta)$ such that $(p, \dot{q}) \Vdash \check{x}_0 = \check{A} \cap \dot{x}$. But then $p = p_\beta$ for some $\beta < \delta$ and thus for $\alpha = 2\beta + 1$, $p^i_\beta \leqslant p_\beta$ for $i < 2$ and $(p^0_\beta, \dot{q}) \leqslant (p^0_\beta, \dot{q}_\alpha) \Vdash \check{a}_\beta \in \dot{x}$ and $(p^1_\beta, \dot{q}) \leqslant (p^1_\beta, \dot{q}_\alpha) \Vdash \check{a}_\beta \notin \dot{x}$, a contradiction. Thus not every δ^+-approximation of x is in V. □

Remark 2.1.17. Since the second step in the two step iteration may be chosen trivial, the above proposition shows that whenever W is a δ-ground, then $W \subseteq V$ has the δ-cover and approximation properties.

Proposition 2.1.18. *The statement "θ is a strong limit cardinal of cofinality $> \kappa$" is downwards absolute to inner models.*

Proof. Let W be an inner model and suppose the statement is true in V. Clearly $cof(\theta)^W \geqslant cof(\theta)^V > \kappa$. Let's show $|\beth_\alpha^W|^V \leqslant \beth_\alpha^V$ by induction on α. The base and limit cases are trivial. If $|\beth_\alpha^W|^V \leqslant \beth_\alpha^V$ then $|\beth_{\alpha+1}^W|^V \leqslant (2^{\beth_\alpha^W})^V \leqslant (2^{\beth_\alpha^V})^V = \beth_{\alpha+1}^V$. Clearly $\beth_\theta^W \geqslant \theta$. On the other hand $|\beth_\alpha^W|^V \leqslant \beth_\alpha^V < \theta$ and thus $\beth_\alpha^W < \theta$ for all $\alpha < \theta$. But then $\beth_\theta^W = sup_{\alpha<\theta}\beth_\alpha^W \leqslant \theta$. □

With the last few lemmata and propositions we have collected all the tools we need to show the definability of grounds.

Proof. (Theorem 2.1.1) We have already mentioned the general idea for this proof. We can make this more precise now: Suppose δ is a regular cardinal. If W is a δ-ground then $W \subseteq V$ satisfies the δ-cover and approximation properties by Proposition 2.1.14 and the same is true for $W_\alpha \subseteq V_\alpha$ for all limit α. Define C_δ as the class of all $\beth$-fixed points of cofinality $> \delta$. Proposition 2.1.11 states that $V_\alpha \models ZFC_\delta$ for all $\alpha \in C_\delta$ and furthermore the same is true for W_α by Proposition 2.1.18. This already shows that every ground W is definable from the parameter $r = ({}^{<\delta}2)^W$ as the union of the unique r-substructures of the V_α for $\alpha \in C_\delta$.
For the uniform definability of grounds, we basically have to do this backwards. That means we start with r and have to reconstruct W. We define W_r via the following steps. The first order formula $\phi(x, r)$ can be extracted from this procedure.

Stage 1: Here, we try to recover δ from r. If r is not a set of $0-1$ sequences with ordinal domain, then this stage fails. Otherwise $\delta = sup\{dom\ f | f \in r\}$. This stage can also fail if δ is not a regular cardinal.

Stage 2: For the mean time, we cache $\overline{W}_r$ as the union over the unique r-substructures of V_α for $\alpha \in C_\delta$. If any one of them does not exists, this stage fails. Notice that for two sets $m \subseteq n$, it is possible to check whether or not m is the r-substructure of n in one first order formula.

Stage 3: We perform the last sanity checks. Firstly, we check in one single first order formula whether $\overline{W}_r$ is an inner model. This can be done via the inner model criterion (Theorem 6.2.4).
Next, if the last step succeeded, we check if $\overline{W}_r$ is a ground of V. This can be done by looking for a forcing $\mathbb{P} \in \overline{W}_r$ and a $G \in V$ which is $\mathbb{P}$-generic over $\overline{W}_r$ such that for all $x \in V$ there is $\dot{x} \in \overline{W}_r$ with $x = \dot{x}^G$. This succeeds if and only if $\overline{W}_r$ is a ground of V.

If all stages were successful, we let $W_r = \overline{W}_r$. If any failed, we just take $W_r = V$.
The first part of this proof shows that if W is a δ-ground, then $W_r = W$ for $r = ({}^{<\delta}2)^W$. On the other hand, if $W_r \neq V$, then r must have passed stage 3 and thus is a ground. Clearly $r \in W_r$ for any r. □

2.2 The Ground Axiom

The uniform definability of grounds allows models of set theory to talk about the structure of their grounds in a first order sense. This can be understood as a dual to the forcing theorem, which allows models to talk about their forcing extensions in a first order sense.

Definition 2.2.1. (The ground axiom) The ground axiom (GA) is the sentence "$\forall r\ W_r = V$".

The ground axiom states that there are no nontrivial grounds. This is of course the most simple structure the grounds can have. The ground axiom holds for example in Gödels constructible universe L, which is a consequence of L being the minimal inner model (and the absoluteness of L). The same is true for other canonical inner models such as $L[0^\#]$ since the existence of $0^\#$ is absolute between grounds and extensions. Clearly, any nontrivial set forcing forces the negation of (GA). On the other hand, using class forcing it is always possible to force the ground axiom, so (GA) is not reserved for small inner models.

Theorem 2.2.2. *There is a class forcing extension $V[G]$ of V which is a model of* (GA)*. Even more, for any given α we can arrange $V[G]_\alpha = V_\alpha$.*

The strategy for the above theorem is to iteratively code sets into the GCH pattern.

Definition 2.2.3. We make precise what this means:

(*i*) A set of ordinals $x \subseteq \alpha$ is coded into the GCH pattern if

$$\exists\beta\forall\gamma < \alpha\ 2^{\aleph_{\beta+\gamma+1}} = \aleph_{\beta+\gamma+2} \leftrightarrow \gamma \in x$$

In other words, to every β, we can define the $0-1$ sequence of length α which corresponds to whether or not the GCH holds at $\aleph_{\beta+\gamma+1}$. The above formula holds if and only if for some β this sequence is the characteristic function of x in α.

(*ii*) The Continuum Coding Axiom (CCA) states that all sets of ordinals are coded into the continuum pattern.

Remark 2.2.4. If (CCA) holds, then in fact every set of ordinals x must be coded into the GCH pattern unboundedly often. This is because every set of ordinals x is a proper initial segment of class many other sets of ordinals, which all must be coded into the GCH pattern. We can choose these sets incompatible with each other, so that the position β at which the coding takes place must vary. Then x is coded at each of these class many positions.

If $x \subseteq \alpha$ is coded into the GCH pattern, then x is definable from α and the corresponding position β, so the (CCA) is essentially a strong form of $V = HOD$. It is useful for us as it does entail the ground axiom.

Lemma 2.2.5. *(CCA) implies (GA).*

Proof. Assume W is a ground of V. Then the continuum functions of W and V coincide eventually by Lemma 1.2.2. If $x \subseteq \alpha$ is a set of ordinals in V then there are arbitrarily large β so that x is coded into the GCH pattern of V at position β. If β is large enough, the GCH pattern of W is the same as in V. But then x is definable in W from β and α as

$$x = \{\gamma < \alpha | 2^{\aleph_{\beta+\gamma+1}} = \aleph_{\beta+\gamma+2}\}^W$$

This means that W contains all sets of ordinals of V. As in Lemma 2.1.9 we can conclude that $V \subseteq W$ and hence $W = V$. □

For Theorem 2.2.2 it is now enough to show that one can extend every model of ZFC to one in which (CCA) holds. To do this we have to code every set of V into the GCH pattern, as well as every new set we add in this way. It is possible to manage this via iterative bookkeeping, however it is more appealing to generically choose each bit of the encoding. The following concept is the perfect fit for this job.

Definition 2.2.6. Given two forcings $(\mathbb{P}_0, \leqslant_0)$ and $(\mathbb{P}_1, \leqslant_1)$, the lottery sum $(\mathbb{P}_0 \oplus \mathbb{P}_1, \leqslant)$ is defined as the coproduct of the two forcings together with a new maximal element. To be more precise:

$$\mathbb{P}_0 \oplus \mathbb{P}_1 := (\mathbb{P}_0 \times \{0\}) \cup (\mathbb{P}_1 \times \{1\}) \cup \{\varnothing\}$$

and $(p, i) \leqslant (q, j)$ iff $i = j$ and $p \leqslant_i q$. $\varnothing$ is the new maximal element.

Notice that if G is generic for $\mathbb{P}_0 \oplus \mathbb{P}_1$, then it is essentially either a generic for $\mathbb{P}_0$ or for $\mathbb{P}_1$. If there are no other restrictions on G, then one can think of this situation as a random binary choice of G to be either generic for the first forcing or the latter. Because of this, $\mathbb{P}_0 \oplus \mathbb{P}_1$ is called the lottery sum.
We will say that G chose $\mathbb{P}_i$ if the second coordinates of (non-maximal) conditions in G are i. Similarly we say that $p \in \mathbb{P}_0 \oplus \mathbb{P}_1$ lies in $\mathbb{P}_i$ if its not maximal and has second coordinate i.
On the other hand, every generic for $\mathbb{P}_0$ or $\mathbb{P}_1$ is easily transformed into a generic for $\mathbb{P}_0 \oplus \mathbb{P}_1$. It seems like one looses a bit of control when using the lottery sum in a forcing construction, but it can be useful in iterations or products if one does not want to make these binary choices by hand. One can let these happen generically.
In the case of forcing (GA) we will use the class iteration with Easton support $\mathbb{P}(\kappa) = (\langle \mathbb{P}_\theta | \kappa \leqslant \theta \rangle, \langle \dot{\mathbb{Q}}_\theta | \kappa \leqslant \theta \rangle)$ where $\dot{\mathbb{Q}}_\theta$ is a $\mathbb{P}_\theta$-name for $Add(\theta^+, \theta^{+++}) \oplus \{\mathbb{1}\}$ (as defined in the extension) if θ is a cardinal and a name for the trivial forcing otherwise. We will see that $\mathbb{P}(\kappa)$ forces (CCA) given that GCH holds in V at and above κ. The construction is a modification of [Rei06, Theorem 10] by using lottery sums.

Lemma 2.2.7. *Let κ be a cardinal such that GCH holds at and above κ and let G be $\mathbb{P} = \mathbb{P}(\kappa)$-generic over V.*

(*i*) *$\mathbb{P}$ preserves ZFC.*

(*ii*) *$\mathbb{P}$ preserves all cardinals.*

(*iii*) *In $V[G]$, if the generic G chose the trivial dorcing at stage $\lambda \geqslant \kappa$ cardinal, then GCH holds at λ^+. Otherwise $2^{\lambda^+} = \lambda^{+++}$.*

(*iv*) *Every bounded subset x of κ is coded into the GCH pattern of $V[G]$.*

Proof. (*i*) In the language of [Rei06], $\mathbb{P}$ is a *progressively closed iteration.* Thus Theorem 95, in the above source, yields that $\mathbb{P}$ preserves ZFC. Note that we may consider V together with it's definable classes as a model of GBC, even with global choice after forcing to add a global choice function without adding sets.
Alternatively, the proof of 2.3.8 (*i*) can be modified (and simplified) to work here.

(*ii*) It is enough to show that every regular cardinal is preserved. We can factor $\mathbb{P}$ at stage λ into $\mathbb{P}_{<\lambda} * \dot{\mathbb{P}}_{\geqslant\lambda}$. The latter is forced to be $< \lambda^+$-closed.
We will show by induction that $\mathbb{P}_{<\lambda}$ has the λ^+-cc if λ is regular and the λ^{++}-cc if it is singular.

<u>$\lambda = \kappa$:</u> This case is trivial.

<u>$\lambda = \theta^+$:</u> By induction, $\mathbb{P}_{<\theta}$ has the $\lambda^+ = \theta^{++}$-cc. Furthermore, we have that $Add(\theta^+, \theta^{+++})$ is $\theta^{++} = \lambda^+$-cc by Lemma 1.1.3 (*i*) as $(\theta^+)^\theta = 2^\theta = \theta^+$ and thus $\dot{\mathbb{Q}}_\theta^{G_{<\theta}} = Add(\theta^+, \theta^{+++}) \oplus \{\mathbb{1}\}$ is λ^+-cc in $V[G_{<\theta}]$. Now $\mathbb{P}_{<\lambda} \cong \mathbb{P}_{<\theta} * \dot{\mathbb{Q}}_\theta$ is λ^+-cc by Lemma 1.5.6.

<u>$\lambda \in Lim(Card)$:</u> As GCH holds above κ, we have that

$$|\mathbb{P}_{<\lambda}| \leqslant \prod_{\theta<\lambda} |\dot{\mathbb{Q}}_\theta| \leqslant \prod_{\delta<\lambda} \lambda = \lambda^\lambda = \lambda^+$$

Hence $\mathbb{P}_{<\lambda}$ has the λ^{++}-cc.
If λ is regular then can use that $\mathbb{P}$ is Easton-supported. If $A \subseteq \mathbb{P}_{<\lambda}$ is a set of size λ^+ then there must be some $A_0 \subseteq A$ of the same size and $\theta < \lambda$ such that $dom(p) \subseteq \theta$ for all $p \in A_0$, since all conditions in $\mathbb{P}_\lambda$ have support bounded in λ. But then we can understand A_0 as a subset of $\mathbb{P}_{<\theta}$ which has the θ^{++}-cc by induction. But then A_0 and in particular A cannot be an antichain. Hence $\mathbb{P}_{<\lambda}$ has the λ^+-cc.

As $\mathbb{P}$ is $\leqslant \kappa$-closed, it preserves all cardinals $\leqslant \kappa$. Assume $\lambda > \kappa$ is a regular cardinal in V, but not in $V[G]$. This implies that there is some $\theta < \lambda$ regular and a cofinal function $f \in V[G]$, $f : \theta \to \lambda$. We know that $\mathbb{P}_{<\theta}$ has the θ^+-cc and that $\mathbb{P}_{\geqslant\theta} = \dot{\mathbb{P}}_{\geqslant\theta}^{G_{<\theta}}$ is $< \theta^+$-closed in $V[G_{<\theta}]$ and thus it cannot have added f. We conclude $f \in V[G_{<\theta}]$, but this means that $\mathbb{P}_{<\theta}$ has destroyed the regularity of λ in contradiction to its $\theta^+ \leqslant \lambda$-cc.

(*iii*) First assume that G chose the trivial forcing at stage λ. Then we can split G into $G_{<\lambda}$, a $\mathbb{P}_{<\lambda}$-generic over V, and $G_{>\lambda}$, a $\mathbb{P}_{>\lambda} = \dot{\mathbb{P}}_{>\lambda}^{(G_{<\lambda} * \{\mathbb{1}\})}$-generic over V, such that $V[G] = V[G_{<\lambda}][G_{>\lambda}]$. We have that $\mathbb{P}_{>\lambda}$ is λ^{++}-closed in $V[G_{<\lambda}]$. Thus every sequence of ordinals of length λ^+ in $V[G]$ is already in $V[G_{<\lambda}]$. In particular $\mathcal{P}(\lambda^+)^{V[G]} = \mathcal{P}(\lambda^+)^{V[G_{<\lambda}]}$. Counting nice names for the forcing $\mathbb{P}_{<\lambda}$ gives:

$$\left(2^{(\lambda^+)}\right)^{V[G]} = \left(2^{(\lambda^+)}\right)^{V[G_{<\lambda}]} \leqslant ((|\mathbb{P}_{<\lambda}|^{\lambda^+})^{\lambda^+})^V \leqslant ((\lambda^{++})^{\lambda^+})^V = \lambda^{++}$$

On the other hand, suppose that G did not choose the trivial forcing at stage λ. Then G adds a generic g for $Add(\lambda^+, \lambda^{+++})$ and thus $\left(2^{(\lambda^+)}\right)^{V[G]} \geqslant \lambda^{+++}$. For the other inequality, we can conclude as above that $\mathcal{P}(\lambda^+)^{V[G]} = \mathcal{P}(\lambda^+)^{V[G_{\leqslant\lambda}]}$ and that GCH holds at λ^+ in $V[G_{<\lambda}]$. By Lemma 1.1.3 (*ii*), $Add(\lambda^+, \lambda^{+++})$ is λ^{+++}-cc in $V[G_{<\lambda}]$. Counting names again yields

$$\begin{aligned}\left(2^{(\lambda^+)}\right)^{V[G]} = \left(2^{(\lambda^+)}\right)^{V[G_{\leqslant\lambda}]} &\leqslant ((|Add(\lambda^+, \lambda^{+++})|^{\lambda^{++}})^{\lambda^+})^{V[G_{<\lambda}]} \\ &\leqslant ((\lambda^{+++})^{\lambda^{++}})^{V[G_{<\lambda}]} = \lambda^{+++}\end{aligned}$$

where the last equality holds as $\lambda^{+++} = 2^{\lambda^{++}}$ in $V[G_{<\lambda}]$.

(*iv*) Every $p \in \mathbb{P}$ defines a "lottery" sequence $l(p) : d(p) \to 2$ that describes the outcome of the lottery for the condition p. We make this precise: $d(p) \subseteq dom(p)$ is the set of cardinals $\lambda \in dom(p)$ so that either

$$p \restriction \lambda \Vdash \text{“}p(\lambda) \text{ lies in the nontrivial part of } \dot{\mathbb{Q}}_\lambda\text{”}$$

or

$$p \restriction \lambda \Vdash \text{“}p(\lambda) \text{ lies in the trivial part of } \dot{\mathbb{Q}}_\lambda\text{”}$$

In the former case, we define $l(\lambda) = 0$ and in the latter $l(\lambda) = 1$. Observe that if two conditions p, q are compatible then their lottery sequences $l(p), l(q)$ coincide on $d(p) \cap d(q)$. In particular, the generic G has its own lottery sequence $l(G) = \bigcup_{p \in G} l(p)$ that corresponds to the choices it has made along the iteration. Suppose $x \subseteq \rho$ is a bounded subset of κ, i.e. $\rho < \kappa$. Notice that V and $V[G]$ have the same subsets of κ as $\mathbb{P}$ is $< \kappa^+$-closed, so that $x \in V$. We have to show that x is coded into the GCH pattern of $V[G]$, which will follow from a density argument. Define a subclass D of $\mathbb{P}$ as:

$$D = \{p \in \mathbb{P} | \exists \beta \ \forall \gamma < \alpha \ \aleph_{\beta+\gamma} \in d(p) \wedge (l(p)(\aleph_{\beta+\gamma}) = 1 \leftrightarrow \gamma \in x)\}$$

If $p \in \mathbb{P}$ is any condition, then we can find β so that $dom(p) \subseteq \aleph_\beta$. As $\mathbb{P}$ is Easton supported and $\rho < \kappa$ we can add conditions $q(\aleph_{\beta+\gamma})$ to p for $\gamma < \rho$ as we like and are guaranteed that the resulting q is again in $\mathbb{P}$. In particular we can define q in such a way that q satisfies the defining property of D with the β we have picked. This shows that $\mathbb{P}$ is dense, so that the characteristic function of x in ρ appears as a block in $l(G)$. By part (iii), $l(G)$ exactly describes the GCH pattern of successor cardinals in $V[G]$, that means x is coded into the GCH pattern.

□

Proof. (Theorem 2.2.2) Let $\kappa = |V_\alpha|^+$. We can assume that GCH holds in V at and above κ as otherwise we would first extend via the canonical Easton support product that forces this. Since this forcing is $< \kappa$-closed, it preserves V_α. Now let G be $\mathbb{P} = \mathbb{P}(\kappa)$-generic over V. First of all $V[G]$ is a model of ZFC with the same cardinals as V by the above lemma. Since $\mathbb{P}$ also has enough closure, $V[G]_\alpha = V[G]$. Now assume that $x \subseteq \lambda$ is a set of ordinals in $V[G]$. Split G as $G_{<\lambda}$, a $\mathbb{P}_{<\lambda}$-generic filter over V, and $G_{\geqslant\lambda}$ a $\mathbb{P}_{\geqslant\lambda} = \dot{\mathbb{P}}_{\geqslant\lambda}^{G_{<\lambda}}$-generic filter over $V[G_{<\lambda}]$. As $\mathbb{P}_{\geqslant\lambda}$ is $< \lambda^+$-closed, x is already an element of $V[G_{<\lambda}]$. But now we have that $\mathbb{P}_{\geqslant\lambda} \cong \mathbb{P}(\lambda)^{V[G_{<\lambda}]}$. Furthermore, the counting names arguments of part (iii) in the above lemma show that GCH holds at and above λ in $V[G_{<\lambda}]$. Part (iv) shows that x is coded into the GCH pattern of $V[G]$. □

2.3 The Mantle

A lot of the motivation of Set-Theoretic Geology comes from the hope to find regular structure beneath the generic sets that are the result of forcing. This takes the perspective that the general set theoretic universe is similar to dry erosion-prone land that once was the home of a flourishing flora. However by digging through the grounds, we should be able to uncover this old and forgotten structure that has been buried by the accumulated dust added by forcing.

It is not unreasonable to propose that such a structure should be digging-minimal, i.e. should satisfy the ground axiom. However, we have seen that one can turn the tables and add new generic sets to extend to such a structure, rather than cleaning up and looking inwards. Since there are extensions that satisfy the ground axiom and coincide with V to an arbitrary degree, $ZFC + GA$ cannot have any Σ_2-consequences that do not already follow from ZFC alone. One can come to this conclusion via the following lemma, a proof can be found in the addendum.

Lemma 6.1.1. *A formula $\phi(x)$ is Σ_2^{ZFC} if and only if*

$$ZFC \vdash \forall x \ (\phi(x) \leftrightarrow \exists \alpha \ x \in V_\alpha \wedge V_\alpha \models \ulcorner \psi(x) \urcorner)$$

for some formula $\psi(x)$.

To round this up, the ground axiom does not fully capture what was actually looked for. The downwards directed perspective motivates the next definition, one that plays a central role in Set-Theoretic Geology and this thesis.

Definition 2.3.1. The mantle $\mathbb{M} = \bigcap_r W_r$ is the intersection of all grounds.

Is the mantle the promised land? Unfortunately, the first answer to this question will be a hard no, in a way that is even somewhat worse than in the case of the ground axiom. Also, if we take a quick glance at the canonical inner models L and $L[0^\#]$ again, we see that since they satisfy (GA), they are their own mantles (observe that trying to capture "$V = \mathbb{M}$" resolves in the ground axiom, so the two ideas are connected in this way). More so, the mantle of every set forcing extension of $L, L[0^\#]$ is again L, $L[0^\#]$ respectively. For L this is clear as it is the minimal inner model of ZFC and is always contained in every ground. For $L[0^\#]$, this holds as every ground of every extension must also contain $0^\#$ since this set refuses to be added by forcing. We will later see that this is no coincidence.

Theorem 2.3.2. *[FHR15, Theorem 66] Any model V of ZFC has a class forcing extension $V[G] \models ZFC$ such that $\mathbb{M}^{V[G]} = V$.*

This has the consequence that in general, the mantle does not have any special properties. In particular, the mantle need not satisfy the ground axiom, which might go a bit against the first unreflected intuition.
Let's get back on track to the theorem above. Our construction will achieve the ground axiom fails badly in the final model. More precisely, there will be no bedroch. Ironically, the strategy is almost identical to forcing the ground axiom. In fact if GCH holds, one can just take the product instead of an iteration. We want to extend a model V using a class forcing to produce a model $V[G]$ with mantle V. Instead of coding iteratively all sets into the GCH pattern that were added in a prior step as in Theorem 2.2.2, we only code the sets of V in the below construction. That is the reason we use a class product instead of an iteration. A new obstacle is that we do not want to start with a model of GCH or eventual GCH, so we have to modify the forcing even more. In Theorem 2.2.2, we only had to force the failure of GCH at certain stages. However in this situation the GCH can fail at a cardinal λ already in V. That means that now we have to produce

instances of GCH and will use $Add(\lambda^{++}, 1)$ instead of the trivial forcing in the second component of the lottery sum to make this happen. We still have one more problem to solve: Recall that a quiet but none the less important part in the argument for forcing (GA) was that the forcing we used did not collapse cardinals. This was necessary in order to ensure that the encoding at each stage was effective. Since both $Add(\lambda^{+}, \lambda^{+++})$ and $Add(\lambda^{++}, 1)$ can collapse cardinals without the assumption of GCH, we will code sets into the GCH pattern relative to a rather spaced out class of cardinals that both makes the construction easy and is robust in the sense that it will not be changed by the forcing we use.

Definition 2.3.3. Suppose $\mathcal{C}$ is a class of cardinals and $\langle \zeta_\gamma | \gamma \in Ord \rangle$ is the increasing enumeration of $\mathcal{C}$. We say that a set of ordinals $x \subseteq \alpha$ is coded into the GCH pattern relative to $\mathcal{C}$ if

$$\exists \beta \forall \gamma < \alpha \ (GCH \text{ fails at } \zeta_\gamma \leftrightarrow \gamma \in x)$$

Note that being coded into the GCH pattern according to Definition 2.2.3 is relative to the class of all successor cardinals. In this case, we choose $\mathcal{C} = \mathcal{C}_\kappa$ as the class of all strong limit cardinals above some cardinal κ that are not itself limits of such cardinals. Furthermore, having already the next chapter in mind, we want to keep some flexibility in the forcing itself. Because of this, we will allow a sequence of forcings on that we only impose mild restrictions to interfere.

Notice that $\oplus$ is associative in the sense that $(\mathbb{P} \oplus \mathbb{Q}) \oplus \mathbb{R} \cong \mathbb{P} \oplus (\mathbb{Q} \oplus \mathbb{R})$, so we will omit the brackets.

As we deal with class forcing, we have to argue that ZFC is preserved. Unfortunately we may not use the reasoning we have applied in Lemma 2.2.2 (i). There we used a quite general preservation theorem for class iterations. Here we use a class product, for which Reitz has elaborated an analogous result, but we will not have the necessary closure properties at hand.

Definition 2.3.4. A class forcing $\mathbb{P}$ is pretame if for any sequence $\langle D_i | i \in I \rangle$ of dense subclasses of $\mathbb{P}$ and any $p \in \mathbb{P}$, there is $q \leqslant p$ and a sequence of subsets $\langle d_i \subseteq D_i | i \in I \rangle$ so that each d_i is predense below q.

Remark 2.3.5. A sequence of classes $\langle C_i | i \in I \rangle$ is to be understood as one single definable class C with the following property:

$$C = \{(x, i) | i \in I \wedge x \in C_i\}$$

Fact 2.3.6. *[Fri00] If a class forcing is pretame then it preserves ZF^-.*

The following is a modificaton of Theorem 66 in [FHR15] that further encapsulates an additional sequence of forcings.

Theorem 2.3.7. *Assume $\langle \mathbb{Q}_\lambda | \lambda \in \mathcal{C}_\kappa \rangle$ is a sequence of forcings so that $\mathbb{Q}_\lambda$ is $< \lambda$-strategically closed and has size less than the next cardinal above λ in $\mathcal{C}$. Let $\mathbb{P}$ be the class product*

$$\prod_{\lambda \in \mathcal{C}_\kappa} Add(\lambda^+, \lambda^{+++}) \oplus Add(\lambda^+, 1) \oplus \mathbb{Q}_\lambda$$

with set-sized support. If $V[G]$ is a $\mathbb{P}$-extension of V then $V[G]$ does not contain any new sequences of ordinals of length $< \kappa$ and $\mathbb{M}^{V[G]} = V$. Moreover, if $\lambda \in \mathcal{C}_\kappa$, $\bar{\lambda}$ is the next strong limit cardinal and g_λ is the generic for stage λ, then the cardinals in the interval $[\lambda, \bar{\lambda})$ are the same in $V[G]$ and $V[g_\lambda]$.

Proof. Let us break up the necessary ingredients of the core argument into several points.

Claim 2.3.8. *Let $\lambda \in \mathcal{C}_\kappa$. The following hold:*

(i) $\mathbb{P}$ *preserves ZFC.*

(ii) The initial factor $\mathbb{P}_{<\lambda}$ has size $< \lambda$.

(iii) The tail segment $\mathbb{P}_{\geqslant\lambda}$ is $< \lambda$-distributive.

(iv) If G chose $Add(\lambda^+, \lambda^{+++})$ at stage λ then GCH fails at λ^+ in $V[G]$.

(v) If G chose $Add(\lambda^{++}, 1)$ at stage λ then GCH holds at λ^+ in $V[G]$.

(vi) The class $\mathcal{C}_\kappa$ is absolute between V and $V[G]$.

Let us first assume that all of the above is true. Most importantly $V[G] \models ZFC$ by (i). To show that $V[G]_\kappa = V_\kappa$ it is enough to prove that $V[G]$ contains no new sequences of ordinals of length $< \kappa$. But this holds by (iii).

Now we will see that every set in V is coded into the GCH pattern of $V[G]$ relative to $\mathcal{C}_\kappa$. We write $\langle \zeta_\alpha | \alpha \in Ord \rangle$ for the increasing enumeration of $\mathcal{C}_\kappa$. Note that $\mathcal{C}_\kappa$ is the same in V and $V[G]$ by (vi). As in Lemma 2.2.7, every $p \in \mathbb{P}$ defines a lottery sequence $l(p) : d(p) \to 3$ which describes the choices p has made in the lottery sums. It is a little more direct to define this here: We let $d(p) \subseteq dom(p)$ be the subset where the decision has already fallen, that is the set of all λ for which $p(\lambda)$ is not the maximal element. Now for $\lambda \in d(p)$ we define:

$$l(p)(\lambda) = \begin{cases} 0 & \text{if } p(\lambda) \text{ lies in } Add(\lambda^+, \lambda^{+++}) \\ 1 & \text{if } p(\lambda) \text{ lies in } Add(\lambda^{++}, 1) \\ 2 & \text{if } p(\lambda) \text{ lies in } \mathbb{Q}_\lambda \end{cases}$$

In the same way as in the proof of Lemma 2.2.7 (iv), the generic filter has its own lottery sequence $l(G) = \bigcup_{p \in G} l(p) : \mathcal{C}_\kappa \to 3$. By (iv) and (v), the 0's and 1's in $l(G)$ describe whether or not GCH holds in $V[G]$ at the respective successor cardinals, i.e. GCH holds at λ^+ in $V[G]$ if $l(G)(\lambda) = 1$ and it fails if $l(G)(\lambda) = 0$. If $l(G)(\lambda) = 2$, this will depend on V and the forcing $\mathbb{Q}_\lambda$, but this will not be a problem. Let $x \in V$ be a subset of some ordinal α. If the characteristic function of x appears as a block in $l(G)$, this would exactly mean that x is coded into the GCH pattern of $V[G]$ relative to $\mathcal{C}_\kappa$. We will see that this is indeed the case. Define a subclass of $\mathbb{P}$:

$$\begin{aligned} D_x = \{p \in \mathbb{P} | \exists \beta \forall \gamma < \alpha\ \zeta_{\beta+\gamma} \in d(p) \wedge (l(p)(\zeta_{\beta+\gamma}) = 1 \leftrightarrow \gamma \in x) \\ \wedge (l(p)(\zeta_{\beta+\gamma}) = 0 \leftrightarrow \gamma \notin x)\} \end{aligned}$$

If $p \in \mathbb{P}$ then we can add conditions to p high enough above its support to make sure the defining property of D_x is satisfied for some β, this is okay since $\mathbb{P}$ has set-sized support. This shows that D_x is dense and hence $G \cap D_x \neq \emptyset$. Thus x is coded into the GCH pattern of $V[G]$ relative to $\mathcal{C}_\kappa$. This is very convenient to show that $V \subseteq \mathbb{M}^{V[G]}$: As usual, we can conclude that indeed every set of ordinals $x \in V$ is coded unboundedly often into the GCH pattern of $V[G]$ relative to $\mathcal{C}_\kappa$. This implies that this also holds for every ground W of $V[G]$ as the continuum functions of $V[G]$ and W eventually coincide, and so the classes $\mathcal{C}_\kappa^W$ and $\mathcal{C}_\kappa^{V[G]}$ coincide eventually, too. Hence $x \in W$ as it is definable in W. As every set in V can be coded as a set of ordinals in V, this implies $V \subseteq W$ and as W was arbitrary, $V \subseteq \mathbb{M}^{V[G]}$.
On the other hand, for every $\beth$-fixed point $\lambda \geqslant \kappa$ we can factor $\mathbb{P} = \mathbb{P}_{<\lambda} \times \mathbb{P}_{\geqslant\lambda}$ and the generic $G = G_{<\lambda} \times G_{\geqslant\lambda}$ accordingly. Now $\mathbb{P}_{\geqslant\lambda}$ is $< \lambda$-distributive by (iii) and as λ is a $\beth$-fixed point, in particular $V[G_{>\lambda}]_\lambda = V_\lambda$. But $V[G_{\geqslant\lambda}]$ is also a ground of $V[G]$ as $V[G_{\geqslant\lambda}][G_{<\lambda}] = V[G]$ and hence $\mathbb{M}_\lambda^{V[G]} \subseteq V_\lambda$.
Next up, we have to show that given $\lambda \in \mathcal{C}_\kappa$ and $\bar{\lambda}$ the next strong limit, the cardinals of $V[G]$ and $V[g_\lambda]$ in the interval $[\lambda, \bar{\lambda})$ coincide. It is enough to show that if $\delta \in [\lambda, \bar{\lambda})$ is a cardinal in $V[g_\lambda]$, then it is in $V[G]$. Notice that we can factor G as $G_{<\lambda} \times g_\lambda \times G_{>\lambda}$. Since $\mathbb{P}_{<\lambda}$ has size $< \lambda$, δ is still a cardinal in $V[g_\lambda][G_{<\lambda}]$. We have seen that every initial segment of $\mathbb{P}_{>\lambda}$ is $< \bar{\lambda}$-strategically closed in V. By Lemma 1.3.5, every such initial segment is still $< \bar{\lambda}$-distributive in $V[g_\lambda][G_{<\lambda}]$. Thus $\mathbb{P}_{>\lambda}$ does not add any new sequences of length $< \bar{\lambda}$. This shows that δ is still a cardinal in $V[g_\lambda][G_{<\lambda}][G_{>\lambda}] = V[G]$.
It remains to prove the claim. The reasoning is quite similar to Lemma 2.2.7. Critically, we have to replace arguments exploiting the iterative nature of the forcing there with arguments that work for products.

Proof. (Claim 2.3.8)

(i) Let us show that $\mathbb{P}$ is pretame. Our argument will be a modification of Lemma 2.23 in [Fri00]. Suppose $\langle D_\alpha | \alpha < \delta \rangle$ is a sequence of dense subclasses of $\mathbb{P}$, $\delta > \kappa$ is a cardinal and $p \in \mathbb{P}$. Let $\lambda > \delta$ so that

$p \in \mathbb{P}_{<\lambda}$. By (ii), we may enumerate $\mathbb{P}_{<\lambda}$ as $\{p_\alpha | \alpha < \rho\}$ for some $\rho < \lambda$. By increasing one or the other, we may assume $\delta = \rho$. It is not necessary that the enumeration of $\mathbb{P}_{<\lambda}$ is injective. Notice that every initial segment of $\mathbb{P}_{\geqslant\lambda}$ is $\leqslant \delta$-strategically closed, as each factor is. The point is that we can build a winning strategy for player II by applying winning strategies coordinate-wise. Since we have to apply a winning strategy only in a large enough initial segment of $\mathbb{P}_{\geqslant\lambda}$, we may assume that σ_{II} is a winning strategy for player II in the game $G(\mathbb{P}_{\geqslant\lambda}, \delta+1)$ (formulated in the canonical way for class forcings). Alternatively, we could assume that global choice holds after forcing to add a global choice function without adding new sets. In that case, we can build the class winning strategy by choosing strategies for each factor. Let $h : \delta \to \delta \times \delta$ be a bijection. Let σ_I be the following strategy for player I in $G(\mathbb{P}_{\geqslant\lambda}, \delta+1)$: Suppose we are at stage $2\beta+1$. Let $h(\beta) = (\beta_0, \beta_1)$. If $\langle q_\beta | \beta \geqslant 2\alpha\rangle$ is the prior play, then let the next move be any $q \leqslant q_{2\beta}$ so that $q \cup p_{\beta_0} \in \mathbb{P}$ is a condition below some $r_\beta \in D_{\beta_1}$, if possible. Otherwise, play $r_{2\beta}$.
Let $\langle q_\beta | \beta \leqslant \delta\rangle$ be the outcome after playing according to σ_I and σ_{II}. Moreover, we can extract the sequence $\langle r_\beta | \beta < \lambda\rangle$ from the resulting play. Note that by using an inductive argument, we can avoid the use of global choice in the construction of σ_I. Let $p_\star = p \cup q_\delta$ and furthermore set

$$d_\alpha = \{r_\gamma | \exists \beta < \delta \; h(\beta) = (\gamma, \alpha)\}$$

for every $\alpha < \delta$. It is left to show that d_α is predense below $p_\star$. Suppose $q \leqslant p_\star$ and find a stronger condition $q' \in D_\alpha$. We can find $\gamma < \delta$ so that $q' \upharpoonright \lambda = p_\gamma$. Find $\beta < \delta$ with $h(\beta) = (\gamma, \alpha)$ and note that $q \leqslant p_{2\beta+1} \cup p_\beta \leqslant r_\gamma \in d_\alpha$.
Thus $\mathbb{P}$ is pretame and so preserves ZF^- by Fact 2.3.6. To show that the powerset axiom holds in an extension $V[G]$, suppose $\lambda \geqslant \kappa$ is a cardinal. We can factor $\mathbb{P}$ at stage λ^+ into $\mathbb{P} = \mathbb{P}_{<\lambda^+} \times \mathbb{P}_{\geqslant\lambda^+}$ (and the generic accordingly) then $\mathbb{P}_{\geqslant\lambda^+}$ is $< \lambda^+$-distributive and thus every subset of λ in $V[G]$ is already contained in $V[G_{\leqslant\lambda^+}]$, which is a set forcing extension. Hence

$$\mathcal{P}(\lambda)^{V[G]} = \mathcal{P}(\lambda)^{V[G_{\leqslant\lambda^+}]} \in V[G_{\leqslant\lambda^+}] \subseteq V[G]$$

which shows that $\mathbb{P}$ preserves the power set axiom.

(ii) Find α so that $\lambda = \zeta_\alpha$. For any given β, our assumption on the size of $\mathbb{Q}_{\zeta_\beta}$ implies that the factor of $\mathbb{P}$ at stage ζ_β has size some $\delta_\beta < \zeta_{\beta+1}$. If $\alpha = \alpha' + 1$ then we can conclude by induction that $|\mathbb{P}_{\leqslant\lambda}| = |\mathbb{P}_{\leqslant\zeta_{\alpha'}}| \cdot \delta_\alpha < \lambda$. If α is a limit, we have

$$|\mathbb{P}_{<\lambda}| \leqslant \prod_{\beta<\alpha} \zeta_{\beta+1} \leqslant \alpha^{(sup_{\beta<\alpha}\zeta_\beta)} = 2^{(sup_{\beta<\alpha}\zeta_\beta)} < \zeta_\alpha$$

where the last inequality holds since we have purposefully excluded limits of strong limits in $\mathcal{C}_\kappa$.

(*iii*) It is enough to show that $\mathbb{P}$ is $< \kappa$-distributive as $\mathbb{P}_{\geqslant\lambda}$ is essentially $\mathbb{P}$ defined using the parameter λ instead of κ. As any sequence of ordinals is already contained in the induced extension $V[G_{<\lambda}]$ of $\mathbb{P}_{<\lambda}$ for some large enough λ, it is enough to show that $\mathbb{P}_{<\zeta_\alpha}$ is $< \kappa$-distributive for any α. Notice that the factor at stage $\lambda \in \mathcal{C}_\kappa$ is $< \lambda$-strategically closed as each summand is in the lottery sum. Now the product $\mathbb{P}_{<\lambda}$ is $< \kappa$-strategically closed, and thus $< \kappa$-distributive, as can be seen by applying winning strategies coordinate wise. In fact the whole product $\mathbb{P}$ would satisfy the class equivalent of $< \kappa$-strategical closure if there were a class sequence of winning strategies.

(*iv*) Let λ' be the successor of λ in $\mathcal{C}_\kappa$ and let $V[G_{\geqslant\lambda'}]$ be the induced extension of V by $\mathbb{P}_{\geqslant\lambda'}$. Using (*iii*), we get $Add(\lambda^+, \lambda^{+++})^{V[G_{\geqslant\lambda'}]} = Add(\lambda^+, \lambda^{+++})^V$ and since $G_{\geqslant\lambda}$ adds a $Add(\lambda^+, \lambda^{+++})$-generic filter by assumption, we conclude that in $V[G_{\geqslant\lambda}]$ GCH fails at λ^+. Now $\mathbb{P}_{<\lambda}$ has size $< \lambda$ by (*ii*) and so the same is true in $V[G]$.

(*v*) The same argument as above works in this case, using that $Add(\lambda^{++}, 1)$ forces GCH at λ. This is true as the new Cohen subset of λ^{++} must contain every subset of λ^+ in V as a block and does itself not add any new subsets of λ^+.

(*vi*) Let $\lambda \in \mathcal{C}_\kappa$. By (*iii*), $\mathbb{P}_{\geqslant\lambda}$ is $< \lambda$-distributive and thus does not destroy any strong limit cardinal $\leqslant \lambda$. Since $\mathbb{P}_{<\lambda}$ has size $< \lambda$ by (*ii*), λ is still a strong limit in $V[G_{\geqslant\lambda}][G_{<\lambda}] = V[G]$. This shows that any cardinal in $\mathcal{C}_\kappa^V$ is a strong limit in $V[G]$. On the other hand, "λ is a strong limit cardinal" is downwards absolute. This is enough to conclude $\mathcal{C}_\kappa^V = \mathcal{C}_\kappa^{V[G]}$.

□

□

Theorem 2.3.2 follows by applying Theorem 2.3.7 with the trivial sequence $\mathbb{Q}_\lambda = \{\mathbb{1}\}$.

2.4 The Structure of Grounds

Now that we have seen that all models of ZFC are the mantle of another model, one can ask if the other direction also holds: Is the mantle always a model of ZFC? This is where we get some redemption for the disappointing lack of special features of the mantle, as the answer turns out to be yes. The mantle is always an inner model of ZFC. Furthermore, this result is highly nontrivial and was only recently solved by Usuba ([Usu17]). The structure of how the grounds align is a the key insight we need to show that the mantle is a nice object. But before we dive into this, we will first investigate a few other geologic questions that are connected to the above one. We have seen that in the case of the constructible universe, L is the mantle of all its extensions, but we needed a crucial property of L to see this. In general, without further assumption on the universe, it is not clear whether V is the mantle of all its extensions if $V \models GA$. A priori, there might be grounds of some extension $V[G]$ not included in V that descend indefinitely. We capture this setting in a definition.

Definition 2.4.1. (*i*) A bedrock W is a local minimum of grounds, that means it has no nontrivial grounds itself. In other words $(GA)^W$ holds.

(*ii*) A solid bedrock W is a global minimum of grounds, i.e. $W = \mathbb{M}$.

The above situation put differently results in the question whether or not all bedrocks are solid. It is not clear whether a ground U exists which does not contain the bedrock.
A more ambitious question is to ask if the mantle is forcing invariant. This would make the above situation where a bedrock W is non-solid impossible, since in this case $\mathbb{M}^W = W \neq \mathbb{M}^V$. Let's quickly introduce the generic multiverse of a model of ZFC. Whenever we mention this, we picture V as a countable set in a large background model $\mathcal{V}$. Then the generic multiverse is the closure of $\{V\}$ under taking grounds and forcing extensions. If the mantle were provably forcing invariant, this would clearly imply that the mantle is constant across the whole generic multiverse, since any point in there can be reached via a finite crisscross between grounds and extension. But how many steps do you need? Is there an upper bound? Observe that the generic multiverse is two-dimensional: In general, one step is not enough, as if $\mathbb{P}$, $\mathbb{Q}$ are nontrivial forcing notions and $G \times H$ is generic for their product, then $V[G]$ and $V[H]$ are in the same generic multiverse, but are neither extension nor grounds of each other as $V[G] \nsubseteq V[H]$ and vice versa. But are two steps enough?

The answers to all of these questions are consequences of a hypothesis that prescribes that the grounds are structured in the arguably simplest way possible.

Definition 2.4.2. The strong Downwards Directed Grounds Hypothesis ($sDDG$) is the assertion $\forall X \exists r\ W_r \subseteq \bigcap_{s \in X} W_s$. This states that for any set-sized collection of grounds, the intersection contains another ground.

This hypothesis is now a theorem.

Theorem 2.4.3. *[Usu17] (Usuba) The sDDG is a consequence of ZFC.*

Originally, there was also a weak Downwards Directed Grounds Hypothesis that states that for any two grounds, their intersection contains a grounds. However this has become obsolete with the above theorem.
First let's show that the *sDDG* is both the link and the answer to all of the above problems.

Corollary 2.4.4. (*i*) *All bedrocks are solid.*

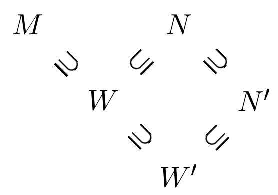
(*ii*) *The mantle is forcing invariant.*

(*iii*) *Two points M, N in the generic multiverse are at most two steps apart from each other. More precisely, N is a ground extension of M, that means it is a forcing extension of some ground of M.*

Proof. (*i*) As we have already discussed, this follows from (*ii*).

(*ii*) It is enough to show that $\mathbb{M}^V = \mathbb{M}^W$ for every ground W of V. By the product lemma, every ground of W is still a ground of V, so that $\mathbb{M}^V \subseteq \mathbb{M}^W$. For the other direction, assume M is another ground of V. Applying the *sDDG* yields a ground $N \subseteq W, M$. By the quotient lemma (Corollary 6.3.10), N is again a ground of W.

(*iii*) Assume N is a ground extension of M, i.e. W is a common ground of M, N. It is enough to show that the same holds for all grounds and extensions of N. It is clear for all extensions of N by the product lemma. If $N' \subseteq N$ is a ground then apply the *sDDG* in N to find a ground $W' \subseteq W, N'$.



Then W' is a ground of W by the quotient lemma and thus a ground of M and so witnesses that N' is a ground extension of M.

□

Remark 2.4.5. The order is important in part (*iii*). It is possible that N is not a ground of an extension of M, despite being in the same generic multiverse. We will see an example for this situation later.
Moreover, (*iii*) allows us to state "everywhere in the generic multiverse φ holds" and "somewhere in the generic multiverse φ holds" as first order $\in$-formulas since we can express "in all ground extension φ holds" and "in some ground extension φ holds" using the definability of grounds and the forcing relation.

We argue that the mantle is, from the perspective of forcing, a canonical object. First let's get back to the first question. Since this one was the most prominent one and a strong driving force in research concerning Set-Theoretic Geology, it deserves its own theorem.

Theorem 2.4.6. $\mathbb{M} \models ZFC$.

Proof. We apply the inner model criterion (Lemma 6.2.4). As the Gödel operations are all absolute between transitive models of ZFC and since all grounds are closed under them, their intersection $\mathbb{M}$ is, too. Assume $x \subseteq \mathbb{M}$ for some set x of V. Let $\alpha = rk(x)$. Since the mantle is forcing invariant, we have that $\mathbb{M} \cap V_\alpha = \mathbb{M}^W \cap W_\alpha$ is definable in every ground W and thus $x \subseteq \mathbb{M} \cap V_\alpha \in \mathbb{M}$. Hence $M \models ZF$. For the axiom of choice, we need the full strength of $sDDG$. Suppose $x \in \mathbb{M}$ has no well-order in $\mathbb{M}$. For every well-order $\preceq$ of x in V, there must be a ground $W_{r_\preceq}$ that does not contain it. Let $X = \{r_\preceq | \preceq \text{ is a well-order of } x\}$. By the $sDDG$, there is a ground $W \subseteq \bigcap_{r_\preceq \in X} W_{r_\preceq}$, but then W does not contain a well-order of x, a contradiction. □

Remark 2.4.7. Earlier we have argued that it is consistent that the mantle does not satisfy the ground axiom. Now that we know that the mantle is a model of ZFC, we can put this differently: It is possible that $\mathbb{M}^{\mathbb{M}} \neq \mathbb{M}$. One could try to iterate this procedure, which first of all gives rise to the n-mantle $\mathbb{M}^n$, the result of taking mantles n times, but only for meta-theoretic natural numbers n. The problem is that the uniform definability of grounds does not give any ground to believe that the n-mantles should be uniformly definable in n. That means that it is not at all clear whether or not $\bigcap_{n<\omega} \mathbb{M}^n$ makes sense or is a definable class even when the natural numbers in the object theory coincide with the meta natural numbers. However, in some models of the second order set theory GBC, there might be a meaningful way to define the α-mantle for any ordinal α, so that the limit mantles are intersections of the previous mantles. It is conjectured in a strong way by Fuchs, Hamkins and Reitz ([FHR15, Conjecture 74]) that every model of ZFC is the α-mantle of an outer model of GBC. Furthermore they expect that similar to the case of iterating HOD (compare [Zad83]), it is in general not possible to define the α-mantle and more precisely that there is a model of GBC in which $\mathbb{M}^n$ is a class for all $n < \omega$, but the ω-mantle is not.

In addition to this, the mantle is not just forcing invariant as we have seen in Corollary 2.4.4 (*ii*), it is the largest class with this property.

Lemma 2.4.8. *If C is a class term such that $ZFC \vdash$ "C is forcing invariant" then $ZFC \vdash C \subseteq \mathbb{M}$.*

Proof. It is enough to show that $C^V \subseteq \mathbb{M}^V$. If W is any ground then applying the assumption in W yields that $C^W = C^V$. But then $C^V \subseteq \bigcap_r W_r = \mathbb{M}$. □

2.5 A Destructibility Result

We apply the Definability of Grounds Theorem to see that a certain class of forcings destroys a correctness property. As a wide variety of large cardinals necessarily satisfy this correctness property, these large cardinals are destroyed by this class of forcings. We follow [BHTU16].

Theorem 2.5.1. *Suppose κ is a strongly inaccessible cardinal and $V_\kappa \prec V_\lambda$ for some $\lambda \geqslant \eta \geqslant \kappa$. If $\mathbb{P}$ is a nontrivial $< \kappa$-strategically closed forcing, $\mathbb{P} \in V_\eta$ and G is $\mathbb{P}$-generic over V, then for all $\theta \geqslant \eta$, $V[G]_\kappa \nprec V[G]_\theta$.*

We first prove an auxiliary result.

Lemma 2.5.2. *If κ is a strongly inaccessible cardinal, $\kappa < \lambda$ and $V_\kappa \prec V_\lambda$ then both κ and λ are $\beth$-fixed points and fixed points of the increasing enumeration of $\beth$-fixed points.*

Proof. The claim holds for κ as κ is strongly inaccessible. Furthermore, $V_\kappa \models$ "for all α, $\beth_\alpha$ exists" and so the same is true in V_λ. As V_λ computes the $\beth$-function correctly, this shows $\beth_\alpha < \lambda$ for all $\alpha < \lambda$. Since the $\beth$-function is continuous and strictly increasing, this implies that λ is a $\beth$-fixed point. The same argument works if we replace the $\beth$-function by the increasing enumeration of $\beth$-fixed points. □

Proof. (Theorem 2.5.1) Assume towards a contradiction that $V[G]_\kappa \prec V[G]_\theta$ for some $\theta \geqslant \eta$. By Lemma 2.5.2, λ and θ are $\beth$-fixed points. Thus we can assume, by increasing η if necessary, that η is a $\beth$-fixed point itself. Moreover, by passing to the κ^+-th $\beth$-fixed point above η if necessary, we can assume that $cof(\eta) > \kappa$. The above Lemma implies that η will still be less than λ and θ. Since $\mathbb{P}$ adds no new sequences of ordinals of length $< \kappa$ and since κ is a $\beth$-fixed point, it follows that $V[G]_\kappa = V_\kappa$. We thus want to conclude a contradiction from the following:

$$\begin{array}{ccc} V_\lambda & & V[G]_\theta \\ & \prec \quad \succ & \\ & V_\kappa & \end{array}$$

Observe that $V_\kappa \models ZFC$ and thus V_λ and $V[G]_\theta$ are models of ZFC, too.

Claim 2.5.3. *The following hold:*

(*i*) $V_\theta \models ZFC$

(*ii*) $V[G]_\lambda = V_\lambda[G]$ *and* $V[G]_\theta = V_\theta[G]$.

Proof. (*i*) Clearly V_θ is closed under the Gödel operations and satisfies the Ord-cover property with respect to $V[G]_\theta$. This means we can apply the inner model citerion (Theorem 6.2.4). Since furthermore the axiom of choice holds in V_θ, as it does so in V, we conclude $V_\theta \models ZFC$.

(*ii*) Let ξ be λ or θ. Notice that $V_\xi \models ZFC$ and so does $V_\xi[G]$. First suppose $x \in V[G]_\xi$. If $x \subseteq \alpha$ then there is a nice name $\dot{x}$ for a subset of α with $\dot{x}^G = x$. Since $\mathbb{P} \in V_\eta \subseteq V_\xi$, one can see that $rk(\dot{x}) < \xi$. Thus $\dot{x}^G \in V_\xi[G]$. Any general $x \in V[G]_\xi$ can be coded as a set of ordinals (compare Remark 2.1.8) and decoded correctly inside $V_\xi[G]$. The other inclusion follows from the simple observation that $rk(\dot{x}^G) \leqslant rk(\dot{x})$ for any $\mathbb{P}$-name $\dot{x}$.

□

Thus, $V[G]_\xi$ sees that it is generated by a $\mathbb{P}$-generic filter G over V_ξ for $\xi = \lambda, \theta$. Let $r = \left({}^{<|\mathbb{P}|^+}2\right)^V$. We want to apply Theorem 2.1.1 to be able to define V_θ inside of $V[G]_\theta$. This only works if V_θ is a ground of $V[G]_\theta$, but this is true by the above claim. Thus:

$$V[G]_\theta \models \text{"for some parameter } s \text{ and } H \text{ some generic over } W_s \text{ for nontrivial forcing } \mathbb{Q}, \text{ the universe is } W_s[H]\text{"}$$

By elementarity, the same is true in V_κ and thus $V_\kappa = W_s^{V_\kappa}[H]$ for some parameter s, nontrivial forcing $\mathbb{Q}$ and $\mathbb{Q}$-generic H. Let $\delta = |\mathbb{Q}|^+$. We can assume that $s = \left({}^{<\delta}2\right)^{W_s^{V_\kappa}}$. It follows from the two elementarity conditions that:

(*i*) $V_\lambda = W_s^{V_\lambda}[H]$

(*ii*) $V[G]_\theta = W_s^{V[G]_\theta}[H]$

$V[G]_\lambda = V_\lambda[G] = W_s^{V_\lambda}[H][G]$ is an extension by a forcing of size $< \delta$ followed by a $< \delta$-strategically closed forcing. By Proposition 2.1.14, it follows that $W_s^{V_\lambda} \subseteq V[G]_\lambda$ has the δ-cover and approximation properties. Notice that $W_s^{V_\lambda}$ has the correct δ^+ since $\mathbb{P}$ is in fact $< \kappa$-strategically closed. The same holds true for $W_s^{V[G]_\theta} \subseteq V[G]_\theta$. By going down to $\eta < \lambda, \theta$ it follows that:

(*i*)′ $V[G]_\eta = W_s^{V_\lambda}[H][G] \cap V[G]_\eta$

(*ii*)′ $V[G]_\eta = W_s^{V[G]_\theta}[H] \cap V[G]_\eta$

Let $W_0 = W_s^{V_\lambda} \cap V[G]_\eta$ and $W_1 = W_s^{V[G]_\theta} \cap V[G]_\eta$. Thus $W_0, W_1 \subseteq V[G]_\eta$ have the δ-cover and approximation properties, the correct δ^+ and the same set of bounded subsets of δ (namely s) and are models of ZFC_δ, by our further assumptions on η and Proposition 2.1.18 as well as 2.1.11. By Lemma 2.1.9, $W_0 = W_1$.

Claim 2.5.4.

$$V[G]_\eta = W_s^{V[G]_\theta}[H] \cap V[G]_\eta = W_s^{V_\lambda}[H] \cap V[G]_\eta$$

Proof. The first equality is just $(ii)'$. From $(i)'$, we can conclude

$$W_s^{V_\lambda}[H] \cap V[G]_\eta \subseteq W_s^{V_\lambda}[H][G] \cap V[G]_\eta = V[G]_\eta$$

so it is left to show that the inclusion

$$W_s^{V[G]_\theta}[H] \cap V[G]_\eta \subseteq W_s^{V_\lambda}[H] \cap V[G]_\eta$$

holds. So suppose $\dot{x}^H$ is in the left hand side, where $\dot{x} \in W_s^{V[G]_\theta}$ is a $\mathbb{Q}$-name. Since η is a strong limit cardinal in $V[G]_\theta$, we can find a regular $\zeta < \eta$ with $\dot{x}^H \in H_\zeta^{V[G]_\theta}$. We can assume without loss of generality that $\mathbb{1}_\mathbb{Q} \Vdash \dot{x} \in H_{\check{\zeta}}$ holds in $W_s^{V[G]_\theta}$. By Lemma 1.5.2, we can furthermore assume that $\dot{x} \in H_\zeta^{(W_s^{V[G]_\theta})} \subseteq W_s^{V[G]_\theta} \cap V[G]_\eta = W_0 = W_1$. Thus $\dot{x}^H \in W_s^{V_\lambda}[H]$ and by absoluteness of the rank, $\dot{x}^H$ is in the right hand side. □

This shows $G \in W_s^{V_\lambda}[H] = V_\lambda$, contradicting the non-triviality of $\mathbb{P}$. □

Remark 2.5.5. The following assumptions can be weakened a bit, see [BHTU16]. For example, a thorough analysis of the complexity of the key formulas used in the above proof yields that it is enough to assume $V_\kappa \prec_3 V_\lambda$ to conclude $V_\kappa \not\prec_2 V_\theta$ for all $\theta \geqslant \eta$.

We will apply the above theorem later on in chapter 4. Arguably, this is quite a heavy gun for that application and more simple arguments could do the trick. However, there is a reason why this section is included in this thesis. Any useful theory should influence another part of mathematics. The result above is not a geologic one, but an insight into the nature of large cardinals. Down the road, we will make use of a concept called Laver indestructibility. It is possible that a supercompact cardinal is preserved by any sufficiently closed forcing. Here, we can conclude that there is no analogue of this for, say, extendible cardinals. This destructibility of extendible cardinals is in line with Theorem 4.2.2, which will be presented right after the proof of the *sDDG*. It states that an extendible cardinal is incompatible with a bottomless, that is bedrock-free, universe. If there were a "Laver indestructible extendible cardinal" than the forcing of Theorem 2.3.7 would produce a universe with an extendible in which there is no bedrock.

3 The Downwards Directed Grounds Hypothesis

The result we present in this chapter is due to Usuba [Usu17]. However, the proof has been improved by Hamkins and thus we follow his version that he presented at the University of Bonn in January 2017 [Ham17a].

3.1 Combinatorial Prerequisites

For $\delta < \kappa$, δ a regular cardinal, we denote the set of ordinals below κ of cofinality δ by E^κ_δ. We will need a variant of Fodor's lemma for singular ordinals.

Lemma 3.1.1. *If $\delta < cof(\lambda)$ is a regular cardinal and $f : E^\lambda_\delta \to \lambda$ is regressive, there a stationary subset $S \subseteq E^\lambda_\delta$ such that $f[S]$ is bounded in λ.*

Proof. Let $\kappa = cof(\lambda)$ and $g : \kappa \to \lambda$ increasing, continuous and cofinal with $g(0) = 0$. By continuity, $g[E^\kappa_\delta] \subseteq E^\lambda_\delta$, so the following function is well-defined: We let $h : E^\kappa_\delta \to \kappa$, $h(\alpha) = max\{\beta < \kappa | g(\beta) \leqslant f \circ g(\alpha)\}$. The maximum always exists as $ran(g)$ is a club in λ and $g(0) = 0$. As f is regressive and g increasing, h must be regressive, too. By Fodor's Lemma, there is a stationary set $T \subseteq E^\kappa_\delta$ and $\rho < \kappa$ such that $h \upharpoonright T \equiv \rho$. I claim that $g[T]$ is stationary in λ. Let D be a club in λ. Then $D \cap ran(g)$ is club and since g is continuous and increasing, $D' = g^{-1}[D \cap ran(g)]$ is club in κ. Find $\alpha \in T \cap D'$. Then $g(\alpha) \in g[T] \cap D$. Now since h is constant on T with value ρ, it must be that $f[g[T]]$ is bounded by $g(\rho + 1)$. □

Lemma 3.1.2. *Assume $\delta < cof(\lambda)$ is a regular cardinal and $\mathbb{T}$ is a tree of height λ with levels of size $< \delta$. Then $\mathbb{T}$ has a cofinal branch, but fewer than δ many.*

Proof. If $\alpha \in E^\lambda_\delta$, then let $f(\alpha) = sup\{\Delta(t, s) | t \neq s \in \mathbb{T}_\alpha\}$. Since $\mathbb{T}_\alpha$ has size $< \delta = cof(\alpha)$, we have $f(\alpha) < \alpha$. Lemma 3.1.1 yields a stationary $S \subseteq E^\lambda_\delta$ and $\rho < \lambda$ such that $f \upharpoonright S$ is bounded by ρ. For every $\alpha \in S$ choose one $t_\alpha \in \mathbb{T}_\alpha$. Since S has size $> \delta$ and $\mathbb{T}_\rho$ has size $< \delta$, there has to be $S_\star \subseteq S$ unbounded and some $t_\star \in \mathbb{T}_\rho$ with $t_\star \leqslant_\mathbb{T} t_\alpha$ for all $\alpha \in S_\star$. Now the sequence $\langle t_\alpha | \alpha \in S_\star \rangle$ must form a cofinal branch: Suppose $\alpha < \beta \in S_\star$. If $t_\alpha \nleqslant_\mathbb{T} t_\beta$, then let t'_β be the unique node on level α that extends to t_β. As $t'_\beta \neq t_\alpha$ are both nodes on level $\alpha \in S$, we have that $\Delta(t'_\beta, t_\alpha) \leqslant f(\alpha) \leqslant \rho$. But this contradicts $t_\star \leqslant_\mathbb{T} t_\alpha, t_\beta$.

If b, c are cofinal branches then b, c differ at all large enough levels, in particular at all large enough levels in S. But then b and c must have parted ways already below level ρ. This yields an injection

$$\{b \subseteq \mathbb{T} | b \text{ is a cofinal branch}\} \to \mathbb{T}_\rho$$

and thus there must be fewer than δ many. □

Lemma 3.1.3. *Suppose W is an inner model of V, $\delta < cof(\lambda)^V$ a regular cardinal in V and $\mathbb{T}$ a tree in W of height λ and levels of size $< \delta$ in W. Then every cofinal branch $b \in V$ of W is already in W.*

Proof. Notice that if $t \in \mathbb{T}$ is a node that can extend to arbitrary large levels, then $\mathbb{T} \upharpoonright t = \{s \in \mathbb{T} | s \leqslant_{\mathbb{T}} t \vee t \leqslant_{\mathbb{T}} s\}$ is also a tree of height λ with levels of size $< \delta$ in W. If $b \in V$ is a cofinal branch that is not contained in W then it differs from all $< \delta$ many cofinal branches W knows about. Since $\delta < cof(\lambda)^V$ we have

$$sup\{\Delta(b, c) | c \in W \text{ cofinal branch in } \mathbb{T}\} < \lambda$$

and thus there is a node $t \in b$ that is not contained in any cofinal branch $c \in W$. Nonetheless W sees that t can extend to arbitrary large levels, so our observation together with Lemma 3.1.2 implies that $\mathbb{T} \upharpoonright t$ has a cofinal branch in W. But $\mathbb{T} \upharpoonright t \subseteq \mathbb{T}$, so this cofinal branch must be a cofinal branch of $\mathbb{T}$ that contains the node t, a contradiction. □

3.2 The Proof

Firstly, let us see that the naive approach is doomed. That would be to try to prove that any intersection $\bigcap_{r \in X} W_r$ is itself a ground. Indeed this would not even work for just two grounds:

Lemma 3.2.1. *[FHR15] If ZFC is consistent, it is consistent that the intersection of two grounds does not satisfy ZFC and thus is not a ground.*

Proof. For simplicity, start with a model $V[c]$ of GCH that already is the result of adding a Cohen real. Consider the finite support product $\mathbb{P} = \prod_{n<\omega} Add(\aleph_n, \aleph_{n+2})^V$ and notice that $\mathbb{P}$ can be factored in $V[c]$ as

$$\left(\prod_{c(n)=0} Add(\aleph_n, \aleph_{n+2})^V \right) \times \left(\prod_{c(n)=1} Add(\aleph_n, \aleph_{n+2})^V \right) = \mathbb{P}_0 \times \mathbb{P}_1$$

Observe that since V and $V[c]$ have the same finite subsets of ω, $\mathbb{P}$ is a member of V even though $\mathbb{P}_0$ and $\mathbb{P}_1$ are not. If G is $\mathbb{P}$-generic over $V[c]$ then G factors accordingly into $G_0 \times G_1$. This shows that $V[c][G_1]$ is a ground of $V[c][G]$. Furthermore by the product lemma, $V[G]$ is a ground of $V[c][G]$, too. Consider their intersection $W = V[c][G_1] \cap V[G]$ and assume for a contradiction that W is a model of ZFC. In the present situation, the counting names argument used in the proof of Claim 2.3.8 applies as well and shows that in $V[c][G_1]$, GCH holds at $\aleph_n$ if and only if $c(n) = 1$ (actually one can understand c as a the lottery sequence of G_1 in the context of the forcing $\prod_{n<\omega}(Add(\aleph_n, \aleph_{n+2})^V \oplus \{\mathbb{1}\})$). In addition, we get that in $V[G]$, $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$ and that $V, V[G], V[c][G_1]$ all have the same cardinals. As $V \subseteq W \subseteq V[G]$, W has exactly these cardinals as well. The point is that as $c \notin V[G]$, $c \notin W$, anyhow one can read c off from the GCH pattern of W: If $c(n) = 1$, the n-th coordinate g_n of G (which is $Add(\aleph_n, \aleph_{n+2})^V$-generic) is also part of G_1. This means that in W, $2^{\aleph_n} \geqslant \aleph_{n+2}$. On the other hand if $c(n) = 0$, there must be a bijection $f : \mathcal{P}(\aleph_n)^W \to \kappa$ in W for some cardinal κ. But then $f \in V[c][G_1]$ and since $\mathcal{P}(\aleph_n)^W \subseteq \mathcal{P}(\aleph_n))^{V[c][G_1]}$ it must be that $\kappa = \aleph_{n+1}$.

□

The dual to the Downwards Directed Grounds Hypothesis would be the Upwards Directed Extensions Hypothesis. It is not clear how one should formulate this on a first order basis, except for countable substructures. In any case, this hypothesis is false and thus we are unable to use the more refined understanding of extensions to tackle the *sDDG*.

Lemma 3.2.2. *[FHR15] It is consistent that there are two extensions that have no common extensions.*

Proof. Here, we have to take the perspective that V is a transitive countable model in a large background universe $\mathcal{V}$. Let $\alpha = Ord^V$. We produce two Cohen extensions of V. The main idea is that the two Cohen reals together uncover the countability of the ordinals if put togehter, so they cannot coexist in any model of ZFC with $Ord = \alpha$.
In $\mathcal{V}$, take a bijection $f : \omega \to \alpha$ and code it as $R \subseteq \omega \times \omega$ so that

$$\langle \omega, R\rangle \cong \langle tc(\{f\}), \in\rangle$$

and let $g : \omega \times \omega \to \omega$ be the Gödel pairing function. Let h be the characteristic function of $g[R]$. Any transitive model of ZFC that contains h can reconstruct f as it contains g and thus knows about the countability of α. Let $\langle D_n | n < \omega\rangle$ be an enumeration of all dense open subsets of Cohen forcing of V. We define Cohen reals c, d over V and sequences $\langle k_n | n < \omega\rangle, \langle l_n | n < \omega\rangle$ simultaneously by induction.
For formal correctness, put $l_{-1} = 0$. First let $c \upharpoonright k_0$ be any condition in D_0 with integer domain. Let $c(k_0) = h(0)$. If $c \upharpoonright (k_n + 1), d \upharpoonright l_{n-1}$ are already defined, we store k_n in d by letting $d \upharpoonright [l_{n-1}, l_{n-1} + k_n) = 0$ and $d(l_{n-1} + k_n) = 1$. Then extend to a condition in D_n with integer domain l_n. If $c \upharpoonright (k_n+1), d \upharpoonright l_n$ are defined, then store l_n in c by letting $c \upharpoonright [k_n+1, k_n+1+l_n) = 0$ and $c(k_n+1+l_n) = 1$. Next extend to a condition with integer domain k_{n+1} in D_{n+1} and let $c(k_{n+1}) = h(n+1)$.
Since $c \upharpoonright k_n, d \upharpoonright l_n \in D_n$, c and d are indeed Cohen reals over V. If both c and d are present in a transitive model of ZFC, one can reverse the above construction to recover the sequence $\langle k_n | n < \omega\rangle$. The first bit can be read off of d immediately, with this information one can find l_0 through c, which in turn yields k_1 as the length of the block of zeros in d starting at l_0, and

so on. But now $h = c \circ k$. Hence $V[c]$ and $V[d]$ can have no common extensions. $\square$

Remark 3.2.3. Having no common extension is just a different way of saying that one model is not a ground of an extension of the other. So the lemma above gives the example promised in Remark 2.4.5.

Above, we have seen that the naive strategy cannot work out. In some way, the *sDDG* suffers from a similar problem as the Goldbach conjecture. Given an even natural number $n > 2$, there tend to be *a lot* of pairs of primes whose sum is n. On the other hand, it seems like there is no best such pair, distinguished from all other pairs uniformly in n. In our case, there tend to be a lot of grounds contained in the intersection of two grounds W_0 and W_1. However, the canonical example for a good common ground would be the intersection. This obstacle leads to a nonconstructive proof of the *sDDG*. The final argument produces a common ground $W_\star$, but this procedure may lead to almost *any* common ground, so $W_\star$ is hardly distinguished from any other one. Notably, we will need a general way to show that $W_\star$ is a ground.

Definition 3.2.4. Suppose $M \subseteq N$ are classes and κ is a cardinal in N. Then $M \subseteq N$ has the κ-global cover property if for any α and any function $f : \alpha \to M$ in N such that $f(\beta) \subseteq M$ and $|f(\beta)|^N < \kappa$ for all $\beta < \alpha$, there is a function $F : \alpha \to M$ such that for all $\beta < \alpha$, $|F(\beta)|^M < \kappa$ and $f(\beta) \subseteq F(\beta)$.

Depending on taste, the κ-global cover property seems to be a bit misnamed. A better fitting name would be the κ-uniform cover property, as it just asserts that for set many instances of the κ-cover property, there always is a uniform way to cover. As it was originally defined with this name, we will keep it. In any case, this is the correct criterion.

Theorem 6.3.1. *(Bukovský's Theorem)[Buk73] Suppose W is an inner model of V and κ is a cardinal. Then W is a ground which extends to V via a κ-cc forcing if and only if $W \subseteq V$ has the κ-global cover property.*

In this chapter, we will only proof the easy direction.

Proof. "$\Rightarrow$" Suppose W extends to V via a κ-cc forcing $\mathbb{P}$. Suppose that $\dot{f}$ is a $\mathbb{P}$-name so that for some α, $\mathbb{1} \Vdash$ "$\dot{f} : \check{\alpha} \to V$" (notice that $\mathbb{1} \Vdash ran(\dot{f}) \subseteq V$ makes sense, as V is definable in the same manner in all its $\mathbb{P}$ extensions from the same parameter). For each $\beta < \alpha$, there is a maximal antichain A_α in the dense set of all $p \in \mathbb{P}$ that decide $\dot{f}(\check{\alpha})$. Finally

$$F(\beta) = \{x | \exists p \in A_\beta \; p \Vdash \dot{f}(\check{\beta}) = \check{x}\}$$

is as desired. $\square$

A proof of the hard and interesting direction can be found in the addendum.

If we have a set of grounds W_r for $r \in X$ present, we want to construct a common inner model that has the κ-global cover property relative to V fo some large regular κ. The construction will in some vague way resemble the proof of the Definability of Grounds Theorem, as we want to approximate this inner model from below. More precisely, we stratify the κ-global cover property and produce common inner models that come closer and closer to the full property in this sense. Only the initial segments of these approximations are good approximations, and in the end if we do this in the right way, enough of these initial segments will cohere to produce the desired common ground.

Definition 3.2.5. If $M \subseteq N$ are classes, κ is a cardinal in N, $\theta \geqslant \kappa$ some ordinal, then $M \subseteq N$ has the (κ, θ)-global cover property if for any $f : \lambda \to \mathcal{P}_\kappa(\lambda)^N$ in N, there is $F : \theta \to \mathcal{P}_\kappa(\theta)^M$ in M with $f(\alpha) \subseteq F(\alpha)$ for all $\alpha < \theta$.

Proposition 3.2.6. *Suppose W is an inner model of V. Then $W \subseteq V$ has the κ-global cover property if and only if it has the (κ, θ)-global cover property for all $\theta \geqslant \kappa$.*

Proof. The forward direction is clear so assume that $W \subseteq V$ has the (κ, θ)-global cover property for all $\theta \geqslant \kappa$. Let $f : \alpha \to V$ be a function in V so that $f(\beta) \subseteq M$ and $|f(\beta)|^V < \kappa$ for all $\beta < \alpha$. Find $\gamma \geqslant \kappa$ large enough so that $ran(f) \subseteq W_\gamma$ and put $\theta = |W_\gamma|^W$. Now take $g \in W$ a bijection between W_γ and θ. We can use g to translate $f(\beta)$ as the subset $g[f(\alpha)] \subseteq \theta$. The (κ, θ)-global cover property yields a function $F' : \theta \to \mathcal{P}_\kappa(\theta)$ that covers $g[f(\_)]$. Translating back using g^{-1} yields a κ-global cover $F \in W$ of f. □

Recall that we want to show that initial segments of certain approximations cohere. The way to go is Lemma 2.1.9, so we have to have the κ-approximation property present. Luckily, we get this for free.

Lemma 3.2.7. *If $W \subseteq V$ has the (κ, θ)-global cover property for regular κ and strong limit θ in V, then also the κ^+-cover and approximation properties hold for subsets of θ.*

Proof. First let us show that $W \subseteq V$ has the κ^+-cover property. Notice that this is not a completely trivial consequence as we have to account for sets of size κ. So let $x \in V$ be a subset of θ of size κ. Pick an enumeration $f : \kappa \to x$ and use the (κ, θ)-global cover property to find a function $F : \kappa \to \mathcal{P}_\kappa(\theta)^W$ in W such that for all $\alpha < \kappa$, $f(\alpha) \in F(\alpha)$. Then $y = \bigcup_{\alpha<\kappa} F(\alpha)$ is a κ^+-cover of x.

Next onto the κ^+-approximation property for subsets of θ: Suppose that $x \subseteq \theta$ is a set of size at most κ in V so that all κ^+ W-approximations of x

are in W. By induction we can assume that the κ^+-approximation property holds for all ordinals below θ, so that $x \cap \alpha \in W$ for all $\alpha < \theta$.

We build a tree $\mathbb{T} \in W$ of height θ and levels of size $< \kappa$ so that the characteristic function χ of x is a maximal branch. Since θ is a strong limit in both V and W, we have $|^{<\theta}2|^W = \theta$. Therefore we can use the (κ, θ)-global cover property to find a function $F : \theta \to \mathcal{P}_\kappa(^{<\theta}2)^W$ so that $\chi \upharpoonright \alpha \in F(\alpha)$ for all $\alpha < \theta$. Furthermore we can assume that $F(\alpha) \subseteq {}^\alpha 2$ for all $\alpha < \theta$ and that $t \upharpoonright \beta \in F(\beta)$ for all $t \in F(\alpha)$, $\beta < \alpha < \theta$, as we could otherwise throw out undesired elements of $F(\alpha)$. Now $\mathbb{T}$ is defined in the natural way as $\bigcup_{\alpha<\theta} F(\alpha)$, ordered by $\leqslant_{\mathbb{T}} = \subseteq$. Since $F(\alpha)$ is nonempty for every α, $\mathbb{T}$ has height θ. Clearly $\mathbb{T}_\alpha = F(\alpha)$ and therefore has size $< \kappa$. As promised, χ is (the union of) a cofinal branch of $\mathbb{T}$.

<u>**Case** $1, cof(\theta)^V > \kappa$:</u> In this case, Lemma 3.1.3 implies $\chi \in W$ and thus $x \in W$.

<u>**Case** $2, cof(\theta)^V \leqslant \kappa$:</u> Notice that $cof(\theta)^W \leqslant \kappa$ as well, since we can κ^+-cover a cofinal subset $C \in V$ of θ of size κ in W. Work in W. Find a cofinal subset $C_0 \subseteq \theta$ of size $\leqslant \kappa$. If C_n is defined let

$$C_{n+1} = C_n \cup \bigcup_{\alpha \in C_n} \{\Delta(t,s) | t \neq s \in \mathbb{T}_\alpha\}$$

Finally $C_\star = \bigcup_{n<\omega} C_n$. $C_\star$ has size $\leqslant \kappa$ and satisfies the following:

For $\alpha \in C_\star, t \neq s \in \mathbb{T}_\alpha$ there is $\beta \in C_\star, \beta < \alpha$ with $t \upharpoonright \beta \neq s \upharpoonright \beta$ $(*)$

Now $\chi \upharpoonright C_\star$ is a κ^+-approximation of χ in W and thus a member of W. Let $\langle \gamma_\alpha | \alpha < \xi \rangle$ be the increasing enumeration of $C_\star$. I claim that there is a unique function $h : \theta \to 2$ in W so that

(*i*) $h \upharpoonright \gamma_0 \in \mathbb{T}_{\gamma_0}$

(*ii*) $h \upharpoonright \gamma_{\alpha+1}$ is the unique node $t \in \mathbb{T}_{\gamma_{\alpha+1}}$ extending $h \upharpoonright \gamma_\alpha$ with $t(\gamma_\alpha) = \chi(\gamma_\alpha)$

(*iii*) for $\alpha \in Lim$, h_{γ_α} is the unique node in $\mathbb{T}_{\gamma_\alpha}$ that extends all $h \upharpoonright \gamma_\beta$ for $\beta < \alpha$

The closure property $(*)$ of $C_\star$ implies that $\mathbb{T}_{\gamma_0}$ contains exactly one node, which must be $\chi \upharpoonright \gamma_0$. If this is true up to and including α, then $(*)$ implies that every node $t \in \mathbb{T}_{\gamma_{\alpha+1}}$ extending $h \upharpoonright \gamma_\alpha$ is uniquely determined by $t(\gamma_\alpha)$, thus there is at most one valid continuation. By induction, $h \upharpoonright \gamma_\alpha = \chi \upharpoonright \gamma_\alpha$. Since $\chi \upharpoonright \gamma_{\alpha+1} \in \mathbb{T}_{\gamma_{\alpha+1}}$ there is such a node with $t(\gamma_\alpha) = \chi(\gamma_\alpha)$. If α is a limit, then again by $(*)$ there can be at most one $t \in \mathbb{T}_{\gamma_\alpha}$ extending $\bigcup_{\beta<\alpha} h \upharpoonright \gamma_\beta$ and as before, $t = \chi \upharpoonright \gamma_\alpha$ does the trick. This shows that $h = \chi$ is definable in W from $\chi \upharpoonright C_\star$ and $\mathbb{T}$ and hence $x \in W$.

□

In the following proposition and subsequent proof, κ^+ will always refer to $(\kappa^+)^V$ even if that is not $(\kappa^+)^W$.

Proposition 3.2.8. *Suppose κ^+ and $\theta > \kappa$ are cardinals and W is an inner model of ZFC. Assume that that for any $f : \theta \to \theta$ in V there is a map $F : \theta \to \mathcal{P}_{\kappa^+}(\theta)$ in W with $f(\alpha) \in F(\alpha)$ and $|F(\alpha)|^W \leqslant \kappa$ for all $\alpha < \theta$. Then $W \subseteq V$ has the (κ^+, θ)-global cover property.*

Proof. Suppose $g : \theta \to \mathcal{P}_{\kappa^+}(\theta)$ is a function in V. Since $\kappa \cdot \theta = \theta$, we can split θ into θ-many blocks $(B_\alpha)_{\alpha<\theta} \in W$ of size κ and in V define a function $f : \theta \to \theta$ with $g(\alpha) \subseteq f[B_\alpha]$ for all $\alpha < \theta$. Let $F \in W$ be a cover of f as in our assumption. In W, we can define $G(\alpha) = \bigcup_{\beta \in B_\alpha} F(\beta)$ for $\alpha < \theta$. Then G is a (κ^+, θ)-global over of g. □

Next up is the key construction.

Lemma 3.2.9. *Assume W_r, $r \in X$ is a collection of grounds that extend to V via $\mathbb{P}_r$. Let κ be a regular cardinal larger then the size of X and $\mathbb{P}_r$ for all $r \in X$. Then for all cardinals $\theta \geqslant \kappa$, there is a set of ordinals A so that $L[A] \subseteq \bigcap_{r\in X} W_r$ and $L[A] \subseteq V$ has the (κ^+, θ)-global cover property.*

Proof. Notice that all $W_r \subseteq V$ have the full κ-global cover property by the easy direction of Bukovský's Theorem. Let θ be a cardinal $\geqslant \kappa$. We want to code (κ, θ)-global covers for all $f : \theta \to \mathcal{P}_\kappa(\theta)$ in a set of ordinals A. On the other hand, we have to achieve $L[A] \subseteq \bigcap_{r\in X} W_r$, which is equivalent to $A \in \bigcap_{r\in X} W_r$. Since A codes covers, we must find covers that are contained in all W_r simultaneously. This is the point where me make the jump from κ to κ^+. Let $\lambda = |\mathcal{P}_\kappa(\theta)|^\theta = 2^\theta$.

Claim 3.2.10. *Let $f : \lambda \to \mathcal{P}_\kappa(\lambda)$ be a function in V. Then there is a (κ^+, λ)-global cover $F \in \bigcap_{r\in X} W_r$ of f.*

Proof. We construct (κ, λ)-covers $(F^r_\alpha)^{r\in X}_{\alpha<\kappa}$ by induction on $\alpha < \kappa$ so that

(*i*) $F^r_\alpha \in W_r$

(*ii*) F^r_0 covers f for all $r \in X$

(*iii*) $\forall \gamma < \lambda \; \bigcup_{s\in X, \beta<\alpha} F^s_\beta(\gamma) \subseteq F^r_\alpha(\gamma)$

F^r_0 is given by (*ii*). If F^s_β is constructed for all $\beta < \alpha$ and $s \in X$, then $F^r_\alpha \in W_r$ is a (κ, λ)-global cover of $\gamma \longmapsto f_\alpha(\gamma) = \bigcup_{s\in X, \beta<\alpha} F^s_\beta(\gamma)$. Since $\kappa \geqslant |X|$ is regular in V, we have that $ran(f_\alpha) \subseteq \mathcal{P}_\kappa(\lambda)$, so this is fine.
Take any $r \in X$ and let $F : \lambda \to \mathcal{P}_{\kappa^+}(\lambda)$, $F(\gamma) = \bigcup_{\alpha<\kappa} F^r_\alpha(\gamma)$ (this definition is in fact independent of $r \in X$ as $F^s_\alpha(\gamma) \subseteq F^r_{\alpha+1}$ for all $s, r \in X$ and $\alpha < \kappa$). F is certainly a (κ^+, λ)-cover of f by (*ii*), so it is left to show that $F \in W_r$.

This is not immediate as we do not know whether $\langle F^r_\alpha | \alpha < \kappa \rangle \in W_r$. On the other hand, we know that $W_r \subseteq V$ has the κ-approximation property as $|\mathbb{P}_r| < \kappa$, which we will make use of. It is enough to show that $\bar{F} = \bigcup_{\gamma<\lambda}\{\gamma\} \times F(\gamma)$ is in W_r. Clearly $\bar{F} \subseteq W_r$. So suppose $a \cap \bar{F}$ is a κ-approximation of $\bar{F}$ in W_r. Then $dom(a \cap \bar{F})$ is a set of size $< \kappa$. In the same manner, the set $(a \cap \bar{F})_\gamma = \{\beta < \lambda | (\gamma, \beta) \in a \cap \bar{F}\}$ has size $< \kappa$ for all $\gamma \in dom(a \cap \bar{F})$. Define $\bar{F}^r_\alpha$ from F^r_α as we have $\bar{F}$ from F. Given $\gamma \in dom(a \cap \bar{F})$, the regularity of κ implies that $(a \cap \bar{F})_\gamma = (a \cap \bar{F}^r_{\alpha_\gamma})_\gamma$ for some $\alpha_\gamma < \kappa$. Again by regularity of κ, $\alpha_\star = sup\{\alpha_\gamma | \gamma \in dom(a \cap \bar{F})\} < \kappa$ and thus $a \cap \bar{F} = a \cap \bar{F}^r_{\alpha_\star} \in W_r$. This shows $\bar{F} \in W_r$. □

We just have to pick the right f. We need one that encapsulates all instances of (κ, θ)-global cover. So define $f : \lambda \to \mathcal{P}_\kappa(\theta)$ so that every $g : \theta \to \mathcal{P}_\kappa(\theta)$ appears as a block in f. This is possible if λ is large enough. Using the above claim, find a cover $F : \lambda \to \mathcal{P}_{\kappa^+}(\lambda)$ for f that works for all W_r simultaneously. We can assume that $ran(F) \subseteq \mathcal{P}_{\kappa^+}(\theta)$. Code F in the usual way as a subset A of λ. This implies $A \in \bigcap_{r\in X} W_r$. It is left to show that $L[A] \subseteq V$ has the (κ^+, θ)-global cover property. As $A \in L[A]$, we can recover F. Any instance $g : \theta \to \mathcal{P}_\kappa(\theta)$ we have to check appears as a block in f and thus the corresponding block in F a (κ^+, θ)-global cover of g in $L[A]$. It seems troublesome that clearly not all maps $h : \theta \to \mathcal{P}_{\kappa^+}(\lambda)$ appear in f, however Proposition 3.2.8 gives that we have more than enough to conclude that $L[A] \subseteq V$ satisfies the full (κ^+, θ)-global cover property. □

We have acquired all tools we need to proof the strong Downwards Directed Grounds Hypothesis.

Proof. (Theorem 2.4.3) Suppose W_r, $r \in X$ is a collection of grounds that extend to V via $\mathbb{P}_r$ respectively. Let κ be a regular cardinal such that $\kappa \geqslant |X|, |\mathbb{P}_r|$ for all $r \in X$. For any $\beth$-fixed point θ with $cof(\theta) > \kappa^{++}$, there is a set $A \subseteq 2^\theta$ such that $L[A_\theta] \subseteq \bigcap_{r\in X} W_r$ and $L[A_\theta] \subseteq V$ has the (κ^+, θ)-global cover property. Notice that this implies that $L[A_\theta]_\theta \subseteq V_\theta$ has the κ^+-global cover property. By Lemma 3.2.7, $L[A_\theta] \subseteq V$ has the κ^{++}-cover and approximation properties for subsets of θ. This shows that $L[A_\theta]_\theta \subseteq V_\theta$ has the κ^{++}-cover and approximation properties. Moreover, $L[A_\theta] \subseteq V$ has the κ^{+++}-cover property for subsets of θ, too. Hence $(\kappa^{++})^+$ is the same in $L[A_\theta]$ and V. By Proposition 2.1.18 and Proposition 2.1.11, $L[A_\theta]_\theta$ and V_θ are models of ZFC_δ. Lemma 2.1.9 shows that $L[A_\theta]_\theta$ is the unique $({}^{<\kappa^{++}}2)^{L[A_\theta]}$-substructure of V_θ. We can conclude:

$V \models$ "for all $\beth$-fixed points of cofinality $> \kappa$ there is $r \subseteq {}^{<\kappa^{++}}2$ such that the unique r-substructure of V_θ exists, is a subset of $\bigcap_{r\in X} W_r$, can compute its Von-Neumann-hierachy and has the κ^+-global cover property relative to V_θ"

Here, we understand the ability of a structure M to compute the Von-Neumann-hierachy as containing the set $M \cap V_\alpha$ for all $\alpha \in M$. As usual, we abbreviate this as M_α. By the pigeonhole principle, there is some $r_\star$ that works for a proper class C of these θ. Let $W_\star$ be the union of the corresponding $r_\star$-substructures $W_\star^\theta$ of V_θ.

Claim 3.2.11. *$W_\star$ satisfies the following:*

(*i*) $W_\star \subseteq \bigcap_{r \in X} W_r$

(*ii*) $(W_\star^\theta)_{\theta \in C}$ *cohere: If $\theta < \theta'$ are both in C then $(W_\star^{\theta'})_\theta = W_\star^\theta$*

(*iii*) *$W_\star$ is an inner model of ZFC*

(*iv*) *$W_\star \subseteq V$ has the κ^+-global cover property.*

Proof. (*i*) By construction, $W_\star^\theta \subseteq \bigcap_{r \in X}$ holds for all $\theta \in C$.

(*ii*) In this case $(W_\star^{\theta'})_\theta \subseteq V_\theta$ is another $r_\star$-substructure of V_θ. By Lemma 2.1.9, the assertion follows.

(*iii*) We apply the inner model criterion (6.2.4). $W_\star$ is closed under all Gödel operations, as each $W_\star^\theta$ is on its own. We have to show that $W_\star \subseteq V$ has the Ord-cover property. Suppose $x \in V$ is a subset of $W_\star$. Let $\alpha = rk(x)$. If $\theta \in C$ is larger than α, then the coherence (*ii*) implies that already $x \subseteq W_\star^\theta$. In particular $x \subseteq (W_\star^\theta)_\alpha \in W_\star^\theta \subseteq W_\star$. By Theorem 6.2.4, $W_\star$ is a model of ZF. The axiom of choice holds in $W_\star$ since it holds in each $W_\star^\theta$ respectively.

(*iv*) For every $\theta \in C$ we have that $W_\star^\theta \subseteq V_\theta$ has the κ^+-global cover property. Thus $W_\star = \bigcup_{\theta \in C} W_\star^\theta \subseteq \bigcup_{\theta \in C} V_\theta = V$ has the κ^+-global cover property.

□

By Bukovskýs Theorem (6.3.1), (*iii*) and (*iv*) imply that $W_\star$ is a ground of W. By (*i*), $W_\star$ is as desired. □

4 Large Cardinals in the Mantle

In this chapter, we investigate the relationship between large cardinals, the mantle and, the generic multiverse.

4.1 Preliminary Considerations

We say that $\varphi(x)$ is a large cardinal axiom if $\varphi(\kappa)$ can only hold for uncountable cardinals. First of all, one can observe that no large cardinal axiom is necessarily (upwards) absolute between the mantle $\mathbb{M}$ and V. Any cardinal κ is countable in some generic extension $V[G]$ and $V[G]$ has the same mantle as V by Corollary 2.4.4 (*ii*). Nonetheless, downwards absoluteness from V to $\mathbb{M}$ is still a reasonable property and in fact we will see an example of nontrivial downwards absoluteness. Regarding the other direction, we will replace the notion of upwards absoluteness with a criterion that is independent of the basepoint V in the generic multiverse. To be precise we will ask the question, whether or not κ having a certain property φ in $\mathbb{M}$ implies that the same is true in dense many grounds, i.e. for every ground W there is a deeper ground $W' \subseteq W$ in which $\varphi(\kappa)$ holds. Observe that this is first order definable by the Definability of Grounds Theorem (2.1.1).

Proposition 4.1.1. *Given a set-theoretic formula φ with parameters in the mantle, the evaluation of the statement "φ holds in dense many grounds" is constant across the generic multiverse.*

Proof. Suppose W is a ground of V. First assume that $V \models$ "φ holds in dense many grounds". Notice that any ground of W is a ground of V, too. Thus the statement is true in W. Now suppose that $W \models$ "φ holds in dense many grounds". Let M be any ground of V. By the *sDDG*, there is a deeper ground $W' \subseteq M, W$. Our assumption on W gives that there is $W'' \subseteq W'$ that satisfies φ. But then, from the perspective of V, this is a deeper ground than M. □

Let us start our analysis with results due to Usuba that show that the existence of very large cardinals has a huge impact on the Set-Theoretic Geology of V. In the end, this will yield the promised nontrivial downwards absoluteness of some very large cardinal.

4.2 Extendibles

Definition 4.2.1. A cardinal κ is extendible iff for every λ there is an elementary embedding $j : V_\lambda \to V_\xi$ for some ξ with $crit(j) = \kappa$ and $\lambda < j(\kappa)$.

The main result of this section is that the existence of an extendible implies the bedrock axiom.

Theorem 4.2.2. *(Usuba)[Usu18] If there is an extendible cardinal κ, then the mantle is a ground.*

Our strategy is to show that the mantle is the intersection of all grounds which are a κ-small forcing away.

Definition 4.2.3. The κ-mantle $\mathbb{M} \upharpoonright \kappa$ is the intersection of all κ-grounds.

Showing $\mathbb{M} = \mathbb{M} \upharpoonright \kappa$ is enough.

Proposition 4.2.4. *[Usu18] For any κ there is a ground W contained in $\mathbb{M} \upharpoonright \kappa$. In particular, if $\mathbb{M} = \mathbb{M} \upharpoonright \kappa$ then the mantle is a ground.*

Proof. Let $X = \mathcal{P}({}^{<\kappa}2)$. By the *sDDG*, there is a ground W contained in $\bigcap_{r \in X} W_r$. If N is a $< \kappa$-ground then $N \subseteq V$ has the κ-approximation and cover properties, thus there is some $r \in X$ with $N = W_r$. □

Let us assume there is an extendible cardinal κ. First of all, this implies that there are class many (strongly) inaccessibles. This is because (strong) inaccessibility of a cardinal λ is decided in $V_{\lambda+1}$ and the extendibility of κ implies the existence of elementary embeddings $j : V_{\kappa+1} \to V_{j(\kappa)+1}$ with arbitrarily large target $j(\kappa)$. Since κ is (strongly) inaccessible, every such $j(\kappa)$ must be (strongly) inaccessible. We will use that almost all (strongly) inaccessible cardinals above κ compute the κ-mantle correctly:

Lemma 4.2.5. *Suppose κ is a cardinal.*

(*i*) *Let W be a κ-ground. If $\lambda > \kappa$ is strongly inaccessible then W_λ is a κ-ground of V_λ and thus $(\mathbb{M} \upharpoonright \kappa)^{V_\lambda} \subseteq (\mathbb{M} \upharpoonright \kappa)_\lambda$.*

(*ii*) *The above inclusion can only be strict in set-many cases, i.e. there is an $\alpha \geqslant \kappa$ such that for all $\lambda > \kappa$ strongly inaccessible, we have $(\mathbb{M} \upharpoonright \kappa)^{V_\lambda} = (\mathbb{M} \upharpoonright \kappa)_\lambda$.*

Proof. (*i*) Let $\mathbb{P} \in W$ be of size $< \kappa$ and let G be $\mathbb{P}$-generic over W such that $W[G] = V$. There is a forcing isomorphic to $\mathbb{P}$ in W_κ (for example take a bijection from $\mathbb{P}$ to its cardinality and consider the induced forcing), so without loss of generality $\mathbb{P}$ has this property. Exactly as in Claim 2.5.3, we can show that $W[G]_\lambda = W_\lambda[G]$ using the strong inaccessibility of λ.

(*ii*) Assume for a contradiction that the class C of strong inaccessibles $\lambda > \kappa$ with $(\mathbb{M} \upharpoonright \kappa)^{V_\lambda} \neq (\mathbb{M} \upharpoonright \kappa)_\lambda$ is unbounded. Lemma 2.1.9 shows that for any $\lambda \in C$ there is some $r \subseteq {}^{<\kappa}2$ and $\mathbb{P}, G \in V_\kappa$ such that the unique r-substructure of V_λ exists, is in fact a ground of V_λ, extends to V_λ via the $\mathbb{P}$-generic filter G and is a proper subset of $(\mathbb{M} \upharpoonright \kappa)_\lambda$. There are only set-many possibilities for such a triple, but class-many λ, hence there is one such triple $(r, \mathbb{P}, G)$ that works for class many λ.

Let W^λ be the unique r-substructure of V_λ for $\lambda \in C$ and $W_\star$ be the union over these. Exactly as in Claim 3.2.11 (ii) and (iii), we see that $W_\star$ is an inner model of ZFC and that $(W_\star)_\lambda = W^\lambda$ for $\lambda \in C$. From $W^\lambda[G] = V_\lambda$ for $\lambda \in C$ we can conclude that $W_\star[G] = V$. But then $W_\star$ is a κ-ground of V that is not contained in $\mathbb{M} \upharpoonright \kappa$, a contradiction. □

We need a tool to characterize the pointwise image $j[\lambda]$ for a given elementary embedding j. This argument is implicit in [Usu17], but apparently goes back to Solovay.

Proposition 4.2.6. *Suppose $N \subseteq M$ are transitive models of ZFC and λ is of uncountable cofinality in M. Assume $\lambda < \xi$ and $j : M_\xi \to N_{\bar{\xi}}$ is an elementary embedding such that $j[\lambda] \in M$. Let $\vec{S} = \langle S_\alpha | \alpha < \lambda \rangle \in M_\xi$ be a sequence of disjoint stationary subsets of E^λ_ω from the perspective of M. Let $\delta = sup\ j[\lambda]$. Then we have that*

$$j[\lambda] = \{\alpha < \delta | \tilde{S}_\alpha \cap \delta \text{ is stationary in } \delta\}^M$$

where $j(\vec{S}) = \langle \tilde{S}_\alpha | \alpha < j(\lambda) \rangle$.

The general idea here is that in fact $\alpha \in j[\lambda]$ iff $\tilde{S}_\alpha$ meets $j[\lambda]$ and that in this context $j[\lambda]$ has enough properties of a club so that "meeting $j[\lambda]$" can be replaced with stationarity.

Proof. Work in M. For the sake of this proof we will call a $D \subseteq \delta$ an ω-club if it is unbounded and contains all its limit points in E^δ_ω. Notice that the standard argument for proving that the intersection of two clubs is a club shows the same statement for ω-clubs (always given that the underlying ordinal has uncountable cofinality). $j[\lambda]$ is an ω-club in δ: Clearly it is unbounded and if $\vec{\gamma} = \langle \gamma_n | n < \omega \rangle \in M$ is an increasing sequence in λ then, since $j(\omega) = \omega$, we have $j(\vec{\gamma}) = \langle j(\gamma_n) | n < \omega \rangle$ and thus by elementarity $j(sup_{n<\omega}\ \gamma_n) = sup_{n<\omega}\ j(\gamma_n)$.
Let us show that $\tilde{S}_{j(\alpha)} \cap \delta$ is stationary in δ in M. Now if $D \in M$ is a club in δ, then $j[\lambda] \cap D$ is an ω-club in M. Notice that $j[\lambda] \in M$ implies that $j \upharpoonright \lambda$, and so also its inverse, is a member of M since this is just the monotone enumeration. As $j \upharpoonright \lambda : \lambda \to \delta$ is increasing, has unbounded range and, as we have seen, is continuous at limit ordinals of cofinality ω, $C = j^{-1}[j[\lambda] \cap D]$ is an ω-club in λ in M. Now S_α meets the club $lim(C)$, but as S_α only consists of ordinals of cofinality ω, $S_\alpha \cap C = S_\alpha \cap lim(C) \neq \emptyset$. Now if $\beta \in S_\alpha \cap C$ then of course $j(\beta) \in j(S_\alpha) \cap j[C] \subseteq \tilde{S}_{j(\alpha)} \cap D$.
Now let us assume that $\tilde{S}_\alpha \cap \delta$ is stationary. By elementarity, $\tilde{S}_\alpha$ consists only of ordinals of cofinality ω. As above this implies that $\tilde{S}_\alpha$ meets $j[\lambda]$, so find some β with $j(\beta) \in \tilde{S}_\alpha$. By elementarity, there is some $\bar{\alpha}$ with $\beta \in S_{\bar{\alpha}}$. Now since $\vec{S}$ is a sequence of disjoint sets

$$M_\xi \models \text{"}\bar{\alpha} \text{ is the unique } \gamma \text{ with } \beta \in S_\gamma\text{"}$$

and thus by elementarity

$$N_{\bar{\xi}} \models \text{“}j(\bar{\alpha}) \text{ is the unique } \gamma \text{ with } j(\beta) \in \tilde{S}_\gamma\text{”}$$

which implies $\alpha = j(\bar{\alpha}) \in j[\lambda]$. □

Remark 4.2.7. We made a slight mistake above. If $\xi = \lambda + 1$ then $\vec{S} \notin dom(j)$ as this sequence has rank $\lambda + 3$. However, we can code $\vec{S}$ as a subset of λ that *is* in $dom(j)$ and reformulate statements about $\vec{S}$ as statements about its code.

Proof. (Theorem 4.2.2) Let W be any ground. We have to show that $\mathbb{M} \upharpoonright \kappa \subseteq W$. It is enough to show $(\mathbb{M} \upharpoonright \kappa)_\lambda \subseteq W$ for all large enough inaccessible λ. Let $\lambda > \kappa$ be strongly inaccessible such that W is a λ-ground. The crucial idea is that W might not be a κ-ground, but using the extendibility of κ we can make W a $j(\kappa)$-ground. Find $\theta > \lambda$ strongly inaccessible such that V_θ computes the κ-mantle correctly according to Lemma 4.2.5 (*ii*). We can also assume that the same holds for all larger strongly inaccessibles. Now let $j : V_{\theta+1} \to V_{j(\theta)+1}$ be an elementary embedding with $crit(j) = \kappa$ and $j(\kappa) > \lambda$. Notice that $j(\theta)$ is inaccessible, too. We have that

$$V_{\theta+1} \models \forall x\ x \in (\mathbb{M} \upharpoonright \kappa)^{V_\theta} \leftrightarrow (\text{“}x \text{ is in all } \kappa\text{-grounds”})^{V_\theta}$$

and by elementarity

$$V_{j(\theta)+1} \models \forall x\ x \in j((\mathbb{M} \upharpoonright \kappa)^{V_\theta}) \leftrightarrow (\text{“}x \text{ is in all } j(\kappa)\text{-grounds”})^{V_{j(\theta)}}$$

and hence

$$j((\mathbb{M} \upharpoonright \kappa)_\theta) = j((\mathbb{M} \upharpoonright \kappa)^{V_\theta}) = (\mathbb{M} \upharpoonright j(\kappa))^{V_{j(\theta)}} = (\mathbb{M} \upharpoonright j(\kappa))_{j(\theta)} \subseteq W_{j(\theta)} \quad (*)$$

where the first and last equality holds by our assumption on θ and the inclusion holds since $W_{j(\theta)}$ is a $\lambda < j(\kappa)$-ground of $V_{j(\lambda)}$ by Lemma 4.2.5 (*i*). For the final argument, we will need that $j[\lambda] \in W$. To see this, we will apply Proposition 4.2.6. So let $\vec{S} = \langle S_\alpha | \alpha < \lambda \rangle \in W$ be a sequence of pairwise disjoint stationary subsets of $(E^\lambda_\omega)^W$ from the perspective of W. Now V is an extension of W by a forcing of size $< \lambda$. Lemma 1.5.4 shows that stationarity in λ is absolute between V and W. In particular in V, $\vec{S}$ is still a sequence of pairwise disjoint stationary subsets of λ which only contain ordinals of cofinality ω. Let $j(\vec{S}) = \langle \tilde{S}_\alpha | \alpha < j(\lambda) \rangle$. We have:

$$j[\lambda] = \{\alpha < \delta | \tilde{S}_\alpha \cap \delta \text{ is stationary in } \delta\}^V$$

If $j(\vec{S}) \in W$, then by absoluteness of stationarity above λ between V and W, we could define $j[\lambda]$ inside W. Indeed, we have $j(\vec{S}) \in j(W_\theta) \subseteq j((\mathbb{M} \upharpoonright \kappa)^{V_\theta}) \subseteq W_{j(\theta)}$ by $(*)$. Hence $j[\lambda] \in W$ which is what we wanted.
We will now show that $(\mathbb{M} \upharpoonright \kappa)_\alpha \subseteq W$ for $\alpha \leqslant \lambda$ by induction on α.

$\underline{\alpha = \kappa}$**:** $(\mathbb{M} \upharpoonright \kappa)_\kappa = ((\mathbb{M} \upharpoonright \kappa)^{V_\theta})_\kappa = j((\mathbb{M} \upharpoonright \kappa)^{V_\theta})_\kappa \subseteq W_\kappa$ where the second equality holds since j has critical point κ and the last inclusion holds by $(*)$.

$\underline{\alpha \rightsquigarrow \alpha + 1}$**:** Assume $(\mathbb{M} \upharpoonright \kappa)_\alpha \subseteq W_\alpha$. Let $X \subseteq (\mathbb{M} \upharpoonright \kappa)_\alpha$. We have to show $X \in W$. The idea is to code the set X and use $(*)$ and $j[\lambda] \in W$. First find a bijection $f : \rho \to (\mathbb{M} \upharpoonright \kappa)_\alpha$ in $W \subseteq \mathbb{M} \upharpoonright \kappa$. By $(*)$, $j(f) \in W$. $j[\rho]$ is an initial segment of $j[\lambda] \in W$ as $\rho < \lambda$ by strong inaccessibility of λ and thus $j[\rho] \in W$. With this we can conclude

$$j[(\mathbb{M} \upharpoonright \kappa)_\alpha] = j(f)[j[\rho]] \in W$$

and hence:

$$j[X] = j(X) \cap j(f)[j[(\mathbb{M} \upharpoonright \kappa)_\alpha]] \in W$$

Now $j^{-1} \upharpoonright j[(\mathbb{M} \upharpoonright \kappa)_\alpha]$ is just the Mostowski collapse and hence in W. Thus $X = j^{-1}[j[X]] \in W$ which we wanted to show.

$\underline{\alpha \in \mathit{Lim}}$**:** This case is clear by continuity of the Von-Neumann-hierarchy.

□

4.3 Hyper-Huge Cardinals

Usuba has given a strengthening of the notion of a 1-superhuge cardinal from which one can conclude even more than from an extendible. All the following large cardinal axioms can be defined in first order logic via demanding the existence of certain ultrafilters (for huge cardinals consult [Kan09], for hyperhuges [Usu17]).

Definition 4.3.1. Let κ be a cardinal.

(*i*) κ is called n-huge for $n < \omega$ if there is some inner model M and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and so that M is closed under $j^n(\kappa)$-sequences.

(*ii*) κ is called n-superhuge if additionally such an embedding can be found with $j(\kappa) > \lambda$ for arbitrarily large λ.

(*iii*) κ is hyper-huge if for every λ there is a nontrivial elementary embedding $j : V \to M$ into some transitive inner model M such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{j(\lambda)}M \subseteq M$.

Remark 4.3.2. Notice that all hyper-huge cardinals are extendible. If $\lambda > \kappa$ is a $\beth$-fixed point and $j : V \to M$ is an elementary embedding with $crit(j) = \kappa$ and ${}^{j(\lambda)}M \subseteq M$ then $V_\lambda \subseteq M$ so that j restricts to an elementary embedding $V_\lambda \to V_{j(\lambda)}$ with critical point κ. In particular, there are unboundedly many strongly inaccessible cardinals above a hyper-huge cardinal.

Corollary 4.3.3. *The existence of a hyper-huge cardinal implies the bedrock axiom.*

More interestingly, the reflective properties of a hyper-huge cardinal are powerful enough to, in a sense, pull down the statement "$j(W)$ is a $j(\kappa)$-ground" to conclude that W is a κ-ground. Of course we will make that precise.

Theorem 4.3.4. *(Usuba)[Usu17] If there is a hyper-huge cardinal κ, then every ground is a κ-ground.*

Remark 4.3.5. Usuba's result regarding extendible cardinals is more recent than the above one and appeared during the time this thesis was written. By that time, this thesis was already focused on hyper-huge cardinals and thus we stay true to that and even present (a modification of) the original proof. The above theorem follows much easier (compared to the proof below) and for smaller large cardinals with a lot less consistency strength than a hyper-huge: It can be shown that a hyper-huge is a limit of extendible cardinals. Suppose κ is the second extendible cardinal (or any large enough cardinal above the first one). Now if λ is the least extendible, then $\mathbb{M} = \mathbb{M} \upharpoonright \lambda$. Since there can be at most λ-many λ-grounds, the proof of the *sDDG* shows that $\mathbb{M} \subseteq V$ has the λ^+-global covering property. The proof of Bukovský's Theorem shows that $\mathbb{M}$ extends to V via a forcing of size well below the next inaccessible above λ. In particular, $\mathbb{M}$ is a κ-ground. The "hyper-huge" in any result of this section can be safely replaced by, for example, 1-superhuge.

Proof. Let W be any ground and find a forcing $\mathbb{P} \in W$ and a $\mathbb{P}$-generic filter G over W such that $W[G] = V$. Pick a strongly inaccessible cardinal $\lambda > \kappa$ large enough so that $\mathbb{P} \in V_\lambda$. Using the hyper-hugeness of κ, we can find some inner model M and a nontrivial elementary embedding $j : V \to M$ with $crit(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{j(\lambda)}M \subseteq M$. Let $j(W) = \bigcup_{\alpha \in Ord} j(W_\alpha)$. Using the definability of grounds, we can find r so that $W = W_r^V$.

Claim 4.3.6. $j(W) = W^M_{j(r)}$.

Proof. We show that $j(W) = \bigcup j[W] = \bigcup j[W_r^V] = W^M_{j(r)}$: If $x \in j(W)$ then we can find some α such that $x \in j(W)_{j(\alpha)} = j(W_\alpha) \in j[W]$. In the same way, if $x \in W^M_{j(r)}$ then there is α with $x \in M_{j(\alpha)} \cap W^M_{j(r)} = j(V_\alpha \cap W_r^V) \in j[W_r^V]$. □

In particular, $j(W)$ is a model of *ZFC* as it satisfies the inner model criterion inside M and the axiom of choice.

Claim 4.3.7. $W_{j(\lambda)} \subseteq j(W_\lambda) \subseteq M_{j(\lambda)} = V_{j(\lambda)}$

Proof. We show one inclusion at a time.

$\underline{W_{j(\lambda)} \subseteq j(W_\lambda)}$**:** Since λ is strongly inaccessible in V, so is $j(\lambda)$ in M. Now M is closed under $j(\lambda)$-sequences and thus $j(\lambda)$ is (strongly) inaccessible in V, too. This shows that $W_{j(\lambda)}$ is a model of ZFC and so is $j(W)_{j(\lambda)} = j(W_\lambda)$. By coding sets as sets of ordinals, it is enough to show that any set of ordinals $x \in W_{j(\lambda)}$ is a member of $j(W_\lambda)$. Note that x is a bounded subset of $j(\lambda)$. Let $\vec{S} = \langle S_\alpha | \alpha < j(\lambda)\rangle \in W$ be a sequence of disjoint stationary subsets of $E_\omega^{j(\lambda)}$ as computed in W. Let $\delta = sup\ j[j(\lambda)]$. Proposition 4.2.6 yields

$$j[j(\lambda)] = \{\alpha < \delta | \tilde{S}_\alpha \cap \delta \text{ is stationary in } \delta\}^V$$

where $j(\vec{S}) = \langle \tilde{S}_\alpha | \alpha < j(\lambda)\rangle$. I claim that

$$\begin{aligned}&\{\alpha < \delta | \tilde{S}_\alpha \cap \delta \text{ is stationary in } \delta\}^V \\ =&\{\alpha < \delta | \tilde{S}_\alpha \cap \delta \text{ is stationary in } \delta\}^{j(W)}\end{aligned}$$

As $j(\vec{S}) \in j(W)$ it is enough to show that stationarity in δ is absolute between V and $j(W)$. As M is closed under $j(\lambda)$-sequences and $cof(\delta) = j(\lambda)$, this is absolute between V and M. We see by elementarity that

$$M \models \text{“}j(W) = W^M_{j(r)} \text{ is a } j(\lambda)\text{-ground”}$$

and hence stationarity in δ is absolute between M and $j(W)$ by Lemma 1.5.4.

This shows that $j[j(\lambda)] \in j(W)$. If x is a set of ordinals in $W_{j(\lambda)}$ then $x \subseteq W_{j(\lambda)} \cap Ord = j(\lambda)$ and $j[x] = j(x) \cap j[j(\lambda)] \in j(W)$. Now $j^{-1} \upharpoonright j[j(\lambda)]$ is the Mostowski collapse of $j[j(\lambda)]$ and thus in $j(W)$. Thus $x = j^{-1}[j[x]] \in j(W)$ and as $rk(x) < j(\lambda)$, $x \in j(W)_{j(\lambda)} = j(W_\lambda)$.

$\underline{j(W_\lambda) \subseteq M_{j(\lambda)}}$**:** This inclusion holds since $j(W_\lambda) \subseteq j(V_\lambda) = M_{j(\lambda)}$.

$\underline{M_{j(\lambda)} = V_{j(\lambda)}}$**:** M is closed under all $j(\lambda)$-sequences. $j(\lambda)$ is strongly inaccessible so in particular a $\beth$-fixed point. Thus $V_{j(\lambda)} \subseteq M$ follows directly by induction.

□

Lemma 4.2.5 (i) shows that $W_{j(\lambda)}$ is a λ-ground of $V_{j(\lambda)}$. Using the quotient lemma (Corollary 6.3.10), we can conclude that $j(W_\lambda)$ is a $\lambda < j(\kappa)$-ground of $j(V_\lambda) = M_{j(\lambda)}$. By elementarity, W_λ is a κ-ground of V_λ. Thus there is a forcing $\mathbb{Q}$ of size $< \kappa$ in W_λ and H a $\mathbb{Q}$-generic filter over W_λ with $W_\lambda[H] = V_\lambda$. It is now enough to show that $W[H] = V$. As the subsets of $\mathbb{Q}$ are the same in W_λ and W, H is $\mathbb{Q}$-generic over W. Moreover, $G \in V_\lambda = W_\lambda[H] \subseteq W[H]$. This shows $V = W[G] \subseteq W[H] \subseteq V$ and hence $W[H] = V$.

□

If we combine this with the following result, we see that hyper-huge cardinals are downwards absolute to the mantle.

Proposition 4.3.8. *If κ is a cardinal and W a κ-ground of V then κ is hyper-huge if and only if it is in W.*

In the "downwards" direction, we will make use of the following result of Hamkins:

Fact 4.3.9. *[Ham03, Corollary 6] Suppose δ is a regular cardinal and W is an inner model of ZFC so that $W \subseteq V$ has the δ-approximation and cover properties. If M is an inner model of V with ${}^{\delta}M \subseteq M$ and $j : V \to M$ an elementary embedding with $\delta < crit(j)$ then $j[W] \subseteq W$, $M \cap W$ is an inner model of W and*

$$j \restriction W : W \to M \cap W$$

is an elementary embedding.

Proof. (Proposition 4.3.8) First suppose κ is hyper-huge in V. Let $\lambda > \kappa$ and $j : V \to M$ an elementary embedding with critical point κ, $j(\kappa) > \lambda$ and ${}^{j(\lambda)}M \subseteq M$. By Proposition 2.1.14, $W \subseteq V$ has the κ-approximation and cover properties. The above fact yields that N is an inner model of W and $k : W \to N$ is an elementary embedding where $k = j \restriction W$ and $N = M \cap W$. We still have $k(\kappa) > \lambda$ and ${}^{j(\lambda)}N \subseteq N$ (from the perspective of W). Thus κ is hyper-huge in W.

Now assume that κ is hyper-huge in W. Let $\lambda > \kappa$ and find $j : W \to M$ an elementary embedding with $crit(j) = \kappa$, $j(\kappa) > \lambda$ where M is an inner model of W with ${}^{j(\kappa)}M \subseteq M$. Find a forcing $\mathbb{P} \in W_\kappa$ and a $\mathbb{P}$-generic G so that $W[G] = V$. Since the critical point of j is κ, $j(\mathbb{P}) = \mathbb{P}$ and thus $j[G] = G$. We can apply Lemma 1.4.2 to lift j to an embedding

$$j^+ : V \to M[G]$$

with $j^+ \restriction W = j$. In particular, $j^+(\kappa) = j(\kappa) > \lambda$ and $j^+(\lambda) = j(\lambda)$. Since W is definable in V and $G \in V$, $M[G]$ is an inner model of V. It is left to show that ${}^{j(\lambda)}M[G] \subseteq M[G]$. As usual, it is enough to verify this closure condition for sequences of ordinals. So let $f : j(\lambda) \to Ord$ be a function in V. By Proposition 1.2.4, there is a $\mathbb{P}$-name $\dot{f}$ for f of size $j(\lambda)$. The closure of M implies that $\dot{f} \in M$ and hence $f = \dot{f}^G \in M[G]$ follows from the absoluteness of interpretation of names. This shows that κ is hyper-huge in V.

□

Lemma 4.3.10. *Hyper-huge cardinals are downwards absolute to the mantle.*

Proof. If κ is a hyper-huge cardinal then by Corollary 4.3.3 there is a bedrock. Thus the mantle is a ground. Furthermore by Theorem 4.3.4, V is a generic extension of $\mathbb{M}$ by a forcing of size $< \kappa$. By Proposition 4.3.8, κ is hyper-huge in $\mathbb{M}$.

□

All in all, if κ is hyper-huge in V we can characterize exactly where in the generic multiverse κ has the same property.

Corollary 4.3.11. *If κ is hyper-huge and W is a point in the generic multiverse of V then κ is hyper-huge in W if and only if W is an extension of $\mathbb{M}$ by a forcing of size $< \kappa$.*

Proof. Find a forcing $\mathbb{P} \in \mathbb{M}$ of minimal size so that $\mathbb{M}$ extends to W via $\mathbb{P}$. We already now that κ is hyper-huge in $\mathbb{M}$. If $|\mathbb{P}|^{\mathbb{M}} < \kappa$ then κ is hyper-huge in W by Proposition 4.3.8. Otherwise,

$$W \models \text{“}\mathbb{M} \text{ is not a } \kappa\text{-ground”}$$

and thus by Theorem 4.3.4, κ is not hyper-huge in W. □

Using the qoutient lemma, we can put this differently:

Corollary 4.3.12. *It is not possible to add hyper-huge cardinals via set forcing.*

Proof. Suppose κ is hyper-huge in an extension $V[G]$. By Theorem 4.3.4, $\mathbb{M}$ is a κ-ground of $V[G]$. Since $\mathbb{M} \subseteq V \subseteq V[G]$, V is a κ-ground of $V[G]$ by Corollary 6.3.10 and thus κ is hyper-huge in V by Proposition 4.3.8. □

In contrast to this, it is possible to add a hyper-huge cardinal via class forcing (relative to some larger cardinal axiom). For this let $\mathbb{P}$ be the canonical Easton supported class iteration that forces GCH, namely the iteration $(\langle \mathbb{P}_\alpha | \alpha \in Ord \rangle, \langle \dot{\mathbb{Q}}_\alpha | \alpha \in Ord \rangle)$ where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the $Add(\alpha^+, 1)$ forcing as defined in the extension if α is an infinite cardinal and for the trivial forcing else. We will prove the following theorem in the addendum.

Theorem 6.5.1.*[Tsa16] After forcing with $\mathbb{P}$, any $(n+1)$-superhuge cardinal remains n-superhuge.*

Since a 2-superhuge cardinal is hyper-huge, any 3-superhuge cardinal remains hyper-huge after forcing with $\mathbb{P}$.

Corollary 4.3.13. *If ZFC plus the existence of a 3-superhuge cardinal is consistent, then it is consistent that some class forcing adds a hyper-huge cardinal.*

Proof. Let κ be 3-superhuge in V and force with the canonical Easton supported class iteration $\mathbb{P}$ for GCH. Thus in the extension $V[G]$, κ is still hyper-huge. However, we can factor $\mathbb{P}$ as $\mathbb{P}_{\leqslant\kappa} * \dot{\mathbb{P}}_{>\kappa}$ into the initial iteration up to stage κ and the tail iteration. If $G_{\leqslant\kappa}$ is the induced generic for $\mathbb{P}_{\leqslant\kappa}$, then κ is not hyper-huge in $V[G_{\leqslant\kappa}]$: Let g be the induced generic for $Add(\kappa^+, 1)^{V[G_{<\kappa}]}$. We can understand g as a subset of κ^+. Note that every bounded subset of g is in $V[G_{<\kappa}]$. That shows that $V[G_{<\kappa}] \subseteq V[G_{\leqslant\kappa}]$ does not have the κ^+-approximation property. Proposition 2.1.14 implies that $V[G_{<\kappa}]$ cannot be a κ-ground of $V[G_{\leqslant\kappa}]$ and so $\mathbb{M}$ cannot be a κ-ground by the quotient lemma (6.3.10). Thus by Theorem 4.3.4, κ is not hyper-huge in $V[G_{\leqslant\kappa}]$. However, the tail iteration $\dot{\mathbb{P}}_{>\kappa}^{G_{\leqslant\kappa}}$ forces κ to be hyper-huge. □

We can also kill off as many hyper-huge cardinals as we want. We will later see the analogous result for supercompacts.

Lemma 4.3.14. *If $ZFC+$ "there are class many hyper-huge cardinals" is consistent, then so is $ZFC+$ "there are class many hyper-huge cardinals in $\mathbb{M}$, but none in the entire generic multiverse".*

Proof. Start with a model V of $ZFC+$ "there are class many hyper-huge cardinals" and add a Cohen real to obtain $V[c]$. By Lemma 4.3.8 all hyper-huge cardinals remain so in $V[c]$. Now obtain any model W of ZFC such that $\mathbb{M}^W = V[c]$ by applying Theorem 2.3.2. Since $V[c]$ has a nontrivial ground, $V[c]$ cannot be a ground of W. Hence, there cannot be a bedrock. By Corollary 4.3.3, there cannot be a hyper-huge anywhere in the generic multiverse of W since the statement "there is a bedrock" is constant across the generic multiverse. □

4.4 Δ_2^{ZFC}-Definable Large Cardinals Axioms

We determine when Δ_2^{ZFC} statements holds in the mantle. This directly gives a criterion when Δ_2^{ZFC}-definable large cardinal axioms hold in the mantle. We say that some property holds for dense-many grounds if for any ground W, there is a deeper ground $W' \subseteq W$ with this property. As grounds are uniformly definable, this is first order expressible if the property is.

Lemma 4.4.1. *For any Δ_2^{ZFC}-formula $\phi(x)$ and any $a \in \mathbb{M}$ the following are equivalent:*

(i) $\mathbb{M} \models \varphi(a)$

(ii) *The grounds W of V with $W \models \varphi(a)$ are dense.*

Σ_2-formulas are exactly the locally verifable statements. Dually, Π_2-formulas are exactly the locally falsifiable statmenets. So Δ_2^{ZFC}-formulas are both and in this sense *local.* We will exploit that the mantle locally coincides with dense many grounds.

Proposition 4.4.2. *For any ordinal α, the grounds W of V for which $W_\alpha = \mathbb{M}_\alpha$ are dense.*

Proof. Let W be any ground of V and work in W. For every $x \in W_\alpha \backslash \mathbb{M}_\alpha$, find some ground N^x such that $x \notin N^x$. By the *sDDG*, there is a ground N which is contained in all N^x for x as above. We have

$$\mathbb{M}_\alpha \subseteq N_\alpha \subseteq N^x_\alpha \subseteq W_\alpha \backslash \{x\}$$

for every $x \in W_\alpha \backslash \mathbb{M}_\alpha$ and thus $N_\alpha = \mathbb{M}_\alpha$. □

Corollary 4.4.3. *If $\varphi(x)$ is a Σ_2-formula and $a \in \mathbb{M}$ such that $\mathbb{M} \models \varphi(a)$ then the grounds W of V for which $W \models \varphi(a)$ are dense.*

Proof. Using Lemma 6.1.1, find a formula $\psi(x)$ such that

$$ZFC \vdash \forall x \ (\varphi(x) \leftrightarrow \exists \alpha \ x \in V_\alpha \wedge \psi(x)^{V_\alpha})$$

holds. Find α with $\mathbb{M}_\alpha \models \psi(a)$. By Proposition 4.4.2, the grounds W of V with $W_\alpha = \mathbb{M}_\alpha$ are dense. Every such ground W satisfies $W \models \psi(a)^{V_\alpha}$ and thus $W \models \varphi(a)$. □

The converse holds for Π_2-formulas.

Proposition 4.4.4. *If $\varphi(x)$ is a Π_2-formula and $a \in \mathbb{M}$ such that the grounds W of V for which $W \models \varphi(a)$ holds are dense then $\mathbb{M} \models \varphi(a)$.*

Proof. Assume that $\mathbb{M} \models \neg\varphi(a)$. As $\neg\phi(x)$ is Σ_2, an application of Corollary 4.4.3 yields that $W \models \neg\varphi(a)$ for dense many grounds. As above, let $\psi(x)$ be a formula that witnesses Lemma 6.1.1 for the formula $\neg\varphi(x)$. Find α with $\mathbb{M}_\alpha \models \psi(a)$. There must be a ground W of V with $W_\alpha = \mathbb{M}_\alpha$ by Proposition 4.4.2. Any deeper ground $W' \subseteq W$ satisfies $W'_\alpha = \mathbb{M}_\alpha$ and thus $W' \models \neg\varphi(a)$, contradicting the assumption. □

Lemma 4.4.1 now follows immediately from Corollary 4.4.3 and Proposition 4.4.4.

Corollary 4.4.5. *Suppose $\kappa \leqslant \gamma$. If X is any of the properties weakly inaccessible, strongly inaccessible, Mahlo, weakly compact, measurable or γ-supercompact, then $\mathbb{M} \models$ "κ is X" if and only if the grounds W for which $W \models$ "κ is X" are dense.*

Proof. All these properties are Δ_2^{ZFC} in κ (and γ).

□

However, the property of supercompactness is not locally verifiable (only locally falsifiable) in contrast to the above properties. We will see later that the Π_2-statement "κ is supercompact" does not in general satisfy the conclusion of Corollay 4.4.3. Certainly, it is easy to build large cardinal axioms that are not Δ_2^{ZFC}, but still satisfy Lemma 4.4.1. For example "κ is supercompact in all grounds" would do the trick. Anyhow, there are natural global large cardinal axioms that still satisfy Lemma 4.4.1. But before we get to this, let us first observe that the Δ_2^{ZFC} statement "$0^\#$ exists" behaves really nice regarding the mantle. By the way, that this is Δ_2^{ZFC} can be observed by noting that the existence of $0^\#$ is equivalent to $\aleph_\omega$ being regular in L ([Jec03, p. 329]). Furthermore, the existence of $0^\#$ cannot be changed via forcing (compare [Jec03, Exercise 18.2]). Together with our characterization of Δ_2^{ZFC}-formulas in the mantle, this yields:

Proposition 4.4.6. *The following are equivalent:*

(*i*) $0^\#$ *exists somewhere in the generic multiverse of* $\mathbb{M}$.

(*ii*) $0^\#$ *exists in* $\mathbb{M}$.

(*iii*) $0^\#$ *exists in dense many grounds.*

(*iv*) $0^\#$ *exists in* V.

(*v*) $0^\#$ *exists somewhere in the generic multiverse of* V.

In that sense, the existence of $0^\#$ transcends through the multiverses of $\mathbb{M}$ and V. Let's get back to the example of a natural global large cardinal notion that does still satisfy our Δ_2^{ZFC}-characterization.

Definition 4.4.7. Let κ be a cardinal.

(*i*) We say that M is a κ-model if M is transitive, of size κ, $\kappa \in M$ and $M \models ZFC^-$.

(*ii*) For $\lambda > \kappa$, κ is called λ-unfoldable if for any κ-model M there is an elementary embedding $j : M \to N$ with $crit(j) = \kappa$ and $j(\kappa) > \lambda$ for some transitive N.

(*iii*) We say that κ is unfoldable if κ is λ-unfoldable for all $\lambda > \kappa$.

Remark 4.4.8. It was carefully not specified whether or not unfoldability is Σ_2-definable. The author is in fact not sure whether this is true or not. There stronger Σ_2-large cardinals axioms that imply unfoldability. One example is measurability. Suppose κ is measurable and $\lambda \geqslant \kappa$. By iterating a nonprincipal $< \kappa$-complete ultrafilter on κ (compare [Jec03, Chapter 19]), we can construct an elementary embedding $V \to M$ with critical point κ and $j(\kappa) > \lambda$. If N is a κ-model, then $j(N)$ is transitive and $j \upharpoonright N : N \to j(N)$ an elementary embedding. Thus κ is unfoldable. The same is true for Ramsey cardinals. However , for example in L, the unfoldable cardinals will not exhibit these stronger properties. On the other hand, Lemma 6.1.1

gives a strategy with which one might try to proof that unfoldability is not Σ_2-definable: Start with κ unfoldable and for any given $\alpha > \kappa$, find a model with the same sets of rank $< \alpha$ in which κ is not unfoldable. This is not possible via forcing as κ will be unfoldable in any forcing extension with the same H_{κ^+} (as all κ-models are in there). Thus we would probably need to employ techniques of inner model theory.

Unfoldability is a strengthening of weak compactness, in a way similar to how supercompactness is a strengthening of measurability.

Lemma 4.4.9. *κ is unfoldable in $\mathbb{M}$ if and only if κ is unfoldable in dense many grounds.*

We first need an auxiliary result.

Proposition 4.4.10. *If M is a κ-model and $\lambda \geqslant \kappa$ a cardinal such that there is an elementary embedding $j : M \to N$ with $crit(j) = \kappa$ and $j(\kappa) > \lambda$ and N transitive, then there is such an embedding for some N' in H_{λ^+}.*

Proof. Let δ be large enough such that $M, N \in H_\delta$. Find an elementary substructure $K \prec H_\delta$ with $N, j \in K$, $tc(\{M\}), \lambda + 1 \subseteq K$ of size λ. Let $\bar{K}$ be the Mostowski collapse of K and π the corresponding collapse map. Let $\varphi(x, y, z, u, v)$ be the statement

$$\text{“}y, z \text{ are transitive and } x : y \to z \text{ is an elementary embedding with } crit(x) = u \text{ and } x(u) > v\text{”}$$

As φ is Σ_0, we have $H_\delta \models \varphi(j, M, N, \kappa, \lambda)$. By elementarity and applying the isomorphism π, we conclude $\bar{K} \models \varphi(\pi(j), \pi(M), \pi(N), \pi(\kappa), \pi(\lambda))$. By choice of K we have that $\pi(M) = M, \pi(\kappa) = \kappa$ and $\pi(\lambda) = \lambda$. As φ is absolute between V and $\bar{K}$, we have that $\pi(j) : M \to \pi(N)$ is an elementary embedding with $crit(\pi(j)) = \kappa$ and $\pi(j)(\kappa) > \lambda$. Since $\bar{K}$ is transitive and of size λ, we get that $\pi(N) \in \bar{K} \subseteq H_{\lambda^+}$. □

Proof. (Lemma 4.4.9) Assume κ is unfoldable in $\mathbb{M}$. As usual, it is enough to show that κ is unfoldable in all grounds W with $W_{\kappa+1} = \mathbb{M}_{\kappa+1}$. So suppose M is a κ-model in W and $\lambda \geqslant \kappa$. Since we can code M as a subset of κ in an absolute way (compare Remark 2.1.8), and since W and $\mathbb{M}$ have the same subsets of κ, we can conclude that $M \in \mathbb{M}$ and M has size κ in $\mathbb{M}$. Now the embedding that witnesses the unfoldability of κ in $\mathbb{M}$ with instance M, λ works in W, too.
For the other direction, observe that being unfoldable is locally falsifiable by Proposition 4.4.10, as λ-unfoldability can be falsified in $H_{\lambda^+} \subseteq V_{\lambda^+}$. Thus “$\kappa$ is unfoldable” is a Π_2-property by (the dual of) Lemma 6.1.1 and hence this direction follows from Proposition 4.4.4. □

On the other hand, it is possible to give an upper bound on the reflective strength a large cardinal axiom that satisfies the conclusion of Lemma 4.4.1 can possibly have.

Lemma 4.4.11. *Suppose $\varphi(x)$ is a large cardinal axiom such that ZFC + GCH plus the existence of a cardinal κ with $\varphi(\kappa)$ is consistent. If such a cardinal provably reflects the failure of GCH, i.e.*

$$ZFC \vdash \forall\kappa\ (\varphi(\kappa) \wedge \neg GCH) \rightarrow \exists\lambda < \kappa\ 2^\lambda > \lambda^+$$

then $\varphi(x)$ does not in general satisfy the conclusion of Proposition 4.4.4.

Proof. Suppose V is a model of $ZFC + GCH$ and a cardinal κ satisfies $\varphi(\kappa)$ in V. Force with the iteration $\mathbb{P}$ constructed in Theorem 2.3.7 where the additional sequence of forcings is chosen trivial and so that $\mathbb{P}$ is $< |V_\kappa|$-distributive. The extension $V[G]$ satisfies $\mathbb{M}^{V[G]} = V$ and $V[G]_\kappa = V_\kappa$ and thus GCH holds in $V[G]$ below κ. The latter is true for any ground W of $V[G]$. On the other hand, GCH fails unboundedly often in $V[G]$ and thus in any ground of $V[G]$, too. As κ reflects the failure of GCH whenever $\varphi(\kappa)$ holds, we can conclude that $W \models \neg\varphi(\kappa)$ for every ground W of $V[G]$. But then, there is in fact no ground in which $\varphi(\kappa)$ holds true. □

Remark 4.4.12. Even though this gives an upper bound on the reflective strength Δ_2^{ZFC}-definable large cardinal axioms exhibit, these can still have exorbitant consistency strength. For example, one of the large cardinal axioms with the highest consistency strength that is not known to be inconsistent is the existence of a nontrivial elementary embedding $j : V_\lambda \rightarrow V_\lambda$. If we phrase this as an axiom in terms of λ instead of the critical point of j, the existence of such an embedding is decided in $V_{\lambda+1}$ and thus is Δ_2^{ZFC}.

4.5 Supercompact Cardinals

Supercompacts are an example a large cardinal axiom that is consistent with GCH and reflects the failure of GCH.

Proposition 4.5.1. *A supercompact cardinal reflects the failure of GCH.*

Proof. Assume GCH fails at $\lambda \geqslant \kappa$. By supercompactness, find some inner model1 M and an elementary embedding $j : V \rightarrow M$ with $crit(j) = \kappa$, $j(\kappa) > \lambda^+$ and ${}^{\lambda^+}M \subseteq M$. Then $(\lambda^+)^M = \lambda^+$ and $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)$. Since there is no surjection from λ^+ to $\mathcal{P}(\lambda)$ in V, there is no such map in M and thus $M \models$ "GCH fails at λ" and thus $M \models$"GCH fails below $j(\kappa)$". By elementarity, $V \models$ "GCH fails below κ". □

It is a standard result that supercomact cardinals are preserved by the canonical Easton supported iteration which forces GCH. In particular, the existence of a supercompact cardinal + GCH is relative consistent to the existence of a supercompact.

Corollary 4.5.2. *If ZFC+ "there is a supercompact cardinal" is consistent, then so is ZFC+ "there is a cardinal κ which is supercompact in $\mathbb{M}$, but in no ground".*

Remark 4.5.3. By Lemma 4.4.3, the analogue of the above result for any Σ_2-definable large cardinal axiom is impossible. As supercompactness is a Π_2-property, this is best possible complexity-wise.

It is easy to see that the same proof shows that the above corollary is true for set-many supercompacts instead of only one. A natural question is now, whether or not we can do this for class many supercompacts to get an analog of Lemma 4.3.14 and the answer is yes, we can. The main result we will prove in the rest of this subsection is thus the following.

Theorem 4.5.4. *If ZFC+"there are class many supercompacts" is consistent then so is ZFC+"there are class many supercompacts in the mantle, but none in the entire generic multiverse".*

The main idea relies on a combinatorial principle we will define next.

Definition 4.5.5. If λ is an uncountable cardinal, then $\Box_\lambda$ denotes the statement that there is a sequence $\langle C_\alpha | \alpha \in Lim \cap \lambda^+ \rangle$ such that the following holds for all $\alpha \in Lim \cap \lambda^+$:

(*i*) C_α is a club in α.

(*ii*) $otp(C_\alpha) \leqslant \lambda$

(*iii*) If $\beta < \alpha$ is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$.

Our result will rely heavily on the failure of $\Box_\lambda$ above a supercompact.

Proposition 4.5.6. *If κ is supercompact, then $\Box_\lambda$ fails for any cardinal $\lambda \geqslant \kappa$.*

Our strategy will thus be to start with a model with class many supercompacts and force it to be the mantle, while simultaneously force $\Box_\lambda$ unboundedly often. Let us first proof the above Proposition.

Definition 4.5.7. Given a cardinal λ and a sequence $\vec{C} = \langle C_\alpha | \alpha < \lambda^+ \rangle$ such that C_α is a club in α, an ω-thread through $\vec{C}$ is an unbounded set $D \subseteq \lambda^+$ such that for all $\alpha \in E^{\lambda^+}_\omega \cap lim(D)$ we have $D \cap \alpha = C_\alpha$.

Proposition 4.5.8. *[SRK78] There are no ω-threads through $\Box_\lambda$-sequences.*

Proof. Suppose $D \subseteq \lambda^+$ is an ω-thread through a $\Box_\lambda$-sequence $\vec{C} = \langle C_\alpha | \alpha < \lambda^+ \rangle$. Since λ^+ is regular and D unbounded in λ^+, there is some $\beta < \lambda^+$ such that $otp(D \cap \beta) = \lambda$. Furthermore, we can find some $\beta < \alpha \in E^{\lambda^+}_\omega \cap lim(D)$. But then $\lambda < otp(D \cap \alpha) = C_\alpha$, contradicting that $\vec{C}$ is a $\Box_\lambda$-sequence. $\Box$

Proof. (Proposition 4.5.6) Let $\lambda \geqslant \kappa$ be a cardinal. Let $j : V \to M$ be a nontrivial elementary embedding witnessing the λ^+-supercompactness of κ. Suppose for a contradiction that $\vec{C} = \langle C_\alpha | \alpha < \lambda^+ \rangle$ is a $\square_\lambda$-sequence in V. Then $j(\vec{C}) = \langle \tilde{C}_\alpha | \alpha < j(\lambda^+) \rangle$ is a $\square_{j(\lambda)}$-sequence in M. Since M is closed under λ^+-sequences, we have that $j[\lambda^+] \in M$. From $j(\lambda^+) > j(\kappa) > \lambda^+$ and the regularity of $j(\lambda^+)$ in M, it follows that $\delta = sup\ j[\lambda^+] < j(\lambda^+)$. Thus we can define $D = j^{-1}[\tilde{C}_\delta]$.

Claim 4.5.9. *D is unbounded in λ^+.*

Proof. First notice that $j[\lambda^+]$ contains all its limit points of cofinality ω: If $\langle \alpha_n | n < \omega \rangle$ is an increasing sequence in λ^+ with limit $\alpha_\star < \lambda^+$ then, as ω does not get moved by j, $j(\langle \alpha_n | n < \omega \rangle) = \langle j(\alpha_n) | n < \omega \rangle$. Thus by elementarity, $sup\langle j(\alpha_n) | n < \omega \rangle = j(\alpha_\star) \in j[\lambda^+]$.
We have $A = j[\lambda^+] \cap \delta \in M$. Clearly, A is unbounded in δ and $cof(\delta) = \lambda^+ > \omega$. To show that $D \subseteq \lambda^+$ is unbounded, let $\alpha < \lambda^+$. In M, construct a sequence $\langle \beta_n | n < \omega \rangle$ such that for all $n < \omega$:

(*i*) $j(\alpha) < \beta_0$

(*ii*) $\beta_n < \beta_{n+1}$

(*iii*) $\beta_{2n} \in A$

(*iv*) $\beta_{2n+1} \in \tilde{C}_\delta$

This is possible as both A and $\tilde{C}_\delta$ are unbounded in δ. Since both A and $\tilde{C}_\delta$ contain its limit points of cofinality ω, $\beta_\star = sup\langle \beta_n | n < \omega \rangle \in A \cap \tilde{C}_\delta$. Now $j^{-1}(\beta_\star) \in D$ is larger than α. □

Claim 4.5.10. *D is an ω-thread through $\vec{C}$.*

Proof. We already now that D is unbounded in λ^+. Suppose γ is the supremum of an increasing sequence $\langle \gamma_n | n < \omega \rangle$ in D. The argument in the above claim shows that $j(\gamma) = sup_{n<\omega} j(\gamma_n) < \delta$. As $j(\gamma_n) \in \tilde{C}_\delta$ for every $n < \omega$, we can conclude that $j(\gamma) \in lim(\tilde{C}_\delta)$. But then $\tilde{C}_\delta \cap j(\gamma) = \tilde{C}_{j(\gamma)}$ and thus:

$$D \cap \gamma = j^{-1}[\tilde{C}_\delta \cap j[\gamma]] = j^{-1}[\tilde{C}_\delta \cap j(\gamma)] = j^{-1}[\tilde{C}_{j(\gamma)}] = C_\gamma$$

Here, the last equality holds as $\beta \in C_\gamma$ is equivalent to $j(\beta) \in \tilde{C}_{j(\gamma)}$, a consequence of the elementarity of j. □

This is a contradiction to Proposition 4.5.8. □

We now have to find a forcing that forces $\square_\lambda$.

Definition 4.5.11. For λ an uncountable cardinal let $\mathbb{P}_\lambda^\Box$ be the following partial order: It consists of conditions which are basically initial segments of a $\Box_\lambda$-sequence. To be more precise, we will have $p \in \mathbb{P}_\lambda^\Box$ iff

$$p = \langle C_\alpha^p | \alpha \leqslant \delta_p, \ \alpha \in Lim \rangle$$

where $\delta_p \in Lim \cap \lambda^+$ and $(i) - (iii)$ hold from the definition above for all $\alpha \in Lim \cap \delta_p$. We also accept the empty sequence as the maximal element. Let $p \leqslant q$ iff $\delta^q \leqslant \delta^p$ and $p \upharpoonright \delta_q = q$.

Lemma 4.5.12. *[CFM01] Let λ be an uncountable cardinal. The forcing $\mathbb{P}_\lambda^\Box$ has the following properties:*

(i) *it adds a $\Box_\lambda$ sequence*

(ii) *it has size at most 2^λ*

To prove this we will show that $\mathbb{P}_\lambda^\Box$ is $\leqslant \lambda$-strategically closed.

Proposition 4.5.13. *[CFM01] Let λ be an uncountable cardinal.*

(i) *$\mathbb{P}_\lambda^\Box$ is $\leqslant \lambda$-strategically closed.*

(ii) *If $p \in \mathbb{P}_\lambda^\Box$ and $\delta_p < \delta < \lambda^+$ then p can be extended to a $q \leqslant p$ with $\delta \leqslant \delta_q$.*

Proof. (i) In the following we will write δ_β instead of δ_{p_β} in all instances. Consider the following strategy for player II:

- At an even successor stage $\gamma = \gamma\prime + 1$ just extend $p_{\gamma\prime}$ nontrivially, for example by p_γ where $p_\gamma \upharpoonright (\delta_{\gamma\prime} + 1) = p_{\gamma\prime}$ and $p_\gamma(\delta_{\gamma\prime} + \omega) = \{\delta_{\gamma\prime} + n | n < \omega\}$.
- At a limit γ, let $\delta_\gamma = sup_{\beta<\gamma} \delta_\beta < \lambda^+$. We define p_γ by $p_\gamma \upharpoonright \delta_\gamma = \bigcup_{\beta<\gamma} p_\beta$ and $p_\gamma(\delta_\gamma) = C_{\delta_\gamma}^{p_\gamma} = \{\delta_\beta | \beta < \gamma\}$.

It is left to show that this is a winning strategy, i.e. that the above play at limit γ is always legal if we have played according to this strategy at prior stages. First of all, δ_γ is strictly larger than all prior δ_β by the choice of play at even successor stages, so p_γ is functional. Next we see that by induction, $C_{\delta_\gamma}^{p_\gamma}$ is closed by our prior choices of δ_β for limit $\beta < \gamma$. Clearly, $otp(C_{\delta_\gamma}^{p_\gamma}) \leqslant \gamma \leqslant \lambda$. Furthermore, if δ_ρ is a limit point of $C_{\delta_\gamma}^{p_\gamma}$ then by our choice of play at stage ρ, we have $C_{\delta_\rho}^{p_\gamma} = C_{\delta_\rho}^{p_\rho} = \{\delta_\beta | \beta < \rho\} = C_{\delta_\gamma}^{p_\gamma} \cap \delta_\rho$.

(ii) For $\delta < \lambda^+$ let $A(\gamma)$ be the statement " if $p \in \mathbb{P}_\lambda^\Box$ and $\delta_p < \delta < \gamma$ then p can be extended to a $q \leqslant p$ with $\delta \leqslant \delta_q$".
Let $\langle \gamma_\alpha | \alpha < \lambda^+ \rangle$ be the increasing enumeration of $Lim \cap \lambda^+$. We show $A(\gamma_\alpha)$ by induction on α.

$\underline{\alpha = 0}$**:** Trivial.

$\underline{\alpha \rightsquigarrow \alpha + 1}$**:** If $p \in \mathbb{P}^{\square}_{\lambda}$ with $\delta_p < \gamma_{\alpha+1}$ then first extend to q_0 with $\delta_{q_0} = \gamma_\alpha$ using $A(\gamma_\alpha)$. We can then extend q_0 as in the successor case of (i) to a q with $\delta_q = \gamma_\alpha + \omega = \gamma_{\alpha+1}$.

$\underline{\alpha \in Lim}$**:** Assume $p \in \mathbb{P}^{\square}_{\lambda}$ has $\delta_p < \gamma_\alpha$. Let $\rho = cof(\gamma_\alpha) \leqslant \lambda$ and find an increasing sequence $\langle \xi_\beta | \beta < \rho \rangle$ in $Lim \cap \gamma_\alpha$ cofinal in γ_α. Let σ_I be the following strategy for player I in the game $G(\mathbb{P}^{\square}_{\lambda}, \lambda+1)$: As a first move play p. At a later odd stage $\beta + 1$ with $\beta < \rho$ extend p_β to any condition $p_{\beta+1}$ with $\delta_{p_{\beta+1}} \geqslant \xi_\beta$ which is possible by $A(\xi_\beta)$. If $\beta \geqslant \rho$, just copy the last move, $p_{\beta+1} = p_\beta$.
Let σ_{II} be the winning strategy from (i). Then

$$O(\sigma_I, \sigma_{II}) = \langle p_\beta | \beta \leqslant \lambda \rangle$$

is a decreasing sequence of conditions of length $\lambda + 1$ since player II must have won. Let $q = p_\lambda$. By the choice of play of player I, we have $q \leqslant p_1 = p$ and $\delta_q \geqslant sup_{\beta<\lambda}\, \delta_{p_\beta} \geqslant sup_{\beta<\rho}\, \xi_\beta = \gamma_\alpha$.

□

Proof. (Lemma 4.5.12)

(i) Let G be $\mathbb{P}^{\square}_{\lambda}$ generic over V. First of all by (ii), $(\lambda^+)^{V[G]} = (\lambda^+)^V$. Since two conditions p, q are compatible if and only if $p \leqslant q$ or $q \leqslant p$, $\vec{C} = \bigcup G$ is functional. Indeed, it is a sequence of length λ^+: For $\delta \in Lim \cap \lambda^+$ define $D_\delta = \{p \in \mathbb{P}^{\square}_{\delta} | \delta \leqslant \delta_p\}$. By Proposition 4.5.13 (ii), D_δ is dense and thus $G \cap D \neq \varnothing$. This shows that $\vec{C}$ is of length λ^+. Any initial segment of $\vec{C}$ belongs to a condition in $\mathbb{P}^{\square}_{\lambda}$ and thus satisfies properties $(i) - (iii)$ of definition 4.5.5, hence $\vec{C}$ itself satisfies these properties.

(ii) We have that

$$|\mathbb{P}^{\square}_{\lambda}| \leqslant \sum_{\delta<\lambda^+} \prod_{\alpha \leqslant \delta} 2^{|\alpha|} \leqslant \lambda^+ \cdot 2^\lambda = 2^\lambda$$

□

We now have all the ingredients we need.

Lemma 4.5.14. *There is a class extension $V[G]$ with $\mathbb{M}^{V[G]} = V$ such that there are no supercompacts anywhere in the generic multiverse of $V[G]$.*

Proof. Finally, the extra work we have put into Theorem 2.3.7 will pay off. Let $\mathbb{P}$ be the forcing constructed in that theorem with $\kappa = \omega_1$ and additional sequence $\mathbb{Q}_\lambda = \mathbb{P}^{\square}_{\lambda}$. We write $\mathcal{C} = \mathcal{C}_\kappa$ (in the notation of that theorem). By Lemma 4.5.12 and Proposition 4.5.13, the sequence $\langle \mathbb{Q}_\lambda | \lambda \in \mathcal{C} \rangle$ meets the requirements. Let $V[G]$ be a corresponding extension. Since $\mathbb{P}^{\square}_{\lambda^+}$ does not collapse λ and λ^+, Theorem 2.3.7 yields the following:

(*i*) $\mathbb{M}^{V[G]} = V$

(*ii*) Whenever $\mathbb{P}_\lambda^\square$ has been chosen, λ and $(\lambda^+)^V$ are still cardinals in $V[G]$.

Since $\mathbb{P}_\lambda^\square$ has been chosen at unboundedly many stages $\lambda \in \mathcal{C}$, we see together with (*ii*) that $\square_\lambda$ holds at any of these cardinals. Suppose W' is another universe in the generic multiverse of $V[G]$. By Corollary 2.4.4 (*iii*), there is a common ground W of $V[G]$ and W'. Find some large enough cardinal δ so that W is a δ-ground of W' and of $V[G]$. Then $W \subseteq V[G]$ has the δ-approximation property by Proposition 2.1.14. If λ is any cardinal above δ in $\mathcal{C}$ such that G chose $\mathbb{P}_\lambda^\square$ at stage λ, then every initial segment of the added $\square_\lambda$-sequence is an element of V and thus of W. The δ-approximation property of $W \subseteq V[G]$ yields that the whole sequence is in W. As λ, λ^+ are cardinals in W, $\square_\lambda$ holds in W. Moreover, λ is still a cardinal in W' and has the same successor there and thus $\square_\lambda$ is true in W' as well. Thus $\square_\lambda$ holds unboundedly often in W'. By Fact 4.5.6, there cannot be any supercompacts in W'. $\square$

Theorem 4.5.4 follows.

This means that, by digging through to the mantle, it is possible to uncover new supercompacts, even class many, that crumbled so badly under the accumulated dust of forcing that they are not resurrectible via set forcing.

4.6 Counterexamples to Downwards Absoluteness

We have answered the questions which large cardinals are upwards absolute from $\mathbb{M}$ to V in our revised sense in a lot of instances. Moreover, we have seen that some very large cardinals are in fact downwards absolute to the mantle. What about smaller large cardinals? Certainly, all Π_1 definable large cardinal axioms such as weakly/strongly inaccessible and Mahlo are trivially downwards absolute to the mantle. Next, we will see that a lot of large cardinal notions are not downwards absolute to the mantle. More precisely, no large cardinal axiom $\varphi(\kappa)$ that is implied by supercompactness and itself implies weak compactness can be downwards absolute to the mantle. This means that there is some kind of sweet spot where the large cardinal axiom is strong enough to not be trivially downwards absolute, but not too strong to cause such drastic consequences for the generic multiverse that imply downwards absoluteness.

Theorem 4.6.1. *Weakly compact, measurable, unfoldable, γ-supercompact, supercompact and any other large cardinals whose defining property is implied by supercompactness and itself implies weak compactness, are in general not downwards absolute between V and $\mathbb{M}$.*

The proof will be a modification of Kunens observation in [Kun78] that these large cardinals can be consistently added by forcing.

Fact 4.6.2. *[Kun78] Given a weakly compact cardinal κ, there is a forcing $\mathbb{Q}$ that adds a Suslin tree T, so that the two step iteration $\mathbb{Q} * \dot{\mathbb{T}}$ is forcing equivalent to $Add(\kappa, 1)$.*

In addition to this, a main ingredient is a tool to make supercompact cardinals indestructible by a large class of forcings.

Definition 4.6.3. A supercompact cardinal κ is Laver indestructible if κ is supercompact in any extension by a $< \kappa$-closed forcing.

We prove in the addendum (Theorem 6.4.1) that any supercompact cardinal can be forced to be Laver indestructible.

Proof. (Theorem 4.6.1) Start with a model V in which κ is supercompact and Laver-indestructible. Now find $\mathbb{Q}*\dot{T}$ as given by Fact 4.6.2. Let $V[T]$ be a $\mathbb{Q}$-generic extension of V. T is a Suslin tree, so in particular an Aronszajn tree and hence the tree property fails at κ so that κ is not even weakly compact in $V[T]$. We want this model to be our mantle, so let $\mathbb{P}$ be the forcing from Theorem 2.3.7 (with trivial additional sequence) starting at κ as defined in $V[T]$, but first we force with with $\mathbb{T}$, the evaluation of $\dot{\mathbb{T}}$ in $V[T]$, to get an extension $V[T][h] = V[g]$, where g is $Add(\kappa, 1)$-generic over V. Since $Add(\kappa, 1)$ is $< \kappa$-directed-closed, κ is again supercompact in $V[g]$. As $V[T]$ is a ground of $V[g]$, this class product is definable in $V[g]$ by the definability of grounds. For any $\alpha > \kappa$, we can factor the generic G as $G_{<\alpha} \times G_{\geqslant\alpha}$, where $G_{<\alpha}$ is the induced generic for the initial factor of $\mathbb{P}$ up to stage α. Note that $\mathbb{P}$ is even $< \kappa^+$-closed and so does not add any new subsets to $\mathbb{T}$, so that h is still generic over $V[T][G_{<\alpha}]$. The product lemma implies that then

$$V[T][G_{<\alpha}][h] = V[T][h][G_{<\alpha}] = V[g][G_{<\alpha}]$$

for any α.

Claim 4.6.4.

$$V[T][G][h] = \bigcup_{\alpha\geqslant\kappa} V[T][G_{<\alpha}][h] = \bigcup_{\alpha\geqslant\kappa} V[g][G_{<\alpha}] = V[g][G]$$

Proof. We only have to show that the first and last equalities hold. For the last equality, this is simply true since any $x \in V[g][G]$ has a $\mathbb{P}$-name $\dot{x}$ in $V[g]$, but $tc(\{\dot{x}\})$ can only contain conditions up to some large stage α so that $x = \dot{x}^G = \dot{x}^{G_{<\alpha}} \in V[g][G_{<\alpha}]$.
The argument for the first equality is similar: Any $x \in V[T][G][h]$ is of the form $\dot{x}^h$ for some T-name $\dot{x} \in V[T][G]$. Thus $\dot{x} \in V[T][G_{<\alpha}]$ for some large enough α and hence $x \in V[T][G_{<\alpha}][h]$. □

We get the following commutative diagram, where arrows represent inclusions:

$$\begin{array}{ccccc} & & V[T][G] & \xrightarrow{\mathbb{T}} & V[g][G] \\ & & \uparrow \mathbb{P} & \circlearrowleft & \uparrow \mathbb{P} \\ V & \xrightarrow{\mathbb{Q}} & V[T] & \xrightarrow{\mathbb{T}} & V[g] \end{array}$$

Note that vertical arrows are class forcing extensions, while horizontal arrows refer to set forcing extensions. We will establish that κ is supercompact in the right column, but fails to be weakly compact in the middle column. I claim that in this universe, κ is supercompact, but not weakly compact in its mantle. First, note that since the mantle is constant across the generic multiverse, $\mathbb{M}^{V[g][G]} = \mathbb{M}^{V[T][G]} = V[T]$, where the second equality holds by Theorem 2.3.7. Thus κ is not weakly compact in $\mathbb{M}^{V[g][G]}$.
Observe that $V, V[T]$ and $V[g]$ all have the same $< \kappa$-sequences of ordinals as $Add(\kappa, 1)$ is $< \kappa$-closed and $V[T]$ is an intermediate model. This implies that $\mathbb{P}$ and each of its factors is $< \kappa$-directed closed in $V[g]$, since it is in $V[T]$. In $V[g]$, κ is still Laver-indestructible, as the two-step iteration of $< \kappa$-directed closed forcings is $< \kappa$-directed closed. We want to apply this to $\mathbb{P}$, but we cannot directly do so, as Laver-indestructibility only takes set forcings into consideration. Let $\gamma \geqslant \kappa$. We show that κ is γ-supercompact in $V[g][G]$. This property only depends on the Von-Neumann-hierarchy up to $\gamma + 2$. Find a $\mathbb{P}$-name $\dot{x}$ for $V_{\gamma+2}^{V[g][G]}$. Let α be large enough so that $\dot{x}$ is an $\mathbb{P}_{<\alpha}$-name. As $\dot{x}^{G_\alpha} = \dot{x}^G = V_{\gamma+2}^{V[g][G]}$, the Von-Neumann-hierarchies of $V[g][G_{<\alpha}]$ and $V[g][G]$ coincide up to $\gamma + 2$. By the prior observation, $\mathbb{P}_\alpha$ is $< \kappa$-directed closed in $V[g]$ and hence by, Laver-indestructibility, κ is supercompact in $V[g][G_{<\alpha}]$, so in particular γ-supercompact. But this means that κ must be γ-supercompact in $V[g][G]$, too. This concludes the proof. $\square$

In the above proof, we still have that κ is not weakly compact in some ground of $V[g][G]$, namely $V[T][G]$, (and in fact in every deeper ground, too) since T is still an Aronszajn tree there (note that $\mathbb{P}$ does not add subsets of T). Since being weakly compact is a Δ_2^{ZFC}-statement, Lemma 4.4.1 shows that this must necessarily be the case whenever κ is weakly compact, but not weakly compact in $\mathbb{M}$.

Remark 4.6.5. The above construction implicitly shows that in contrast to extendible cardinals, the existence of supercompact cardinals does not imply the bedrock axiom.

We give another example of a large cardinal axiom that is not downwards absolute to the mantle.

Definition 4.6.6. A cardinal κ is superstrong if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ and $V_{j(\kappa)} \subseteq M$.

Superstrong cardinals are measurable and thus weakly compact. However, even though the existence of a supercompact cardinal implies the existence of a superstrong cardinal, a supercompact cardinal need not be superstrong itself. Indeed, adding a κ-Cohen subset to a Laver indestructible supercompact cardinal κ will preserve it's supercompactness, but Theorem 2.5.1 shows that κ cannot be superstrong in the extension. Therefore Theorem 4.6.1 does not apply in this case.

Theorem 4.6.7. *Superstrong cardinals are in general not downwards absolute to the mantle.*

The general strategy to construct a model in which κ is superstrong, but in which the superstrongness of κ fails in $\mathbb{M}$, is to apply Theorem 2.5.1 to kill the superstrongness, make the resulting model the mantel and then to restore the superstrongness. To apply Theorem 2.5.1, we need the following auxiliary result to meet the necessary assumptions.

Proposition 4.6.8. *If κ is superstrong and $j : V \to M$ an embedding witnessing this, then $V_\kappa \prec V_{j(\kappa)}$.*

Proof. Let $\varphi(x_0, \ldots x_{n-1})$ be a $\in$-formula and let $a_0, \ldots a_{n-1}$ be parameters in V_κ. Since κ is the critical point of j, the a_i are not moved by j. Assume $V_\kappa \models \varphi(a_0, \ldots a_{n-1})$. Then

$$V \models V_\kappa \models \ulcorner\varphi(a_0, \ldots a_{n-1})\urcorner$$

and hence by elementarity,

$$M \models M_{j(\kappa)} \models \ulcorner\varphi(a_0, \ldots a_{n-1})\urcorner$$

from which we can conclude, by transitivity of M, that

$$V_{j(\kappa)} = M_{j(\kappa)} \models \varphi(a_0, \ldots a_{n-1})$$

□

We will force with a product forcing that preserves superstrongness, but has a factor that destroys it.

Definition 4.6.9. Given a superstrong cardinal κ, we call $j : V \to M$ a superstrong extender embedding if it is the embedding induced by the $(\kappa, k(\kappa))$-extender of an embedding witnessing that κ is superstrong. Note that Fact 1.4.6 implies that $V_{j(\kappa)} \subseteq M$ so that j is superstrong as well.

Lemma 4.6.10. *Suppose κ is superstrong and $j : V \to M$ is a superstrong extender embedding. Then after forcing with the Easton supported product forcing*

$$\mathbb{Q} = \prod_{\lambda < j(\kappa)} Add(\lambda^+, 1)$$

the embedding j lifts to an embedding $j^{++} : V[G] \to M[H]$ witnessing that κ is superstrong in $V[G]$.

Proof. Factor the generic as $G_{<\kappa} \times G_{\geqslant\kappa}$. Notice that since $\mathbb{P}$ is Easton supported and since κ is inaccessible, every $p \in \mathbb{Q}_{<\kappa}$ has bounded domain and thus is a member of V_κ. Furthermore

$$j(\mathbb{Q}_{<\kappa}) = \left(\prod_{\lambda < j(\kappa)} Add(\lambda^+, 1) \right)^M = \mathbb{Q}$$

where the last equality holds since $V_{j(\kappa)} \subseteq M$. Clearly, G is $j(\mathbb{Q}_{<\kappa})$-generic over M and $j[G_{<\kappa}] = G_{<\kappa} \subseteq G$. Thus we can lift j to:

$$j^+ : V[G_{<\kappa}] \to M[G]$$

Observe that $\mathbb{Q}_{\geqslant\kappa}$ fails to be $< \kappa^+$-closed in $V[G_{<\kappa}]$, but is still $< \kappa^+$-distributive by Lemma 1.3.5. But now, as j is a derived extender embedding, so is j^+ by Fact 1.4.8 and thus the upwards closure H of $j[G_{\geqslant\kappa}]$ in $j(\mathbb{Q}_{\geqslant\kappa})$ is generic over $M[G]$ by Lemma 1.4.9. Hence we can lift j^+ to

$$j^{++} : V[G] \to M[G][H]$$

and it is left to verify $V[G]_{j(\kappa)} \subseteq M[G][H]$. But as in Claim 2.5.3, we find that $V[G]_{j(\kappa)} = V_{j(\kappa)}[G]$ and this is certainly a subset of $M[G] \subseteq M[G][H]$. □

Proof. (Theorem 4.6.7) Start with a model V in which some cardinal κ is superstrong and find $j : V \to M$ a superstrong extender embedding for κ. Let g be $Add(\kappa^+, 1)$-generic over V. It follows from Theorem 2.5.1 and Proposition 4.6.8 that κ is not superstrong in $V[g]$. Let G be $\mathbb{P}$-generic over $V[g]$. Now, in $V[g]$, define $\mathbb{P}$ to be the forcing from Theorem 2.3.7 with trivial additional sequence, starting high enough to be $|V[g]_{j(\kappa)}|$-closed. Let G be $\mathbb{P}$-generic over $V[g]$. We find that $\mathbb{M}^{V[g][G]} = V[g]$, where κ is not superstrong. Now let $\mathbb{Q}$ be the forcing from above as defined in V, and let $\hat{\mathbb{Q}}$ be the modification of $\mathbb{Q}$ where the factor at stage κ is trivial. Notice that $Add(\kappa^+, 1) \times \hat{\mathbb{Q}} \cong \mathbb{Q}$ (in fact we could just use $\mathbb{Q}$ instead of $\hat{\mathbb{Q}}$). Let h be $\hat{\mathbb{Q}}$-generic over $V[g][G]$. Exactly as in Claim 4.6.4, it follows that $V[g][G][h] = V[g][h][G]$. By the above lemma, κ is superstrong in $V[g][h]$ and so in $V[g][h][G]$, too, since their Von-Neumann-hierachy coincides up to rank $j(\kappa)$ by closure of $\mathbb{P}$. Again by Corollary 2.4.4 (ii), $\mathbb{M}^{V[g][h][G]} = \mathbb{M}^{V[g][G]} = V[g]$. □

In the above construction, it was significantly easier to make the step up to the class forcing extension that has the right mantle, compared to Theorem 4.6.1. The reason for this is that being superstrong is a Σ_2-property, while being supercompact is not, and thus we were able to choose the class forcing closed enough to not interfere with the destruction and resurrection of this large cardinal property. In this way we can generalize. Lets say that a large cardinal property $\varphi(\kappa)$ is always destructible and resurrectible by set forcing if ZFC proves that if $\varphi(\kappa)$ holds, then there is a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \neg\varphi(\check{\kappa})$, but $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \varphi(\check{\kappa})$.

Corollary 4.6.11. *Suppose $\varphi(\kappa)$ is a Σ_2 large cardinal property that is always destructible and resurrectible by set forcing. If $ZFC + \exists\kappa\ \varphi(\kappa)$ is consistent, then so is $ZFC + \exists\kappa\ \varphi(\kappa) \wedge \neg\varphi(\kappa)^{\mathbb{M}}$.*

Anyhow, the example in the case of superstrongness was included in this thesis as the destruction and resurrection of this property, as we have seen, is a nice application of Theorem 2.5.1, which itself is proven using Set-Theoretic Geology.
The same reasoning applies to worldly cardinals, that are cardinals κ for which $V_\kappa \models ZFC$, even though this is not definable as an $\in$-formula $\varphi(\kappa)$, as in many cases V will not be able to put everything together to see $V \models V_\kappa \models \ulcorner ZFC \urcorner$. In his blog post titled "Worldly cardinals are not always downwards absolute" [Ham17b], Hamkins shows that singular worldly cardinals can always be destroyed and further resurrected via set forcing. Since κ being worldy only depends on V_κ, this property is Σ_2 in a meta-theoretic sense: Whenever $V \subseteq W$ is an outer model with the same sets of rank $\leqslant \kappa$, κ is worldly in W. Thus we can apply the same reasoning to find a model V in which some κ is worldly, but fails to be so in $\mathbb{M}$.

5 Conclusion and Questions

As we have already discussed, Set-Theoretic Geology was not successful at its original task. Theorem 2.3.2 shows that in general, the mantle has no interesting properties. If that were the case, this would have, for example, opened up the possibility of achieving lower bounds in consistency strength, by trying to show that certain cardinals are large in a ground or in $\mathbb{M}$. The current reach of inner model theory is below a supercompact cardinal and thus the expected equiconsistency of PFA and the existence of a supercompact is still open. On the other hand, the mantle can satisfy any large cardinal axiom that is consistent with ZFC. A naive approach would have been to try and reverse the usual forcing construction which gives PFA from a supercompact. There, a supercompact κ is collapsed to $\aleph_2$. Naively, one could hope that in general under PFA, $\aleph_2$ is supercompact in $\mathbb{M}$, or at least in some ground. However, PFA is indestructible under $< \aleph_2$-directed closed forcing ([Lar00]) and thus using Theorem 2.3.7, one can show that PFA is consistent together with the ground axiom.
However, certain canonical inner models *do* have non-trivial grounds, in contrast to L or $L[0^{\#}]$. The mantle of these models will again admit a regular structure. For a starting point, consult [FS16].
Moreover, Set-Theoretic Geology has proven to be an interesting topic with an outreach beyond its own scope. In section 2.5, we have applied the uniform definability of grounds to see that a lot of large cardinals are always destructible by quite mild forcings. To be precise, this is true for all large cardinals κ for which V_κ is provably a $(\Sigma_3$-)elementary substructure of a higher initial segment of the universe. As a consequence, the analogue of Laver indestructibility is impossible for all of these large cardinals. These are already non-geologic statements. Thus Set-Theoretic Geology fulfills the arguably most important property of an interesting theory, it has implications that are not subject of their own nature.

In the present thesis, there was an emphasis on the interplay between large cardinals, the mantle and the generic multiverse discussed in chapter 4. We begun with results due to Usuba, the existence of an extendible implies the bedrock axiom. Even more is true for a hyper-huge (or some smaller large cardinal, compare Remark 4.3.5). If κ is such a cardinal, $\mathbb{M}$ is a κ-ground. Whether the same is true for extendibles was already asked by Usuba in [Usu18].

Question 5.0.1. Is the existence of an extendible cardinal κ already enough to conclude that the mantle is a κ-ground?

Similar questions can be postulated for other consequences we have found to be true for hyper-huges.

Question 5.0.2. Are extendible cardinals downwards absolute to $\mathbb{M}$?

If the prior question can be answered positively then the latter can be shown to be true by similar methods as in the case of a hyper-huge. We have seen that if κ is extendible then the mantle is the intersection of all κ-grounds. If $\lambda > \kappa$ is another extendible cardinal, it follows (compare Remark 4.3.5) that $\mathbb{M}$ is a λ-ground. It can be shown, similar to Proposition 4.3.8, that λ must be extendible in $\mathbb{M}$. Thus all, but maybe the least, extendibles are extendible in $\mathbb{M}$. It seems plausible that the least extendible is downwards absolute to $\mathbb{M}$ as well.
Subsequently, we investigated large cardinals at and below the level of a supercompact. Once again, supercompacts have proven to be very flexible. They inhabit a sweet spot where they are too weak to have drastic effects on the generic multiverse, but are strong (and/or complex) enough to not fall prey to Lemma 4.4.1, a significant restriction on possible configurations in the generic multiverse for Δ_2^{ZFC}-definable large cardinal axioms. To go further into this, similar to hyper-huge cardinals, it is possible by Theorem 4.5.4 to uncover a class of supercompacts while there are none in the entire generic multiverse of V. The same is impossible for the Σ_2-definable large cardinal axioms by the Σ_2-direction of Lemma 4.4.1, e.g. if κ is measurable in $\mathbb{M}$ then it is in dense-many grounds.
We have seen in Theorem 4.6.1 that the large cardinal hierachy starts to gain a bit of flexibility in our context at about the level of a weakly compact (below that, a lot of large cardinals are trivially downwards absolute). Between that and a supercompact (in the sense of direct implication), downwards absoluteness to $\mathbb{M}$ fails. The proof cannot be directly modified for class many supercompacts.

Question 5.0.3. Is it possible that there is a class of supercompact cardinals, all of which are not supercompact (or not even weakly compact) in $\mathbb{M}$?

Central to our proof was the Laver indestructibility. A proof of this generalized statement may involve a global version of Laver indestructibility. We have seen that superstrong cardinals can fail to be superstrong in $\mathbb{M}$ as well. It could not be determined in this thesis whether superstrong cardinals satisfy the Π_2-direction of Lemma 4.4.1.

Question 5.0.4. Is it possible that there is a cardinal κ superstrong in dense-many grounds, but not in $\mathbb{M}$?

By Lemma 4.4.2, it can be seen that if there is such an example, then the minimal target $j(\kappa)$ of a superstrong embedding with critical point κ must get arbitrarily large by passing to deeper and deeper grounds. A construction answering this question positively would thus likely start with an assumption of higher consistency strength than merely one superstrong cardinal. Possibly, one could start with a superstrong cardinal that has arbitrarily large targets of the superstrong embedding, i.e. $j(\kappa)$ can be made larger than any given λ. Next one could destroy the superstrongness of κ, which is quite easy thanks to Theorem 2.5.1, and make this model the mantle. While simultaneously forcing that model to be the mantle, one has to revive κ with target $j(\kappa) > \lambda$ one by one. How the last part could be done is unclear. It seems difficult to revive the "larger instances" of superstrongness while avoiding the "smaller" ones. Variations of this questions can be investigated for other Σ_2-definable large cardinals such as huge cardinals.

6 Addendum

In the main part of this thesis, there was a clear focus on Set-Theoretic Geology. In order to avoid distractions, we skipped the proofs of a few necessary results. However, as the goal of this thesis is to be as complete as possible, we make up for this here.

6.1 A Characterization of Σ_2^{ZFC} Formulas

In Chapters 2 and 4, we used that the satisfaction of Σ_2^{ZFC}-formulas is locally verifiable.

Lemma 6.1.1. *[Rei06, implicit in Corollary 14] A formula $\varphi(x)$ is Σ_2^{ZFC} if and only if*

$$ZFC \vdash \forall x\ (\varphi(x) \leftrightarrow \exists \alpha\ x \in V_\alpha \wedge V_\alpha \models \ulcorner \psi(x) \urcorner)$$

for some formula $\psi(x)$.

Proposition 6.1.2. *If κ is an uncountable cardinal then $H_\kappa \prec_1 V$, that is all Σ_1-formulas with parameters in H_κ are absolute between H_κ and V.*

Proof. Let $\varphi(y_0, \dots y_{n-1}) = \exists x \theta(x, y_0, \dots y_{n-1})$, where θ is Σ_0. We may assume that $n = 1$. If $a \in H_\kappa$ and $\varphi(a)$ is true in H_κ then it is certainly true in V, so assume x is such that $V \models \theta(x, a)$. Find $\lambda \geqslant \kappa$ with $x \in H_\lambda$ so that $H_\lambda \models \theta(x, a)$. Let $M \prec H_\lambda$ be an elementary substructure of size $< \kappa$ with $x \in M$ and $tc(\{a\}) \subseteq M$. Let N be the transitive collapse and $\pi : M \to N$ the corresponding collapse map. Then $N \models \theta(\pi(x), \pi(a))$ and by our choice of M, $\pi(a) = a$. Furthermore, as N is transitive and of size $< \kappa$, $N \subseteq H_\kappa$. Finally, because θ is Σ_0, we can conclude that $H_\kappa \models \theta(\pi(x), a)$. $\square$

Proposition 6.1.3. *If $\varphi(x)$ is Σ_2 then*

$$ZFC \vdash \forall x\ (\varphi(x) \leftrightarrow \exists \kappa \in Card \backslash \omega_1\ x \in H_\kappa \wedge \varphi(x)^{H_\kappa})$$

Proof. Write $\varphi(x) = \exists y \forall z\ \theta(x, y, z)$. If $\varphi(a)$ holds in V then we can find an uncountable cardinal κ with $a \in H_\kappa$ so that $H_\kappa \models \varphi(a)$.
On the other hand, assume that for some uncountable cardinal κ and $a \in H_\kappa$, we have $H_\kappa \models \varphi(a)$. Find $b \in H_\kappa$ so that $H_\kappa \models \forall z\ \theta(a, b, z)$. As $H_\kappa \prec_1 V$, we now know that $V \models \forall z\ \theta(a, b, z)$ and thus $\varphi(a)$ holds in V. $\square$

Proof. (Lemma 6.1.1) "$\Rightarrow$" Without loss of generality, we can suppose that $\varphi(x)$ is already Σ_2. Let $\psi(x) = \exists \kappa \in Card \backslash \omega_1 \wedge x \in H_\kappa \wedge \varphi(x)^{H_\kappa}$. Since any V_α is correct about its uncountable cardinals and whenever $\kappa \in V_\alpha$ is an uncountable cardinal we have $H_\kappa^{V_\alpha} = H_\kappa$, we can conclude with Proposition 6.1.3 that $\psi(x)$ is as desired.
"$\Leftarrow$" The formalized predicate "$A \models k$" is Σ_0 in A and k. The term V_α is Π_1 (in α) and hence "$\exists \alpha\ x \in V_\alpha \wedge V_\alpha \models \ulcorner \psi(x) \urcorner$" is Σ_2. $\square$

6.2 The Inner Model Criterion

In the Definability of Grounds Theorem, it is essential to be able to check whether certain definable subclasses of V are models of ZFC or not in a single first order $\in$-formula. We will show here that this is possible. We also used this quite simple criterion in a few other places to easily verify that a given transitive class models ZFC. We follow [Jec03, pp. 177-183]

Definition 6.2.1. A transitive subclass $M \subseteq V$ is an inner model of $ZF(C)$ if $M \models ZF(C)$ and contains all ordinals.

Theorem 6.2.2. *[Jec03, Theorem 13.9] (The Inner Model Criterion) Suppose M is a definable subclass of V from a parameter r. Then there is a first order formula $\psi(r)$ with the property that M is an inner model of ZF if and only if $V \models \psi(r)$.*

The formula ψ will state (apart from the obvious part "M is transitive and contains all ordinals") that M is closed under certain very basic operations and satisfies a cover property with respect to V. We now introduce these operations.

Definition 6.2.3. The Gödel operations $(G_i)_{i<10}$ are defined as follows:

$G_0(x, y) = \{x, y\}$	$G_1(x, y) = x \times y$
$G_2(x, y) = \{(u, v) \mid u \in v \in y \wedge u \in x\}$	$G_3(x, y) = x \backslash y$
$G_4(x, y) = x \cap y$	$G_5(x) = \bigcup x$
$G_6(x) = dom(x)$	$G_7(x) = \{(u, v) \mid (v, u) \in x\}$
$G_8(x) = \{(u, v, w) \mid (u, w, v) \in x\}$	$G_9(x) = \{(u, v, w) \mid (v, w, u) \in x\}$

One necessary ingredient we need is that closure under the Gödel operations is (under a very weak fragment of ZF) equivalent to satisfying separation for Δ_0-formulas.

Lemma 6.2.4. *If $\phi(v_0, \ldots, v_{n-1})$ is a Δ_0-formula then there is a composition G of Gödel operations such that*

$$G(x_0, \ldots, x_{n-1}) = \{(u_0, \ldots, u_{n-1}) \in \prod_{i<n} x_i \mid \phi(u_0, \ldots, u_{n-1})\}$$

Proof. We proof the statement by induction over the complexity of φ. We can suppose that ϕ is build up by only using the logical connectives $\neg$, $\wedge$ and $\exists v_i \in v_j$. We can do without $v_i = v_j$ as we can replace this by

$$(\forall x \in v_i \; x \in v_j) \wedge (\forall y \in v_j \; y \in v_i)$$

and then substitute the restricted $\forall$ quantifiers by $\neg, \exists$ as usual.

$\underline{\phi = v_i \in v_j}$**:** If $i = j$, ϕ can never be satisfied and we let $G(x) = G_3(x, x)$. First, we assume $n = 2$. If $i = 0, j = 1$, then let $G = G_2$. Otherwise,

if $i = 1, j = 0$, set $G = G_7 \circ G_2$.
If $n > 2$, there are two cases. If $i, j \neq n-1$, then by induction there is such a composition H for $\phi(v_0, \ldots, v_{n-2})$. In this case $G(x_0, \ldots, x_{n-1}) = G_1(H(x_0, \ldots, x_{n-2}), x_{n-1})$ works. Hence suppose $i = n-1$ or $j = n-1$. If $i, j \neq n-2$ then by induction we can find a composition H for $\phi(v_0, \ldots, v_{n-3}, v_{n-1}, v_{n-2})$ and so $G = G_8 \circ H$ suffices. We are left with the cases $i = n-2, j = n-1$ and $i = n-1, j = n-2$. In the former case, let H be the following composition of G_1 and G_2:

$$H(x_0, \ldots, x_{n-1}) = G_2(x_{n-1}, x_{n-2}) \times \left(\prod_{i<n-2} x_i \right)$$

Then $G = G_9 \circ H$ is as desired since

$$((u_{n-2}, u_{n-1}), (u_0, \ldots u_{n-3})) = (u_{n-2}, u_{n-1}, (u_0, \ldots, u_{n-3}))$$

and

$$((u_0, \ldots, u_{n-3}), u_{n-2}, u_{n-1}) = (u_0, \ldots, u_{n-3}, u_{n-2}, u_{n-1})$$

The latter case follows from the former by applying G_8.

$\phi = \neg\theta$: Here, we let $G(x_0, \ldots, x_{n-1}) = G_3(\prod_{i<n} x_i, H(x_0, \ldots x_{n-1}))$ where H is the composition of Gödel operations corresponding to θ. Notice that $\prod_{i<n} x_i$ is the result of successive nesting of G_1.

$\phi = \theta_0 \wedge \theta_1$: Find compositions H_0 and H_1 corresponding to θ_0 and θ_1 respectively. The following works out:

$$G(x_0, \ldots x_{n-1}) = G_4(H_0(x_0, \ldots, x_{n-1}), H_1(x_0, \ldots, x_{n-1}))$$

$\phi = \exists v_n \in v_i \ \theta(v_0, \ldots v_n)$: Let $H(x_0, \ldots, x_n)$ be a composition corresponding to

$$\theta(v_0, \ldots, v_n) \wedge v_n \in v_i$$

We now have the following:

$$\begin{aligned}
&\{(u_0, \ldots, u_{n-1}) \in \prod_{i<n} x_i | \phi(u_0, \ldots, u_{n-1})\} \\
&= \{(u_0, \ldots, u_{n-1}) \in \prod_{i<n} x_i | \exists v \in u_i \ \theta(u_0, \ldots, u_{n-1}, v)\} \\
&= dom(H(x_0, \ldots, x_{n-1}, \bigcup x_i)) = G_6(H(x_0, \ldots, x_{n-1}, G_5(x_i)))
\end{aligned}$$

□

Next, we define the correct cover property. The nomenclature we use is in line with the δ-cover property defined in chapter 2. A common name found in the literature is "almost universality".

Definition 6.2.5. We say that $M \subseteq V$ has the *Ord*-cover property if for any $x \in V$, $x \subseteq M$ there is $y \in M$ with $x \subseteq y$.

Proof. (Theorem 6.2.4) Let $\psi(r)$ be the following statement:

"W is transitive, contains all ordinals, is closed under all Gödel operations and has the Ord-cover property"

As the Gödel operations are absolute between transitive models, any inner model of ZF is closed under them. Moreover, if M is an inner model and $x \in V$, $x \subseteq M$, the absoluteness of the rank function implies that $x \subseteq M_\alpha$ for some α large enough. This shows that M has the *Ord*-cover property.

For the other direction, assume that $\psi(r)$ holds. We have to show that $M \models ZF$. The extensionality and set existence axioms as well as the foundation scheme hold in M as it is a nonempty transitive class. Pairing and union hold as M is closed under G_0 and G_5 respectively. The infinity axiom holds as $\omega \in M$.

Let us show that the separation scheme holds in M. As M is closed under the Gödel operations, M is also closed under all compositions of Gödel functions, so the above lemma shows that separation holds for all Δ_0-formulas. Every ϕ is equivalent to a formula of the form $Q_0 x_0 \dots Q_{n-1} x_{n-1} \psi$ for ψ some Δ_0-formula and $Q_i \in \{\exists, \forall\}$. This means we only have to show that the set of formulas for which separation holds is closed under $\exists$ and $\forall$.

$\phi = \exists x \psi$**:** We neglect additional parameters. Given $b \in M$, we must show $\{a \in b | \exists x \ \psi(a, b, x)\}^M \in M$. For any $a \in b$, if there is $x \in M$ with $\psi(a, b, x)^M$, then there is some minimal α_a with the property that there is such a witness in V_{α_a}. Taking $\alpha_\star = sup_{a \in b}\ \alpha_a$ yields a uniform α with this property. Now find $y \in M$ with $V_{\alpha_\star} \cap M \subseteq y$ using the *Ord*-cover property. The key idea is that $(\exists x \ \psi(a, b, x))^M$ is now equivalent to $(\exists x \in y \ \psi(a, b, x))^M$ for all $a \in b$. As M is closed under G_1, $b \times y \in M$. We can now use the inductive hypothesis for ψ to see that $b_0 = \{(a, x) \in b \times y | \psi(a, b, x)\}^M \in M$. We conclude:

$$\{a \in b | \exists x \ \psi(a, b, x)\}^M = dom(b_0) = G_6(b_0) \in M$$

$\phi = \forall x \psi$**:** Notice that separation also holds for $\neg\psi$ by closure of M under G_3. Then this case follows from the $\exists$-case:

$$\{a \in b | \forall x \psi(a, b, x)\} = b \backslash \{a \in b | \exists x \neg\psi(a, b, x)\}$$

The rest is easy now: To see that the power set axiom holds, let $x \in M$. Then $\mathcal{P}(x) \cap M \in V$ and $\mathcal{P}(x) \cap M \subseteq M$, so by the *Ord*-cover property, there is $y \in M$ with $\mathcal{P}(x) \cap M \subseteq y$. By Separation, $\mathcal{P}(x) \cap M = \{u \in y | u \subseteq x\} \in M$. If F is a class term such that F^M is a class function in M, and $x \in M$, then $F^M[x] \in V$ and $F^M[x] \subseteq M$, so there is $y \in M$ with $F^M[x] \subseteq y$. Again by separation, $F^M[x] \in M$. Thus the replacement scheme holds true in M as well. ☐

6.3 Bukovský's Theorem

In the proof of the strong Downwards Directed Grounds Hypothesis, we skipped the hard direction of Bukovskys Theorem, so let's do it here. The result was originally published in [Buk73]. We follow [FFS18] instead.

Theorem 6.3.1. *(Bukovský's Theorem) Suppose W is an inner model of ZFC and κ is a cardinal. Then W is a ground which extends to V via a κ-cc forcing if and only if $W \subseteq V$ has the κ-global cover property.*

Proposition 6.3.2. *If $W \subseteq M \subseteq V$ are inner models such that $W \subseteq V$ has the κ-global cover property, then $W \subseteq M$ and $M \subseteq V$ have the κ-global cover property, too.*

Proof. By Proposition 3.2.6 it is enough to show that $W \subseteq M$ and $M \subseteq V$ have the (κ, θ)-global cover property for every $\theta \geqslant \kappa$. For the first inclusion, any function we have to consider is a member of V and so this follows from the (κ, θ)-global cover property of $W \subseteq V$. For the latter, if $F \in W$ is a (κ, θ)-global cover of $f : \theta \to \mathcal{P}_\kappa(\theta)^V$, then the same is true for $F \in M$. □

First, we show that in the situation prescribed in the theorem, every set of ordinals in V is contained in a generic extension of W.

Lemma 6.3.3. *If W is an inner model of ZFC such that $W \subseteq M$ has the κ-global cover property and A is a set of ordinals in V, then there is a generic extension $W[G]$ of W with $A \in W[G] \subseteq V$.*

The proof presented here relies on infinitary logic, so we need a few tools first.

Definition 6.3.4. Let μ be an ordinal and κ a cardinal.

- (i) The language S_μ consists of one unary relation symbol $\dot{A}$ and constant symbols $\dot{\alpha}$ for $\alpha < \mu$.
- (ii) We define a logic, that we will call $\mathcal{L}_\kappa$, in languages that only contain symbols for constants and unary relations. The following are the rules of producing formulas:
  - (a) The only atomic formulas are $R(c)$ for constants c and unary relations R.
  - (b) $\neg\phi$ is a formula for any formula ϕ.
  - (c) $\bigvee \Phi$ is a formula for Φ a set of formulas of size $< \kappa$.

  We will write $c \in R$ instead of $R(c)$. We let $\mathcal{L}_\kappa(S)$ be the minimal set that contains all atomic formulas and is closed under the above rules. Since we are mainly interested in the languages S_μ, we write $\mathcal{L}_\kappa(\mu)$ for the set of formulas in the language S_μ.

(*iii*) Let $\mathbb{B}$ be the canonical boolean algebra on $0, 1$, which we understand as the set of truth values. We define satisfaction $\mathcal{M}(\phi) \in \mathbb{B}$ of $\phi \in \mathcal{L}_\kappa(\mu)$ in appropriate S structures $\mathcal{M}$ by induction:

(*a*) $\mathcal{M}(c \in R) = 1$ iff $c^{\mathcal{M}} \in R^{\mathcal{M}}$

(*b*) $\mathcal{M}(\neg\phi) = -\mathcal{M}(\phi)$

(*c*) $\mathcal{M}(\bigvee \Phi) = sup\{\mathcal{M}(\phi)|\phi \in \Phi\}$

Write $\mathcal{M} \models \phi$ instead of $\mathcal{M}(\phi) = 1$.

(*iv*) For any $B \subseteq \mu$ we can canonically define a S_μ-structure $\mathcal{M}_B$ on μ by putting $\dot{\alpha}^{\mathcal{M}_B} = \alpha$ and $\dot{A}^{\mathcal{M}_B} = B$. For readability we will write $B \models \phi$ instead of $\mathcal{M}_B \models \phi$.

Remark 6.3.5. We define the conjunction $\bigwedge \Phi$ as $\neg \bigvee\{\neg\phi|\phi \in \Phi\}$. Furthermore, we let $\phi \vee \psi$ abbreviate $\bigvee\{\phi, \psi\}$ and similarly $\phi \wedge \psi$. As usual, we define $\rightarrow$ and $\leftrightarrow$ from $\vee$ and $\neg$.
We should also be precise for formal correctness and define $\mathcal{L}_\kappa$-formulas as specific sets:

(*i*) "$c \in R$" $= \langle(c, R), 0\rangle$

(*ii*) $\neg\phi = \langle\phi, 1\rangle$

(*iii*) $\bigvee \Phi = \langle\Phi, 2\rangle$

The key steps heavily depend on a formal deductive system for the logic $\mathcal{L}_\kappa$, very similar to the sequent calculus for finitary first order logic, which we will also denote as $\vdash$.

Definition 6.3.6. We define base rules for $\vdash$. To make life easy, we take all rules that follow from the sequent calculus that do not involve equality and quantifiers (and the symbol for contradiction). In addition to this, we introduce the following infinitary deductive rules for any set of formulas $\Gamma, \Phi \subseteq \mathcal{L}_\kappa(S)$ with Φ of size $< \kappa$:

$$(\bigvee\text{-Introduction})\ \frac{\Gamma\ \vdash\ \phi}{\Gamma\ \vdash\ \bigvee \Phi \text{ for any } \Phi \text{ with } \phi \in \Phi}$$

$$(\bigwedge\text{-Introduction})\ \frac{\Gamma\ \vdash\ \phi \text{ for all } \phi \in \Phi}{\Gamma\ \vdash\ \bigwedge \Phi}$$

$$(\text{Infinite De Morgan})\ \frac{\Gamma\ \vdash\ \bigwedge\{\neg\phi|\phi \in \Phi\}}{\Gamma\ \vdash\ \neg\bigvee \Phi}$$

We write $\Gamma \vdash \phi$ for a theory Γ and formula ϕ if there is a formal proof of ϕ from Φ. In this context, a formal proof of ϕ from Φ is a sequence $\langle\phi_\alpha|\alpha \leqslant \gamma\rangle$ for some ordinal γ such that

(a) $\phi_\gamma = \phi$

(b) For each $\alpha \leqslant \gamma$, ϕ_α is either an axiom of Φ or is the result of applying a basic rule to a subset of $\{\phi_\beta | \beta < \alpha\}$.

Proposition 6.3.7. *The deductive system $\vdash$ has the following properties:*

(i) *$\vdash$ is correct, i.e. if $\Gamma \vdash \phi$ then $\mathcal{M} \models \phi$ for any S-structure $\mathcal{M} \models \Gamma$.*

(ii) *$\vdash$ is upwards absolute, i.e. if $M \subseteq N$ are transitive models of ZFC then $(\Gamma \vdash \phi)^M$ implies $(\Gamma \vdash \phi)^N$*

Proof. (i) The correctness of a rule purely depends on its syntactic properties and not on the logical context and formulas allowed in that rule, as long as satisfaction and structures are defined in the right way. In particular, every rule that is deductible from the sequent calculus is correct in our context, since the sequent calculus is correct. Suppose $\mathcal{M} \models \Gamma$.
($\bigvee$-Introduction) If $\Gamma \vdash \phi$ for some $\phi \in \Phi$ then by induction, $\mathcal{M} \models \phi$ and hence $\mathcal{M} \models \bigvee \Phi$.
($\bigwedge$-Introduction) Assume $\Gamma \vdash \phi$ for all $\phi \in \Phi$. It is easy to see that $\mathcal{M}(\bigwedge \Phi) = inf\{\mathcal{M}(\phi) | \phi \in \Phi\}$. By induction $\mathcal{M} \models \phi$ for all $\phi \in \Phi$ and so $\mathcal{M} \models \bigwedge \Phi$.
(Infinite De Morgan) Assume $\Gamma \vdash \bigwedge\{\neg\phi | \phi \in \Phi\}$. By induction $\mathcal{M} \models \bigwedge\{\neg\phi | \phi \in \Phi\}$ and thus $\mathcal{M} \models \neg\phi$ for all $\phi \in \Phi$. If $\mathcal{M} \models \bigvee \Phi$ then there must be $\phi \in \Phi$ with $\mathcal{M} \models \phi$, a contradiction. Hence $\mathcal{M} \models \neg \bigvee \Phi$.

(ii) By induction on the construction of $\mathcal{L}_\kappa$-formulas, it follows directly that $\mathcal{L}_\kappa(S)^M \subseteq \mathcal{L}_\kappa(S)^N$ for any appropriate language $S \in M$. Assume $(\Gamma \vdash \phi)^M$. Let $\langle \phi_\alpha | \alpha \leqslant \gamma \rangle$ be a formal proof of ϕ from Φ in M. By induction on $\beta \leqslant \gamma$ we see that $\langle \phi_\alpha | \alpha \leqslant \beta \rangle$ is a formal proof in N, since M, N agree on what axioms are in Γ, what the basic rules of $\vdash$ are and N contains all sets of formulas that are members of M.

□

Proof. (Lemma 6.3.3) Let $A \subseteq \mu$ be a set of ordinals in V. The idea is that we can approximate the $\mathcal{L}_\kappa(\mu)$ theory of $\mathcal{M}_A$ inside of W using the κ-global cover property. In a natural way, A will then define a generic for the forcing consisting of these approximations.
We will work in the logic $\mathcal{L}_{(2^\kappa)^+}$. Notice that $(2^\kappa)^+$ is the same in W and V as a consequence of the κ-global cover property. Furthermore, $\mathcal{L}_\kappa(\mu) \subseteq \mathcal{L}_{(2^\kappa)^+}(\mu)$. In V, we can find a choice function f on $\mathcal{P}(\mathcal{L}_\kappa(\mu))^W \setminus \{\varnothing\}$ so that $A \models \bigvee \Phi$ implies $A \models f(\Phi)$. By the κ-global cover property, there is a global cover F of f in W. We can assume that $F(\Phi) \subseteq \Phi$. Let

$$\Gamma = \{\bigvee \Phi \rightarrow \bigvee F(\Phi) | \Phi \in dom(F)\}$$

and notice that $A \models \Gamma$ by definition of F and f. Work in W. We define the forcing as all $\mathcal{L}_\kappa(\mu)$-formulas that cannot be falsified from Γ in the deductive system $\vdash$

$$\mathbb{P} = \{\phi \in \mathcal{L}_\kappa(\mu) | \Gamma \nvdash \neg\phi\}$$

ordered by $\phi \leqslant \psi$ iff $\Gamma \vdash \phi \to \psi$.

Claim 6.3.8. *$\phi, \psi \in \mathbb{P}$ are compatible if and only if $\Gamma \nvdash (\phi \to \neg\psi)$ (which is equivalent to $\Gamma \nvdash \neg(\phi \wedge \psi)$).*

Proof. If $\theta \leqslant \phi, \psi$, then $\Gamma \vdash (\theta \to \phi)$ and $\Gamma \vdash (\theta \to \psi)$. Assuming $\Gamma \vdash (\phi \to \neg\psi)$, we can conclude $\Gamma \vdash \neg\theta$ as we have adapted all syntactical rules of the first order sequent calculus. But this is a contradiction to $\theta \in \mathbb{P}$.

On the other hand, if $\phi, \psi \in \mathbb{P}$ are incompatible then $\phi \wedge \psi \notin \mathbb{P}$. Thus $\Gamma \vdash \neg(\phi \wedge \psi)$. □

Let's show that $\mathbb{P}$ has the κ-cc. Suppose that $\Phi \subseteq \mathbb{P}$ is an antichain. Since $F(\Phi)$ has size $< \kappa$ it is enough to show $\Phi = F(\Phi)$, so suppose $\phi \in \Phi$. Combining $(\bigvee \Phi \to \bigvee F(\Phi)) \in \Gamma$ with the rule $\vdash (\phi \to \bigvee \Phi)$ yields

$$\Gamma \vdash (\phi \to \bigvee F(\Phi))$$

If there is no $\psi \in F(\Phi)$ compatible with ϕ then

$$\begin{array}{rcl|l}
\Gamma & \vdash & (\phi \to \bigvee F(\Phi)) & \\
\Gamma & \vdash & (\phi \to \neg\psi) \text{ for all } \psi \in F(\Phi) & \\
\hline
\Gamma, \phi & \vdash & \neg\psi \text{ for all } \psi \in F(\Phi) & \text{(Assumption Rule \& Modus Ponens)} \\
\Gamma, \phi & \vdash & \bigwedge\{\neg\psi | \psi \in F(\Phi)\}) & (\bigwedge\text{-Introduction}) \\
\Gamma, \phi & \vdash & \neg \bigvee F(\Phi) & \text{(Infinite De Morgan)} \\
\Gamma & \vdash & (\phi \to \neg \bigvee F(\Phi)) & (\to\text{-Introduction}) \\
\hline
\Gamma & \vdash & \neg\phi & (\neg\text{-Introduction with line 1})
\end{array}$$

contradicting $\phi \in \mathbb{P}$. So there is $\psi \in F(\Phi) \subseteq \Phi$ compatible with ϕ, but Φ is an antichain, so $\phi = \psi \in F(\Phi)$. We want to find a generic for $\mathbb{P}$ from which A is definable. The canonical choice is $G_A = \{\phi \in \mathbb{P} | A \models \phi\}$.

Claim 6.3.9. *G_A is a $\mathbb{P}$-generic filter over W.*

Proof. If $\phi \in G_A$ and $\phi \leqslant \psi$ then $\Gamma \vdash (\phi \to \psi)$ in W. As $A \models \Gamma \cup \{\phi\}$, we have $A \models \psi$ and thus $\psi \in G_A$. If $\phi, \psi \in G_A$, then $A \models (\phi \wedge \psi)$ and hence by upwards absoluteness and correctness of $\vdash$ we get $\Gamma \nvdash \neg(\phi \wedge \psi)$ in W and so by the above claim, ϕ and ψ are compatible witnessed by $\phi \wedge \psi \in G_A$. This shows that G_A is a filter. To show that G_A is generic over W, suppose that $\Phi \in W$ is a maximal antichain of $\mathbb{P}$ in W. We have already seen that $|\Phi| < \kappa$ so that $\bigvee \Phi \in \mathcal{L}_\kappa(\mu)$. In addition to this, it is the case that $\bigvee \Phi \in \mathbb{P}$: For every $\phi \in \Phi$ we have $\vdash (\phi \to \bigvee \Phi)$ as one of our deduction rules. As $\Gamma \nvdash \neg\phi$, it follows from contraposition and modus ponens that $\Gamma \nvdash \neg \bigvee \Phi$. Moreover, $\Gamma \vdash \bigvee \Phi$. Otherwise, $\neg \bigvee \Phi \in \mathbb{P}$ by elemination of double negation. Using the deductive rule $\vdash (\phi \rightarrow \bigvee \Phi)$ again together with contraposition, it follows that $\neg \bigvee \Phi$ is incompatible with every $\phi \in \Phi$, contradicting its maximality. Finally, we can conclude $A \models \bigvee \Phi$ using $A \models \Gamma$ as well as the correctness and upwards absoluteness $\vdash$. By definition of f, $A \models f(\Phi)$. Hence $f(\Phi) \in G_A \cap \Phi$. □

As $G_A \in V$, we have that $W[G_A] \subseteq V$. Observe that "$\dot{\alpha} \in \dot{A}$" $\in \mathbb{P}$ if $\alpha \in A$ and "$\neg\dot{\alpha} \in \dot{A}$" $\in \mathbb{P}$ if $\alpha \notin A$, as this follows from upwards absoluteness of $\vdash$ from W to V, the correctness of $\vdash$ and $A \models \Gamma$. This implies

$$A = \{\alpha < \mu | \text{“}\dot{\alpha} \in \dot{A}\text{”} \in G_A\}$$

and thus $A \in W[G_A]$. □

Proof. (Bukovský's Theorem)" $\Leftarrow$ " Using the notation of Lemma 6.3.3, it is enough to show that there is an A with $W[G_A] = V$. Let A be a set of ordinals that codes $({}^{<\kappa^+}2)^V$. Then $A \in W[G_A] \subseteq V$ and $({}^{<\kappa^+}2)^{W[G_A]} = ({}^{<\kappa^+}2)^V$. Now $W[G_A] \subseteq V$ still has the κ-global cover property by Proposition 6.3.2. A consequence of this is that $W[G_A]$ and V have the same cardinals above κ. Otherwise, there is a surjection $f : \gamma \to \lambda$ for some $\gamma < \lambda$ cardinals in $W[G_A]$. Now we can find a κ-global cover $F \in W[G_A]$ of f. But then $\lambda \subseteq F[\kappa]$, contradicting $|F[\gamma]|^{W[G_A]} \leqslant \gamma \cdot \kappa < \lambda$. In particular, κ^{++} is absolute between $W[G_A]$ and V. Combining Lemma 3.2.7 with Proposition 3.2.6 yields that $W[G_A] \subseteq V$ has the κ^+-cover and approximation properties for all sets of ordinals, and thus the full properties. This was the last ingredient we need to conclude $W[G_A] = V$ using Lemma 2.1.9. □

The quotient lemma, that we have also used in a prior chapter, is a nice consequence of the above:

Corollary 6.3.10. *Suppose that W is a ground and M an inner model such that $W \subseteq M \subseteq V$. Then W is a ground of M and M is a ground of V. Moreover, if λ is (strongly) inaccessible and W is a λ-ground of V then so is M.*

Proof. One can see that W is a ground of M and M is a ground of V by combining Proposition 6.3.2 with Bukovský's Theorem. For the second part, find $G \in W_\lambda$ such that $W[G] = V$. Code G as a bounded subset A of λ. By applying Lemma 6.3.3 inside V_λ, we find that there is a forcing $\mathbb{P} \in M_\lambda$ and $G_A \in V_\lambda$ $\mathbb{P}$-generic over M_λ such that $A \in M_\lambda[G_A]$. As M and M_λ contain the same subsets of $\mathbb{P}$, G_A is $\mathbb{P}$-generic over M. Thus $A \in M[G_A]$ and hence by decoding, $G \in M[G_A]$. As $W \subseteq M$, $V = W[G] \subseteq M[G_A] \subseteq V$. We conclude that M is a λ-ground. □

Remark 6.3.11. As this result is in some way the dual of the product lemma and since the standard way to proof it, is to show that if W extends to V via a separative forcing $\mathbb{P}$, then M extends to V via a quotient of the unique boolean algebra that $\mathbb{P}$ can be densely embedded into, it is referred to as the quotient lemma in this thesis. The result is originally due to Serge Grigorieff [Gri75]. By proving it with boolean algebras, it is not necessary to assume that λ is inaccessible. With a more thorough analysis of inner models that satisfy the κ-global cover property it is also possible to eliminate this assumption without using boolean algebras.

6.4 Laver Indestructibility

In this section we will prove the following result:

Theorem 6.4.1. *If κ is supercompact then there is a forcing extension in which κ is Laver indestructible.*

The result is due to Laver [Lav78], hence the name. We follow [Cum10, Chapter 24]. We will make quite extensive use of the ultrafilter definition of a supercompact cardinal.

Definition 6.4.2. Suppose $\kappa \leqslant \lambda$ and U is an ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

(*i*) U is called uniform if $\{X \in \mathcal{P}_\kappa(\lambda) | Y \subseteq X\} \in U$ for all $Y \in \mathcal{P}_\kappa(\lambda)$.

(*ii*) U is normal if it is $< \kappa$-complete, uniform and moreover, for any sequence $(A_\alpha)_{\alpha<\lambda}$ in U, the diagonal intersection

$$\triangle_{\alpha<\lambda} A_\alpha = \{X \in \mathcal{P}_\kappa(\lambda) | X \in \textstyle\bigcap_{\alpha \in X} A_\alpha\}$$

is again in U.

Fact 6.4.3. *[Kan09, Theorem 22.7] For any cardinals $\kappa \leqslant \lambda$, κ is λ-supercompact if and only if there is a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.*

The proof of the above fact proceeds as follows: If there is a normal ultrafilter U on $\mathcal{P}_\kappa(\lambda)$, then we may build the ultrapower embedding $j_U : V \to M = Ult(V, U)$. As usual, M is wellfounded and will be identified with its transitive collapse. The fineness condition implies that j_U is non-trivial and one can compute that $crit(j) = \kappa$. Using the normality condition, one can show that $j_U[\lambda] \in M$ and thus (similar to Lemma 1.4.10) that ${}^\lambda M \subseteq M$. On the other hand, given an elementary embedding $j : V \to M$ with critical point κ and ${}^\lambda M \subseteq M$, it is not difficult to prove that

$$U = \{X \subseteq \mathcal{P}_\kappa(\lambda) | j[\lambda] \in j(X)\}$$

is a normal ultrafilter. Below, we will take advantage of the above arguments.

Lemma 6.4.4. *If κ is supercompact then there is a function $f : \kappa \to V_\kappa$ so that for any $\lambda \geqslant \kappa$ and $x \in H_{\lambda^+}$ there is a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$ such that if j_U is the associated elementary embedding then $j(f)(\kappa) = x$.*

Proof. We take a "least counterexample" approach. Let $\prec$ be a wellorder of V_κ. Define f by induction on $\alpha < \kappa$. Suppose $f \upharpoonright \alpha$ is already defined. If there is some $\alpha \leqslant \lambda < \kappa$ and $x \in H_{\lambda^+} \subseteq V_\kappa$ so that there is no normal ultrafilter U on $\mathcal{P}_\alpha(\lambda)$ so that with $j_U(f \upharpoonright \alpha)(\alpha) = x$ then we let $f(\alpha)$ be the $\prec$-least such x. Otherwise $f(\alpha) = 0$.
Suppose that f does not have the desired property. Let $\lambda \geqslant \kappa$ and $x \in H_{\lambda^+}$ so that there is no normal ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ with $j_U(f)(\kappa) = x$. Let $\delta = 2^{(2^\lambda)}$ and find a δ-supercompactness embedding

$$j : V \to M$$

with critical point κ. The closure condition of M yields that $(H_{\lambda^+})^M = H_{\lambda^+}$ and $\mathcal{P}(\mathcal{P}_\kappa(\lambda))^M = \mathcal{P}(\mathcal{P}_\kappa(\lambda))$. By assumption, M and j are definable in V and since V is definable in every extension by a forcing in M, M and its corresponding extension are two, as well as all further embeddings we construct.

Claim 6.4.5. *M believes that there is no normal ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ with $j_U(f)(\kappa) = x$.*

Proof. Suppose otherwise that there is such a U. Then U is a normal ultrafilter in V, too. We thus may construct embeddings $j_K : K \to Ult(K, U)$ for $K = V, M$ and compare the two. Since M is closed under sequences of length δ, there are the same functions $\mathcal{P}_\kappa(\lambda) \to H_{\lambda^+}$. Furthermore, if g, h are such functions then

$$V \models g \sim_U h \Leftrightarrow M \models g \sim_U h$$

and hence the embeddings j_U and j_M coincide on H_{λ^+}. In particular, $j_V(f) = j_M(f)$ and even more, $j_V(f)(\kappa) = j_M(f)(\kappa) = x$, a contradiction. □

By elementarity, $j(f)$ is defined in M over the same induction as in V, only with parameters replaced by their images under j. Since $j(f) \upharpoonright \kappa = f$, $j(f)$ is defined non-trivially at κ in M. We now find the value of $j(f)(\kappa)$. Let μ be minimal so that there is $y \in H_{\mu^+}$ so that for no normal ultrafilter U on $\mathcal{P}_\kappa(\mu)$ we have $j_U(f)(\kappa) = y$. Notice that necessarily $\mu \leqslant \lambda$ and thus $y \in H^M_{\mu^+}$. Let z be the $j(\prec)$-minimal such y. As in the above claim, M, too, is of the opinion that for no normal measure U on $\mathcal{P}_\kappa(\mu)$ we have $j_U(f)(\kappa) = y$. Thus in the inductive definition of $j(f)$, $j(f)(\kappa) = z$. We define a filter on $\mathcal{P}_\kappa(\mu)$:

$$U = \{X \subseteq \mathcal{P}_\kappa(\mu) | j[\mu] \in j(X)\}$$

The standard arguments show, using ${}^{\mu}M \subseteq M$, that U is a normal ultrafilter. Let

$$i : V \to Ult(V, U) = N$$

be the induced elementary embedding. We now need to be able to go from N to M, therefore we define the following map:

$$k : N \to M,\ k([g]_U) = j(g)(j[\mu])$$

k is well-defined since if $g \sim_U h$ then $j(g)$ and $j(h)$ coincide on input $j[\mu]$.

Claim 6.4.6. *k has the following properties:*

(i) *k is elementary.*

(ii) $k \circ i = j$

(iii) $j[\mu] \in ran(k)$

(iv) $k \upharpoonright H_{\mu^+} = id_{H_{\mu^+}}$

Proof. *(i)* Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula. We may suppose that $n = 1$ and let $[g]_U \in M$. Using Łoś's Theorem, we compute:

$$N \models \varphi([g]_U) \Rightarrow X := \{s \in \mathcal{P}_\kappa(\mu) | \varphi(g(s))\} \in U \Rightarrow j[\mu] \in j(X)$$
$$\Rightarrow M \models \varphi(j(g)(j[\mu])) \Rightarrow M \models \varphi(k([g]_U))$$

Thus k is elementary.

(ii) If $a \in V$, then $k(i(a)) = k([c_a]_U) = j(c_a)(j[\mu]) = j(a)$ where c_a is the constant function on $\mathcal{P}_\kappa(\mu)$ with value a. Thus $k \circ i = j$.

(iii) Let g be the identity on $\mathcal{P}_\kappa(\mu)$. Then $k(g) = j(g)(j[\mu]) = j[\mu]$ by elementarity of j.

(iv) We will show that $H_{\mu^+} \subseteq ran(k)$. If we have this, then k "cannot skip a set in H_{μ^+}" and we can see easily by induction that the claim holds. Since we can code every $A \in H_{\mu^+}$ as a subset of μ in an absolute way, it is enough to show $\mathcal{P}(\mu) \subseteq ran(k)$. Given $X \subseteq \mu$, we will show that

$$X = \{otp(\gamma \cap j[\mu]) | \gamma \in j[\mu] \cap j(X)\}$$

First, suppose $\alpha \in X$. Then $j^{-1} : j(\alpha) \cap j[\mu] \to \alpha$ is the transitive collapse and hence $\alpha = otp(j(\alpha) \cap j[\mu])$. On the other hand, if $\gamma \in j[\mu] \cap j(X)$ we can write $\gamma = j(\alpha)$ for some $\alpha \in X$. By $(i), (ii)$ and (iii), the above representation of X shows that $X \in ran(k)$.

□

In particular, $k(z) = z$ and $k(\kappa) = \kappa$ and thus:

$$k(i(f)(\kappa)) = k \circ i(f)(k(\kappa)) = j(f)(\kappa) = z$$

and by applying (iv), we have $k(i(f)(\kappa)) = i(f)(\kappa)$ as $z \in H_{\mu^+}$ But then $i(f)(\kappa) = z$, a contradiction to the choice of z and μ.

□

With such a so called Laver function f, we can prove the main theorem.

Proof. (Theorem 6.4.1) Let f be a Laver function for κ. We define a Easton supported iteration $(\langle \mathbb{P}_{<\alpha} | \alpha < \kappa \rangle, \langle \dot{\mathbb{Q}}_\alpha | \alpha < \kappa \rangle)$ by induction on α. Suppose $\mathbb{P}_{<\alpha}$ is already defined. Assume the following conditions are met:

(i) $f(\alpha)$ is a pair of the form $(\gamma, \dot{\mathbb{Q}})$ where $\dot{\mathbb{Q}}$ is a $\mathbb{P}_{<\alpha}$-name for a $<\alpha$-directed closed forcing.

(ii) For all $\beta < \alpha$, if $f(\beta)$ is a pair with first coordinate an ordinal δ, then $\delta < \gamma$.

In this case, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}$. Otherwise, $\dot{\mathbb{Q}}_\alpha$ is the name for the trivial forcing. f acts as a bookkeeping function. We use the first coordinates to make sure that we have large intervals at which the forcing is trivial. This will come in handy later on.
We must show that $\mathbb{P} = \mathbb{P}_{<\kappa}$ forces κ to be Laver indestructible. Let G be $\mathbb{P}$-generic and suppose that $\mathbb{Q} \in V[G]$ is a $<\kappa$-directed closed forcing. Find a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ in V. Suppose g is $\mathbb{Q}$-generic over $V[G]$. It is enough to show that κ is λ-supercompact in $V[G][g]$ for all cardinals λ large enough so that $\dot{\mathbb{Q}} \in H_\lambda^V$. Set $\delta = 2^{(2^\lambda)}$ and let U be a normal ultrafilter on $\mathcal{P}_\delta(\kappa)$ so that the induced embedding $j : V \to Ult(V, U) = M$ satisfies $j(f)(\kappa) = (\mu, \dot{\mathbb{Q}})$. Again, $j(f) \upharpoonright \kappa = f$ and since $ran(f) \subseteq V_\kappa$, the first coordinate of $j(f)(\alpha) = f(\alpha)$ so that this is a pair with first coordinate an ordinal, is less than μ. Notice that $j(\mathbb{P})$ is an iteration of length $j(\kappa) > \kappa$ with $j(\mathbb{P})_{<\kappa} = \mathbb{P}$ and thus by elementarity, $\dot{\mathbb{Q}}$ is the forcing in $j(\mathbb{P})$ at stage κ. We can find a canonical name $\dot{\mathbb{R}}$ so that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \cong j(\mathbb{P})$ and let $\mathbb{R}$ be the evaluation of $\dot{\mathbb{R}}$ in $V[G * g]$. Suppose H is $\dot{\mathbb{R}}$-generic over $V[G * g]$. Since $\mathbb{P}$ is Easton supported and κ inaccessible in V, we see that $\mathbb{P} \subseteq V_\kappa$ (recall that $ran(f) \subseteq V_\kappa$) and hence $j[G] = G \subseteq G * g * H$. Lemma 1.4.2 shows that we may lift j to

$$j^+ : V[G] \to M[G * g * H]$$

Next up, we have to further lift j^+ to an embedding with domain $V[G * g]$.

Claim 6.4.7. *There is h a $j^+(\mathbb{Q})$-generic filter over $V[G * g * H]$ so that $j^+[g] \subseteq h$.*

Proof. Notice that exactly as in the proof of Proposition 4.3.8, we can proof that

$$^{\mu}M[G] \subseteq M[G]$$

from the perspective of $V[G]$ and

$$^{\mu}M[G * g] \subseteq M[G * g]$$

from the perspective of $V[G * g]$ and last but not least,

$$^{\mu}M[G * g * H] \subseteq M[G * g * H]$$

from the perspective of $V[G*g*H]$. In particular, already $M[G]$ can see that $\mathbb{Q}$ is of size $< \lambda$ and $j \upharpoonright \mathbb{Q} \in M[G*g*H]$. Now we have $j^+[g] \in M[G*g*H]$ and moreover, $j^+[g]$ is directed there and of size $< j^+(\kappa)$. By elementarity, $j^+(\mathbb{Q})$ is $< j^+(\kappa)$-directed closed in $M[G*g*H]$ and thus there is a condition $p \in j^+(\mathbb{Q})$ below $j^+[g]$. Thus any $j^+(\mathbb{Q})$-generic filter with $p \in h$ suffices. $\square$

In $V[G * g * H * h]$, we may lift j^+ to

$$j^{++} : V[G * g] \to M[G * g * H * h]$$

The problem that we have to deal with is that j^{++} lives in $V[G * g * H * h]$ as opposed to $V[G * g]$. To solve this, work in $V[G * g * H * h]$ and let

$$U' = \{X \in \mathcal{P}(\mathcal{P}_\kappa(\lambda))^{V[G*g]} | j^{++}[\lambda] \in j^{++}(X)\}$$

If we are able to show $U' \in V[G * g]$ then we are done as U' would be a λ-normal ultrafilter there. Our strategy will be to show that the two further forcings we have used to get from $V[G * g]$ to $V[G * g * H * h]$ are sufficiently closed so they could not have added U'. First, we deal with the later extension. Observe that $j(\mathbb{Q}) = j^+(\mathbb{Q})$ is $< \mu$-closed in $M[G*g*H]$ by elementarity of j^+ and thus also has this property in $V[G*h*H]$ since they have the same sequences of length μ. This shows that this extension did not add new subsets of $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$. In particular not U'. Now, we can finally make use of $j(f)(\kappa)$ having first coordinate μ. This implies that the stages in the interval $(\kappa, \mu]$ in $j(\mathbb{P})$ are all trivial and since all stages after that are at least $< \mu$-closed, $\mathbb{R}$ could not have added U', too. We can conclude $U' \in V[G * g]$ and hence κ is λ-supercompact in $V[G * g]$.

$\square$

6.5 Preserving n-Superhuge Cardinals

We follow section 6 in [Tsa16]. Let $\mathbb{P}$ be the canonical Easton supported class iteration that forces GCH. The goal of this section is to prove the following theorem:

Theorem 6.5.1. *After forcing with $\mathbb{P}$, any $(n+1)$-superhuge cardinal remains n-superhuge.*

We lose one degree of superhugeness in the proof as we isolate a property $P(\kappa, n)$ that is implied by $(n+1)$-superhugeness, but itself only entails n-superhugeness of κ. Our strategy will be to show that $\mathbb{P}$ preserves the property $P(\kappa, n)$. It is an open question whether or not one degree of superhugeness can possibly be lost, i.e. if $\mathbb{P}$ can destroy the n-superhugeness of a n-superhuge cardinal.

Lemma 6.5.2. *Suppose κ is a cardinal.*

(i) If κ is $(n+1)$-superhuge then for all $\lambda > \kappa$ there is an elementary embedding

$$j : H_{j^n(\kappa)^+} \to H_{j^{n+1}(\kappa)^+}$$

with critical point κ and $j(\kappa) > \lambda$.

(ii) If $n \geqslant 1$ and the conclusion of (i) holds then κ is n-superhuge.

Proof. (i) Suppose κ is $(n+1)$-superhuge. Let $\lambda > \kappa$. Find an embedding

$$j : V \to M$$

for a transitive inner model M with $crit(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{j^{n+1}(\kappa)}M \subseteq M$. Let k be the restriction of j to $H_{j^n(\kappa)^+}$. As a consequence of the elementarity of j,

$$k : H_{j^n(\kappa)^+} \to \left(H_{j^{n+1}(\kappa)^+}\right)^M$$

is an elementary embedding with critical point κ and $j(\kappa) > \lambda$. Now notice that the closure condition of M implies that M computes the successor of $j^{n+1}(\kappa)$ correctly and contains all transitive sets of size $j^{n+1}(\kappa)$. Thus:

$$\left(H_{j^{n+1}(\kappa)^+}\right)^M = H_{j^{n+1}(\kappa)^+}$$

As moreover $k^i(\kappa) = j^i(\kappa)$ for all $i \leqslant n+1$, k is as desired.

(ii) Let $\lambda > \kappa$ and suppose that

$$j : H_{j^n(\kappa)^+} \to H_{j^{n+1}(\kappa)^+}$$

is an elementary embedding with critical point κ and $j(\kappa) > \lambda$. For notational ease, we let $\theta = j^n(\kappa)$. As in the case of elementary embeddings from the universe into a transitive class, we can build $\mathcal{E}$ the $(\kappa, j(\theta))$-extender derived from j. Note that not just the induced embeddings

$$Ult(H_{\theta^+}, E_a) \to Ult(H_{\theta^+}, E_b)$$

are directed for $a \subseteq b \in j(\theta)^{<\omega}$, but so are

$$j_{ab} : Ult(V, E_a) \to Ult(V, E_b)$$

We thus can construct the extender embedding

$$j_{\mathcal{E}} : V \to M_{\mathcal{E}}$$

as the directed limit of this system. We denote by E the element relation on $M_{\mathcal{E}}$.

Claim 6.5.3. *$\langle M_{\mathcal{E}}, E \rangle$ is wellfounded.*

Proof. Observe that for any $a \in j(\theta)^{<\omega}$, $Ult(V, E_a)$ is wellfounded as E_a is countably closed. Suppose that there is a decreasing sequence $(x_n)_{n<\omega}$ with $x_{n+1} E x_n$ for all $n < \omega$. Note that Fact 1.4.6 (i) is still true in our case so that we can find $a_n \in j(\theta)^{<\omega}$ and functions $f_n : \theta^{|a_n|} \to V$ so that $j_{\mathcal{E}}(f_n)(a_n) = x_n$. Since $j_{\mathcal{E}}(f_n)(a_n) = j_{a_n \mathcal{E}}([f_n]_{E_{a_n}})$, this implies that for all $n < \omega$ we have

$$[f_{n+1}]_{E_{a_{n+1}}} \in j_{a_n a_{n+1}}([f_n]_{E_{a_n}})$$

and thus:

$$A_n := \{c \in \theta^{|a_{n+1}|} | f_{n+1}(c) \in f_n \circ \pi_{a_n a_{n+1}}(c)\} \in E_{a_{n+1}}$$

Observe that the existence of a sequence of functions $(g_n)_{n<\omega}$ with

$$A_n = \{c \in \theta^{|a_{n+1}|} | g_{n+1}(c) \in g_n \circ \pi_{a_n a_{n+1}}(c)\}$$

is a Σ_1-statement with parameters $(A_n)_{n<\omega}, (a_n)_{n<\omega} \in H_{\theta^+}$. Proposition 6.1.2 shows that there is such a sequence $(g_n)_{n\in\omega}$ in H_{θ^+}. The elementarity of j together with the definition of the extender ultrafilters gives

$$j(g_{n+1})(a_{n+1}) \in j(g_n \circ \pi_{a_n a_{n+1}})(a_{n+1})$$

or equivalently

$$j(g_{n+1})(a_{n+1}) \in j(g_n)(a_n)$$

for all $n < \omega$, a contradiction. □

Thus we may assume that $M_{\mathcal{E}}$ is transitive and $E = \in \cap (M_{\mathcal{E}} \times M_{\mathcal{E}})$.

Claim 6.5.4. $j[\theta] \in M_{\mathcal{E}}$

Proof. We compare $j_{\mathcal{E}}$ to the derived embedding

$$j'_{\mathcal{E}} : H_{\theta^+} \to M'_{\mathcal{E}}$$

on the domain of j.

It is standard to see that

$$j'_a : H_{\theta^+} \to Ult(H_{\theta^+}, E_a)$$

is the restriction of

$$j_a : V \to Ult(V, E_a)$$

to H_{θ^+} for any $a \in \theta^{<\omega}$. By going to directed limits, it follows that $j'_{\mathcal{E}}$ is the restriction of $j_{\mathcal{E}}$ to H_{θ^+}. Since κ is strongly inaccessible, $H_{j(\theta)^+}$ believes that $j(\theta)$ is strongly inaccessible and thus it is really strongly inaccessible. Let $\delta = sup\ j[\lambda]$. Since $j(\theta)$ is regular, $\delta < j(\theta)$. Now, it follows from Fact 1.4.6 (ii) that $j[\theta] \in V_{\delta+1} \subseteq M'_{\mathcal{E}}$. Find $a \in j(\theta)^{<\omega}$ and $f : \theta^{|a|} \to H_{\theta^+}$ so that $j[\theta] = j'_{\mathcal{E}}(f)(a)$. Then as $j'_{\mathcal{E}}(f) = j_{\mathcal{E}}(f)$, we have

$$j[\theta] = j_{\mathcal{E}}(f)(a) \in M_{\mathcal{E}}$$

□

Now the same argument as in the above claim shows that Fact 1.4.6 (ii) implies

$$j(x) = j'_{\mathcal{E}}(x) = j_{\mathcal{E}}(x)$$

holds for any $x \in H_{j(\theta)} = V_{j(\theta)}$. In particular, $j^i_{\mathcal{E}}(\kappa) = j^i(\kappa)$ for all $i \leqslant n$ and $j_{\mathcal{E}}[\theta] = j[\theta] \in M_{\mathcal{E}}$ so that Proposition 1.4.10 yields ${}^{\theta}M_{\mathcal{E}} \subseteq M_{\mathcal{E}}$. Thus κ is n-superstrong.

□

The following proof is a slight modification of the argument in [Tsa16] where the above theorem was proven for $n = 2$ since this was the only instance relevant for the main interest of that paper. In contrast to this, we applied it with $n = 3$ and thus will go ahead and show it for all $n < \omega$. By $\circledast_{i<n}$ we will denote the operator for iterated two step iterations. We will make use of the weak homogeneity of $Add(\kappa, 1)$. This property means that for any p, q there is an automorphism f of the forcing so that $f(p)$ is compatible with q. For $Add(\kappa, 1)$, this can be seen by finding a suitable permutation of κ such that the image of $dom(p)$ is disjoint from $dom(q)$ and taking the induced automorphism on $Add(\kappa, 1)$. It is not hard to see that this weak homogeneity property still holds for (every inital segment of) $\mathbb{P}$.

Proof. (Theorem 6.5.1) Suppose κ is $(n+1)$-superhuge and G is $\mathbb{P}$-generic. We wand to show that the conclusion of Lemma 6.5.2 (i) holds. We know that it holds in V. So given $\lambda > \kappa$, let

$$j : H_{j^n(\kappa)^+} \longrightarrow H_{j^{n+1}(\kappa)^+}$$

be an elementary embedding with critical point κ and $j(\kappa) > \lambda$.

We want to apply Lemma 1.4.2 to j. Note that the exact same proof works out in the present setting even though the domain of j is only a model of ZFC^-. Basically, the only additional thing we need there is that we can apply the forcing theorem in both the domain and the target model which holds true by Fact 1.5.3. Later on, we will apply that result to elementary embeddings whose domain and target models are the H_δ of some intermediate forcing extension of V and $V[G]$, however not of the same one. It is not difficult to see that we still may apply this lemma in that situation.

For notational simplicity we let $\kappa_i = j^i(\kappa)$ for $i \leqslant n$ (where $\kappa_0 = \kappa$). For Δ an interval of ordinals with minimum α, we denote by $\dot{\mathbb{P}}_\Delta$ the $\mathbb{P}_{<\alpha}$-name for the iteration with stages in Δ. $\mathbb{P}_\Delta$ denotes the evaluation of $\dot{\mathbb{P}}_\Delta$ in $V[G_{<\alpha}]$. We let $\Delta_i = [\kappa_i, \kappa_{i+1})$. Factor $\mathbb{P}$ as

$$\mathbb{P}_{<\kappa} * \left(\mathop{\ast}_{i<n} \dot{\mathbb{P}}_{\Delta_i} \right) * \dot{\mathbb{P}}_{\geqslant \kappa_n}$$

and the generic accordingly. We construct elementary embeddings $(j_i)_{i \leqslant n}$ by induction on i. Since $\mathbb{P}$ is Easton supported and κ inaccessible, $\mathbb{P}_{<\kappa} \subseteq V_\kappa$ and so $j[G_{<\kappa}] = G_{<\kappa} \subseteq G_{<\kappa_1}$ and thus we can lift j to

$$j_0 : H_{\kappa_n^+}[G_{<\kappa}] \rightarrow H_{j(\kappa_n)^+}[G_{<\kappa_1}]$$

Now suppose that the embedding

$$j_m : H_{\kappa_n^+}\Big[G_{<\kappa}\Big]\Big[\mathop{\ast}_{i<m} G_{\Delta_i}\Big] \rightarrow H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<m} G_{\Delta_{i+1}}\Big]$$

is already constructed for some $m < n$. The strategy is the following: Notice that $A_m = j[G_{\Delta_m}] \in H_{\kappa_n^+}$ is directed and of size κ_{m+1} which is a consequence of κ_{m+1} being strongly inaccessible and $\mathbb{P}$ being Easton supported. Since

$$j(\mathbb{P}_{\Delta_m}) = \mathbb{P}_{\Delta_{m+1}}$$

is $\leqslant j^{m+1}(\kappa)$-directed closed, there is a condition p_m below A_m. However, p_m need not be in the generic $G_{\Delta_{m+1}}$. We will be able to solve this problem using the weak homogeneity of the forcing $\mathbb{P}_{\Delta_{m+1}}$ in

$$H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<m} G_{\Delta_{i+1}}\Big]$$

That is, the conditions q for which there is an automorphism f of the forcing, so that $f(q) \leqslant p_m$ is dense. Hence there must be such a q in $G_{\Delta_{m+1}}$ witnessed by some automorphism f. Let H be the generic filter generated by $f[G_{\Delta_{m+1}}]$. Now, $p_m \in H$ which implies

$$j_m[G_{\Delta_m}] \subseteq H$$

and thus we may lift j_m:

$$j_{m+1} : H_{\kappa_n^+}\Big[G_{<\kappa}\Big]\Big[\mathop{\ast}_{i<m+1} G_{\Delta_i}\Big] \to H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<m} G_{\Delta_{i+1}}\Big]\Big[H\Big]$$

Observe that

$$H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<m} G_{\Delta_{i+1}}\Big]\Big[H\Big] = H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<m+1} G_{\Delta_{i+1}}\Big]$$

since the latter can compute H using the automorphism f and the former $G_{\Delta_{m+1}}$ using f^{-1}.
In the end, we have an embedding

$$j_n : H_{\kappa_n^+}\Big[G_{<\kappa}\Big]\Big[\mathop{\ast}_{i<n} G_{\Delta_i}\Big] \to H_{j(\kappa_n)^+}\Big[G_{<\kappa_1}\Big]\Big[\mathop{\ast}_{i<n} G_{\Delta_{i+1}}\Big]$$

which is in fact of the desired form, as the domain of j_n is already $H_{\kappa_n^+}^{V[G]}$ and the target is $H_{\kappa_n^+}^{V[G]}$ as the tail of the iteration is sufficiently closed.

□

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