How to force $(*)$ from less

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By a recent result of Asperó-Schindler [?], Woodin's axiom $(*)$ is a consequence of MM^{++}. As is well-known, MM^{++} can be forced from a supercompact cardinal using a semiproper RCS-iteration and hence so can $(*)$. It is widely believed that the supercompact is necessary for MM^{++}, however $(*)$ has consistency strength of merely ω-many Woodin cardinals. Clearly, $(*)$ can be obtained by forcing over $L(\mathbb{R})$ if AD holds there, but it remains a mystery if $(*)$ is forceable over models of ZFC from large cardinal hypotheses reasonably close to its consistency strength. In these notes we make partial progress to this problem and show that $(*)$ can be forced using a semiproper iteration from an assumption strictly weaker than a κ^{+++}-supercompact cardinal $\kappa(+\mathrm{GCH})$. One way to reduce the large cardinal is to note that the argument of [?] actually shows that $(*)$ follows from $\mathrm{MM}^{++}_{\omega_3} + \diamond_{\omega_3}$ and this can be achieved from less than a full supercompact, but this seems to need more than what we will assume here. Instead, our strategy will be to iterate the forcing from [?]. In order to not collapse ω_1, we only want to use this forcing if it is semiproper. Our argument to force this needs a κ^{++}-supercompact cardinal κ. This is the only part of our construction that needs partial supercompactness, the rest will work if enough Woodin cardinals are at our disposal. The large cardinal assumption we end up with is an inaccessible limit of κ^{++}-supercompact cardinals κ with ω-many Woodin cardinals plus a measurable on top.

We will silently assume GCH in V throughout these notes. This is not a necessary assumption, but will make the notation both in the proof in and the statement of the main theorem easier. Furthermore we fix a set $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1^V$. Given a dense set $D \subseteq \mathbb{P}_{\max}$ in $L(\mathbb{R})$ let $\mathbb{P}_D$ be (one choice of) the forcing used in the proof that $(*)$ follows from MM^{++} using the fixed A.

Proposition 1. *Suppose* NS_{ω_1} *is saturated. Let* $D \in \mathcal{P}(\mathbb{P}_{\max}) \cap L(\mathbb{R})$ *and* $X \prec H_{\omega_4}$ *countable with* $\mathbb{P}_D \in X$*. If* $p \in X \cap \mathcal{P}_D$ *then* $p \cup \{``\alpha \mapsto \xi''\}$ *is a condition in* $\mathbb{P}_D$ *whenever* $\xi \in X \cap \omega_3$ *is large enough where* $\alpha = X \cap \omega_1$*.*

Proof. We will show that

$$X \models T = \{\beta < \omega_1 \mid p \cup \{''\beta \mapsto \delta''\} \in \mathbb{P}_D\} \text{ contains a club}$$

The assertion then follows immediately. We will show that the same statement is true in H_{ω_4}. We will make use of the notation in the $MM^{++} \Rightarrow (*)$-paper. Let $\lambda < \omega_3$ be large enough such that $p \in \mathbb{P}^D_\lambda$. Let g be $\mathrm{Col}(\omega, \omega_3)$-generic over V and find some $h \in V[g]$ that is $\mathbb{P}^D_\beta$-generic over Q_β with $p \in h$. This gives rise to a certificate

$$(M_i, \pi_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j < \omega_1^V), ((k_n, \alpha_n) \mid n < \omega), (\lambda_\delta, X_\delta \mid \delta \in K)$$

Let $\rho = \omega_1^{V[g]}$ and write $N_{\omega_1^V} = (N_{\omega_1^V}, A, I)$. Given any $S \in (\mathrm{NS}^+_{\omega_1})^V \subseteq I^+$, we extend $(N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1^V)$ to a generic iteration $(N_i^S, \sigma^S_{i,j} \mid i \leqslant j \leqslant \rho)$ so that $\omega_1^V \in \sigma_{\omega_1^V, \omega_1^V+1}(S)$. We then let

$$(M_i^S, \sigma^S_{i,j} \mid i \leqslant j \leqslant \rho) = \sigma^S_{0,\rho}((M_i, \pi_{i,j} \mid i \leqslant j \leqslant \omega_1^V))$$

and since $M_{\omega_1^V} = (H^V_{\omega_2}, \mathrm{NS}^V_{\omega_1}, A)$ and NS_{ω_1} is saturated, we may lift this to a generic iteration of V

$$(M_i^{S,+}, \pi^{S,+}_{i,j} \mid \omega_1^V \leqslant i \leqslant j \leqslant \rho)$$

Put $M^S = M^{S,+}_\rho$ and $\pi^S = \pi^{S,+}_{\omega_1^V, \rho}$.

Claim 2. *If $\lambda \leqslant \xi < \omega_1^V$ then $M^S \models \pi^S(p) \cup \{''\omega_1^V \mapsto \pi^S(\xi)''\} \in \pi^S(\mathbb{P}^D)$*

Proof. The semantic certifiacte is given by

$$(M_i^S, \pi^S_{i,j}, N_i^S, \sigma^S_{i,j} \mid i \leqslant j \leqslant \rho), ((k_n, \pi^S(\alpha_n) \mid n < \omega), (\lambda^*_\delta, X^*_\delta \mid \delta \in K^*)$$

where $K^* = K \cup \{\omega_1^V\}$ and $\lambda^*_\delta = \pi^S(\lambda_\delta)$, $X^*_\delta = \pi^S(X_\delta)$ for $\delta \in K$ and $\lambda^*_{\omega_1^V} = \pi^S(\beta)$ and $X_{\omega_1^V} = \pi^S(Q_\beta)$. □

Thus we have $\omega_1^V \in \pi^S(T) \cap \pi^S(S)$, which means $S \cap T \neq \emptyset$. Recall that S was an arbitrary stationary subset of ω_1 in V. Hence T meets any stationary subset of ω_1. This can only happen if T contains a club. □

We will need some combinatorial principle that implies that the forcings of the form $\mathbb{P}_D$ are semiproper. The following one works:

Definition 3. $\mathrm{CC}^*(\omega_4)$ is the statement that

$$\{X \prec H_{\omega_4} \mid X \text{ is countable} \wedge \exists Y\, X \sqsubseteq Y \prec H_{\omega_4} \exists a \in Y\, |a| = \omega_1 \wedge X \cap H_{\omega_4} \subseteq a\}$$

contains a club of $[H_{\omega_4}]^\omega$. Here $X \sqsubseteq Y$ means $X \subseteq Y$ and $X \cap \omega_1 = Y \cap \omega_1$.

Lemma 4. *Assume that NS_{ω_1} is saturated and that $\mathrm{CC}^*(\omega_4)$ holds. Then $\mathbb{P}_D$ is semiproper for any dense set $D \in \mathcal{P}(\mathbb{P}_{\max}) \cap L(\mathbb{R})$.*

Proof. Let

$$\mathcal{C} = \{X \prec H_{\omega_4} \mid X \text{ is countable} \wedge \exists Y\, X \sqsubseteq Y \prec H_{\omega_4} \exists a \in Y\, |a| = \omega_1 \wedge X \cap H_{\omega_4} \subseteq a\}$$

Let $X \in \mathcal{C}$ with $\mathcal{P}_D \in X$ and find Y and $a \in Y$ that witness X to be an element of $\mathcal{C}$. Let $p \in X \cap \mathcal{P}$. Let $b = \langle \dot{x}_\alpha \mid \alpha < \omega_1 \rangle$ be an enumeration in Y of all $\mathbb{P}_D$-names for countable ordinals that are in a. Note that any $\mathbb{P}^D$-name for a countable ordinal that is in X is some $\dot{x}_\beta$ for $\beta < \alpha := X \cap \omega_1 = Y \cap \omega_1$. Now find $\lambda < \omega_3$ in Y such that $p \in \mathbb{P}^D_\lambda$ and

$$(Q_\lambda; \in, \mathbb{P}^D_\lambda, A_\lambda) \prec (H_{\omega_3}; \in, \mathbb{P}^D, \{(q, \beta, \gamma) \mid q \in \mathbb{P}^D \wedge q \Vdash \dot{x}_\beta = \check{\gamma}\})$$

By the proposition above we find that $q = p \cup \{\text{“}\alpha \mapsto \lambda\text{“}\} \in \mathbb{P}^D$. I claim that q is $\mathbb{P}^D$-semigeneric for X. Let $r \leqslant q$ and suppose that $\dot{x} \in X$ is a name for a countable ordinal and $r \Vdash \dot{x} = \check{\gamma}$. Find $\beta < \alpha$ such that $\dot{x} = \dot{x}_\beta$. We must show $\gamma < \alpha$. Let

$$\mathfrak{C} = (M_i, \pi_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1), (k_n, \alpha_n \mid n < \omega), (\lambda_\delta, X_\delta \mid \delta \in K)$$

be a certificate for r. Then $\alpha \in K$ and $\lambda_\alpha = \lambda$ so that

$$X_\alpha \prec (\mathbb{Q}_\lambda; \in, \mathbb{P}^D_\lambda, A_\lambda)$$

and $X_\alpha \cap \omega_1 = \alpha$. Let Σ be the syntactical certificate corresponding to $\mathfrak{C}$. Thus $X_\alpha \cap F \cap [\Sigma]^{<\omega} \neq \varnothing$ whenever $F \subseteq \mathbb{P}^D_\lambda$ is a dense subset definable in $(\mathbb{Q}_\lambda; \in, \mathbb{P}^D_\lambda, A_\lambda)$ from parameters in X_α. In particular this is true for

$$F = \{s \in \mathbb{P}^D_\lambda \mid \exists \xi (s, \beta, \xi) \in A_\lambda\}$$

Note that F is dense as

$$\{s \in \mathbb{P}^D \mid \exists \xi\; s \Vdash \dot{x}_\beta = \check{\xi}\}$$

is. So let $s \in F \cap [\Sigma]^{<\omega} \cap X_\alpha$ and let $\xi \in X_\alpha$ be so that $(s, \beta, \xi) \in A_\lambda$. Thus $s \Vdash_{\mathbb{P}^D} \dot{x}_\beta = \check{\xi}$. Furthermore s and r are compatible as both s and r are elements of $\in [\Sigma]^{<\omega}$. We can conclude $\gamma = \xi < \alpha$ as desired.

□

Lemma 5. *Let κ be κ^{++}-supercompact and assume $\mathbb{P}$ is a semiproper forcing of size κ collapsing κ to ω_2 with the κ-cc such that there is a κ^{++}-supercompact embedding $j : V \to M$ with critical point κ with*

$$j(\mathbb{P}) = \mathbb{P} * \mathrm{Col}(\omega_1, \kappa^{++}) * \dot{\mathbb{Q}}$$

such that $M^{\mathbb{P} * \mathrm{Col}(\omega_1, \kappa^{++})} \models$ “$\dot{\mathbb{Q}}$ is semiproper”. Then $V^{\mathbb{P}} \models \mathrm{CC}^*(\omega_4)$.

Proof. Suppose not. Let j be the embedding assumed to exist. Let g be $\mathbb{P}$-generic over V and h $\mathrm{Col}(\omega_1, \kappa^{++})^{V[g]}$-generic over $V[g]$. Then by picking a bijection $f : \omega_1 \to H^{V[g]}_{\omega_4}$, we can turn $\mathcal{C}^c$ into a stationary subset of ω_1 in $V[g,h]$, simply put

$$S = \{\alpha < \omega_1 \mid f\text{"}\alpha \in \mathcal{C}^c\}$$

In $V[g,h]$ we will build a continuous increasing chain $\langle X_i \mid i < \omega_1\rangle$ of countable elementary substructures of $H^{M[g,h]}_\theta$ for θ large enough. Just make sure $f, \dot{\mathbb{Q}}^{g*h}, H^{V[g]}_{\omega_4}, \prec, j \upharpoonright \kappa^{++}, g, h \in X_0$ and $i \in X_{i+1}$ as well as $X_i \cap H^{V[g]}_{\omega_4} = f\text{"}X_i \cap \omega_1$ for all $i < \omega_1$. Here, $\prec$ is some wellorder of $H^V_{\kappa^{++}}$ in V of length κ^{++}. Find some $\alpha \in S$ with $X_\alpha \cap \omega_1 = \alpha$ and let $X = X_\alpha$. Note that $M[g,h]$ is still countably closed from the point of view of $V[g,h]$ (even ω_1-closed). Thus $X \in M[g,h]$. Now we may find k $\dot{\mathbb{Q}}^{g*h}$-generic such that k contains a X-semigeneric condition. Hence $X \sqsubseteq X[k]$. We may lift j to an elementary embedding $j^+ : V[g] \to M[g,h,k]$ with $j^+(g) = g * h * k$. Let

$$X_* = X \cap H^{V[g]}_{\omega_4} \prec H^{V[g]}_{\omega_4}$$

Note that $f\text{"}\alpha \sqsubseteq X_*$ and $\alpha \in S$ and hence $X_* \in \mathcal{C}^c$ (note that X_* is countable and thus $X^* \in V[g]$). This shows

$$j^{+}\text{"}X_* = j^+(X_*) \in j^+(\mathcal{C}^c)$$

Claim 6. $j^+ \upharpoonright H^{V[g]}_{\omega_4} \in X[k]$.

Proof. As $\mathbb{P}$ is of size κ and collapses κ to ω_2, $\omega_4^{V[g]} = \kappa^{++}$ and $H^{V[g]}_{\omega_4} = H^V_{\kappa^{++}}[g]$. X can define a wellorder of $H^{V[g]}_{\omega_4}$ from $\prec$ by setting $x \prec' y$ iff the $\prec$-least name $\dot{x}$ with $\dot{x}^g = x$ is $\prec$ the least such name for y. In the same way $X[k]$ can define a wellorder $\prec''$ on $H^{M[g*h*k]}_{\omega_4}$ from $j(\prec)$ with $g*h*k \in X[k]$ in place of g. It follows that $j^+(\prec') = \prec''$. Now for $j^+ \upharpoonright H^{V[g]}_{\omega_4}$ maps the α-th point w.r.t. $\prec'$ to the $j(\alpha)$-th set w.r.t. $j(\prec')$. As $j \upharpoonright \kappa^{++} \in X$ and as $\prec'$ has ordertype κ^{++}, $j^+ \upharpoonright H^{V[g]}_{\omega_4}$ can be defined in $X[k]$. □

Finally, $j^+(X_*) \subseteq \mathrm{ran}(j^+ \circ f) \in X[k]$. This shows that $j^+(X_*) \sqsubseteq X[k] \cap H^{M[g,h,k]}_{\omega_4}$ witnesses $j^+(X_*)$ to be in $\mathcal{C}$, a contradiction.

□

Suppose $\mathbb{P} = \mathbb{P}_\kappa = \mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\rangle$ is an iteration. For $\alpha \leqslant \lambda$ a limit, we denote the direct limit along $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha\rangle$ by $\mathbb{P}_{<\alpha}$.

A semiproper iteration of length λ is a RCS iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\rangle$ such that for all $\alpha < \lambda$ we have:

(*i*) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is semiproper"

(*ii*) there is $\beta < \lambda$ so that $\mathbb{P}_\beta \Vdash |\mathbb{P}_\alpha| \leqslant \omega_1$

Proposition 7. *Suppose λ is inaccessible and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda \rangle$ is a semiproper iteration so that $\mathbb{P}_\alpha \Vdash |\dot{\mathbb{Q}}_\alpha| < \check{\lambda}$ for all $\alpha < \lambda$. Then $\mathbb{P}$ is $\lambda - cc$.*

Proof. Clearly all $\mathbb{P}_\alpha$ are of size $<\lambda$ for $\alpha < \lambda$ and $\mathbb{P}_\lambda$ has size λ. Let $h : \lambda \to \mathbb{P}$ be a bijection. We can easily find a club $C \subseteq \lambda$ so that $h[\alpha] \cong \mathbb{P}_{<\alpha}$ (note that $\mathbb{P}_{<\alpha}$ essentially is $\bigcup_{\beta<\alpha} \mathbb{P}_\beta$). It is well-known that $\mathbb{P}$ is semiproper and hence preserves ω_1. Thus by the definition of RCS iteration, a direct limit is taken at any $\alpha \in E^\lambda_{\omega_1} = \{\gamma < \lambda \mid \mathrm{cof}(\gamma) = \omega_1\}$, so that $\mathbb{P}_\alpha = \mathbb{P}_{<\alpha}$ for such α.

Now suppose $A \subseteq \mathbb{P}$ is of size λ, say $A = \{p_\beta \mid \beta < \lambda\}$. We will define a regressive function on the stationary set $S = C \cap E^\lambda_{\omega_1}$ by setting

$$f(\alpha) = h^{-1}(p_\alpha \restriction \alpha)$$

As $\alpha \in S$ we have $p_\alpha \restriction \alpha \in \mathbb{P}_{<\alpha}$ by our prior considerations and as $\alpha \in C$ we get that indeed $f(\alpha) < \alpha$. Fodor tells us that there is a stationary set $T \subseteq S$ and some $\gamma < \lambda$ so that f is constant on T with value γ. Note that the final limit taken in the construction of $\mathbb{P} = \mathbb{P}_\lambda$ is a direct limit as no $q \in \mathbb{P}_\alpha$ could force λ to be of countable cofinality for $\alpha < \lambda$. Hence any condition in $\mathbb{P}$ has support bounded in λ. This means that me may find $\alpha < \beta$ both in T so that the support of p_α is contained in β. We have

$$p_\alpha \restriction \alpha = h(\gamma) = p_\beta \restriction \beta$$

and thus $p_\alpha = p_\beta \restriction \beta$. We have found two compatible conditions in A. □

Lemma 8. *If κ is κ^{++}-supercompact then there is a semiproper iteration $\mathbb{P}$ of length κ such that in $V^{\mathbb{P}}$ both NS_{ω_1} is saturated and $\mathrm{CC}^*(\omega_4)$ hold true.*

Proof. (Sketch) There are multiple ways of arranging this. We sketch on of them where we take advantage of the following fact.

Claim 9. *Let $j : V \to M$ witness that κ is κ^{++}-supercompact. Then for any $B \subseteq \kappa$, κ is $j(B)$-strong up to $j(\kappa)$ in M.*

Proof. There is a standard argument that produces, internal to M, a superstrong embedding

$$i : M \to N$$

with critical point κ and $i \restriction V_{\kappa+1} = j \restriction V_{\kappa+1}$. The proof that shows superstrong cardinals to be Woodin yields that κ is C-strong up to $i(\kappa)$ in N for any $C \in \mathcal{P}(i(\kappa))^N$. Now the extenders witnessing κ to be $i(B) = j(B)$-strong up to $i(\kappa) = j(\kappa)$ lie in $V^N_{i(\kappa)} = V^M_{j(\kappa)}$. Thus these extenders witness κ to be $j(B)$-strong up to $j(\kappa)$ in M. □

The construction of a Laver function applied to the given κ^{++}-supercompact cardinal κ yields a map $f : \kappa \to V_\kappa$ so that for any $x \in H_{\kappa^{++}}$ there is a κ^{++}-supercompact embedding $j_x : V \to M_x$ with $j(f)(\kappa) = x$. We produce a RCS-iteration $\mathbb{P} = \langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \mid \alpha < \kappa\rangle$ by inductively defining $\dot{\mathbb{Q}}_\alpha$ so that

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{\mathbb{Q}}_\alpha \text{ is the antichain sealing forcing for } f(\alpha)\text{”}$$

if $f(\alpha)$ happens to be a $\mathbb{P}_\alpha$-name for a maximal antichain of stationary subsets of ω_1 and the corresponding sealing forcing is forced to be semiproper and

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha = \mathrm{Col}(\omega_1, 2^{\omega_2})$$

otherwise.
Let G be $\mathbb{P}$-generic over V and assume towards a contradiction that there is a maximal antichain

$$a = \{S_\alpha \mid \alpha < \kappa\} \in V[G]$$

of stationary subsets of ω_1. Note that $\omega_1^{V[G]} = \omega_1^V$ and $\omega_2^{V[G]} = \kappa$. Let $\dot{a}$ be a nice $\mathbb{P}$-name for a. Look at the embedding

$$j_a : V \to M_a$$

and let U be the normal measure on κ derived from j_a. As $j_a(f)(\kappa) = a$,

$$\{\alpha < \kappa \mid f(\alpha) = \dot{a} \upharpoonright \alpha \text{ is a } \mathbb{P}_\alpha\text{-name for a maximal antichain of } \mathrm{NS}_{\omega_1}^+\} \in U$$

as well as

$$\{\alpha < \kappa \mid \alpha \text{ is } < \kappa - \dot{a} \oplus \mathbb{P} - \text{strong}\} \in U$$

by the claim above, so that their intersection is non-empty. The argument from [?] then carries over to the situation here and shows that NS_{ω_1} is saturated in $V[G]$. It remains to see that $\mathrm{CC}^*(\omega_4))$ holds true there as well. But this is an easy consequence of the way we have set up the iteration and Lemma **??**: Consider

$$j_\varnothing : V \to M_\varnothing$$

Then in $M_\varnothing$, $j_\varnothing(f(\kappa))$ is clearly not a name for a maximal antichain so that

$$\mathbb{1}_{\mathbb{P}_\kappa} \Vdash^M \dot{\mathbb{Q}}_\kappa = \mathrm{Col}(\omega_1, 2^{\omega_2})$$

As $j(\mathbb{P})$ is a semiproper iteration in $M_\varnothing$, any tail of the iteration is forced to be semiproper. Thus $j_\varnothing(\mathbb{P})$ factors as desired to be able to apply Lemma **??**.

□

We are now ready to prove the main theorem.

Theorem 10. *Assume there is a proper class of Woodin cardinals. Let λ be an inaccessible limit of κ^{++}-supercompact cardinals κ. Then there is a semiproper iteration $\mathbb{P}$ of length λ with the λ-cc collapsing λ to ω_2 and forcing $(*)$.*

Proof. Our strategy will be to force with $\mathbb{P}_D$ for any $D \in L(\mathbb{R})$ that is dense in $\mathbb{P}_{\max}$. Of course we must anticipate that $L(\mathbb{R})$ may change along our iteration, so we will have to do some bookkeeping. The inaccessibility of λ is used to make sure that our iteration will have the λ-cc, so that we "catch our tail" in the sense that in the end we really forced with $\mathbb{P}_D$ for any appropriate D that is in the final $L(\mathbb{R})$. All the κ^{++}-supercompact κ will be used to take advantage of Lemma **??** so that the next $\mathbb{P}_D$ we want to force with is really semiproper by Lemma **??**. We are also obliged to make NS_{ω_1} saturated to satisfy the other assumption of latter lemma. We will proceed to show how this can be organised.

We will need a bookkeeping function h that maps a κ^{++}-supercompact cardinal κ to some $\mathbb{P}_\kappa$-name $\dot{D}$ for a dense subset of $\mathbb{P}_{\max}^{V^{\mathbb{P}_\kappa}}$ that is in $L(\mathbb{R})^{V^{\mathbb{P}_\kappa}}$. We will explicitly construct h alongside our iteration.

We will inductively make sure that if $\kappa < \lambda$ is κ^{++}-supercompact then in $V^{\mathbb{P}_\kappa}$ both NS_{ω_1} is saturated and $\mathrm{CC}^*(\omega_4)$ hold true. Suppose $\mathbb{P}_\beta$ is defined for all $\beta \leqslant \alpha$. Suppose $\mathbb{P}_\beta$ is defined for all $\beta < \alpha$. If α is α^{++}-supercompact then the next step of the forcing will be $\mathbb{P}_h(\alpha)$, or more precisely we make sure that

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha = \mathbb{P}_{\dot{D}}$$

where $\dot{D} = h(\alpha)$. It does not matter which tree witnessing $\dot{D}$ to be κ^+-universally Baire we choose in the construction of $\mathbb{P}_{\dot{D}}$ in $V^{\mathbb{P}_\kappa}$ so we will not make them explicit.

If $\alpha < \lambda$ is either 0, $\kappa + 1$ for a κ^{++}-supercompact cardinal κ or a limit of such cardinals then let κ' be the next such cardinal above α. Note that $|\mathbb{P}_\alpha| < \kappa'$ so that κ' is still κ'^{++}-supercompact in $V^{\mathbb{P}_\alpha}$. We then define $\mathbb{P}_\gamma$ for all $\alpha < \gamma \leqslant \kappa'$ so that $\mathbb{P}_{\kappa'} \cong \mathbb{P}_\alpha * \dot{\mathbb{R}}$ where $\mathbb{R}$ is the forcing given by Lemma **??** in $V^{\mathbb{P}_\alpha}$.

We now make h precise. Let $l : \lambda \to \lambda \times \lambda$ be bijective on the κ^{++}-supercompact $\kappa < \lambda$ such that if $l(\alpha) = (\beta, \gamma)$ then $\beta \leqslant \alpha$. If κ is κ^{++}-supercompact then we enumerate (up to equivalence of names) all $\mathbb{P}_\kappa$-names for dense subsets of $\mathbb{P}_{\max}$ in $L(\mathbb{R})$ as $\langle \dot{D}^\kappa_\gamma \mid \gamma < \lambda \rangle$ (with repetitions). If $l(\kappa) = (\beta, \gamma)$ and β is not β^{++}-supercompact then we let $h(\kappa)$ be arbitrary. Otherwise, $\dot{D} = \dot{D}^\beta_\gamma$ is defined and is a $\mathbb{P}_\beta$-name for a subset of $\mathbb{P}_{\max}$. As $|\mathbb{P}_\beta| < \lambda$ and λ is a limit of Woodin cardinals,

$$V^{\mathbb{P}_\beta} \models \text{“}\dot{D} \text{ is } {<}\check{\lambda}\text{-universally Baire”}$$

Choose any trees $\dot{T}, \dot{S}$ witnessing this, where $\dot{T}$ projects to $\dot{D}$. Then we let $h(\kappa)$ be a name for the projection of $\dot{T}$ in $V^{\mathbb{P}_\kappa}$. We have more than enough large cardinals at our disposal to ensure $L(\mathbb{R})$-absoluteness in the sense that

$$(L(\mathbb{R}); \in, \dot{D})^{V^{\mathbb{P}_\beta}} \equiv (L(\mathbb{R}); \in, h(\kappa))^{V^{\mathbb{P}_\kappa}}$$

so that $h(\kappa)$ is indeed a name for a dense subset of $\mathbb{P}_{\max}$ that lives in $L(\mathbb{R})$.

Let G be $\mathbb{P} = \mathbb{P}_\lambda$-generic over V. We will show that $(*)$ holds in $V[G]$. Recall that we fixed a set $A \subseteq \omega_1^V = \omega_1^{V[G]}$ with respect to which we set up all our forcings of the form $\mathbb{P}_D$ we used. We propose the following filter:

$$g = \{(M, I, a) \in \mathbb{P}_{\max} \,|\text{there is a generic iterate } (M_{\omega_1}, I_{\omega_1}, a_{\omega_1}) \text{ of } (M, I, a) \text{ with } I_{\omega_1} = \mathrm{NS}_{\omega_1} \cap M_{\omega_1} \text{ and } a_{\omega_1} = A\}$$

g is indeed filter, cf. [?]. We show that it is generic over $L(\mathbb{R})^{V[G]}$, so let $D_3 \in L(\mathbb{R})^{V[G]}$ be dense in $\mathbb{P}_{\max}$. We will write $\mathbb{R}_3$ for $\mathbb{R}^{V[G]}$. Now $\mathbb{R}_3^\sharp$ exists in $V[G]$, so there are $L(\mathbb{R}_3)$-indiscernibles $\xi_0, \ldots, \xi_n$ and reals $x_0, \ldots, x_m$ as well as a first order formula φ so that $y \in D$ if and only if

$$L(\mathbb{R}_3) \models \varphi(y, x_0, \ldots, x_m, \xi_0, \ldots, \xi_n)$$

By Proposition **??**, $\mathbb{P}$ is λ-cc so that there must be some κ^{++}-supercompact $\kappa < \lambda$ so that $x_0, \ldots, x_m \in V[G_\kappa]$ (where G_κ is the induced $\mathbb{P}_\kappa$-generic filter). Let's put $\mathbb{R}_0 = \mathbb{R}^{V[G_\kappa]}$. We may assume that $\xi_0, \ldots, \xi_n$ are $L(\mathbb{R}_0)^{V[G_\kappa]}$-indiscernibles as well. Now for some γ, $\dot{D}^\kappa_\gamma$ is a $\mathbb{P}_\kappa$-name for the set

$$D_0 = \{y \in \mathbb{R}_0 \mid L(\mathbb{R}_0) \models \varphi(y, x_0, \ldots, x_n, \xi_0, \ldots, \xi_m)\}$$

Let κ' be κ'^{++}-supercompact so that $l(\kappa') = (\kappa, \gamma)$ and put $D_1 = h(\kappa')^{G_{\kappa'}}$. The forcing we use at step κ' is the $\mathbb{P}_{D_1}$ of $V[G'_\kappa]$, so that there is $y \in V[G_{\kappa'+1}]$ with $y \in D_2 \cap g$, where D_2 is the version of D_1 in $V[G_{\kappa'+1}]$ (i.e. the projection of a tree witnessing D_1 to be sufficiently universally Baire in $V[G_{\kappa'}]$). Our large cardinal hypothesis implies that in a forcing extension of $V[G]$ there is some set of reals $\mathbb{R}_*$ and some D_* as well as elementary embeddings

$$j_i : (L(\mathbb{R}_i); \in, D_i) \to (L(\mathbb{R}_*); \in, D_*)$$

for all $i < 4$ which map indiscernibles to indiscernibles. Using j_1 and j_2, we see that

$$L(\mathbb{R}_*) \models \varphi(y, x_0, \ldots, x_n, \xi^*_0, \ldots \xi^*_m)$$

where $\xi^*_0 < \cdots < \xi^*_m$ are (any) $\mathbb{R}^*$-indiscernibles. Using j_3, we see that

$$L(\mathbb{R}_3) \models \varphi(y, x_0, \ldots, x_n, \xi_0, \ldots, \xi_m)$$

and hence $y \in D_3 \cap g$.

$\square$

Remark 11. A look at the argument above reveals that we do not need the whole proper class of Woodin cardinals, only a limit of Woodins $< \lambda$ with a measurable above it. They are only used to find the embeddings j_i used in the end.

Question 12. Can $(*)$ be forced from a proper class of Woodin cardinals that has an inaccessible limit?

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References

[AS19] David Asperó and Ralf Schindler. MM^{++} implies $(*)$. 2019.

[Sch11] Ralf Schindler. On NS_{ω_1} being saturated, September 2011.