The Problem with Exercise 34.5

Or Why Collapses Don't Fit Together

Andreas Lietz

Abstract

In this short note we address a problem with Exercise 34.5 in [Jec03] and fix the proof of Theorem 34.6. Furthermore we show that for regular uncountable λ and inaccessible $\kappa > \lambda$, $\mathrm{Col}(\omega, \lambda) \times \mathrm{Col}(\lambda, {<}\kappa)$ and $\mathrm{Col}(\omega, {<}\kappa)$ are not forcing equivalent. [1]

Let $\mathbb{B}$ be an atomless complete Boolean algebra and κ an inaccesible cardinal. To keep notation consistent with [Jec03], we call the following property (34.6):

(*i*) $\mathbb{B}$ has the κ-cc

(*ii*) $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ where $\mathbb{B}_\alpha <_{\mathrm{reg}} \mathbb{B}_\beta$ and $|\mathbb{B}_\alpha| < \kappa$ for $\alpha < \beta < \kappa$

(*iii*) every $\gamma < \kappa$ is countable in $V^{\mathbb{B}_\alpha}$ for some $\alpha < \kappa$

Note that the property (34.6) doesn't change if one requires all the $\mathbb{B}_\alpha$ to be complete subalgebras of $\mathbb{B}$ and as usual it is convenient to assume that this is the case.

Exercise 34.5 states that (34.6) implies that $\mathbb{B} \cong \mathrm{Col}(\omega, {<}\kappa)$ and this is again used in the proof of Theorem 34.6. We will show that, as stated, this is not quite true. To conclude $\mathbb{B} \cong \mathrm{Col}(\omega, {<}\kappa)$, one further continuity assumption on $\vec{\mathbb{B}}$ is sufficient, and necessary as well:

Lemma 1. *Suppose $\mathbb{B}$ satisfies* (34.6) *and that $\vec{\mathbb{B}}$ witnesses this. Then $\mathbb{B} \cong \mathrm{Col}(\omega, {<}\kappa)$ if and only if*

$$\Delta(\vec{\mathbb{B}}) = \{\beta < \kappa \mid \bigcup_{\alpha<\beta} \mathbb{B}_\alpha <_{\mathrm{reg}} \mathbb{B}\} = \{\beta < \kappa \mid \bigcup_{\alpha<\beta} \mathbb{B}_\alpha <_{\mathrm{reg}} \mathbb{B}_\beta\}$$

contains a club.

[1] Thanks to Stefan Hoffelner for valuable discussions on this topic

Remark 2. In this note, $\mathrm{Col}(\omega, \alpha)$ refers to the Boolean completion of the partial order ${}^{<\omega}\alpha$ ordered by reverse inclusion. Similarly, $\mathrm{Col}(\omega, {<}\alpha)$ refers to the Boolean completion of the partial order consisting of finite partial functions

$$p : \omega \times \alpha \to \alpha$$

satisfying $p(n, \gamma) < \gamma$ whenever $(n, \gamma) \in \mathrm{dom}(p)$, ordered by reverse inclusion.

Proof. " $\Leftarrow$ " : Suppose $\pi : \mathbb{B} \to \mathrm{Col}(\omega, {<}\kappa)$ is an isomorphism. We can write $\mathrm{Col}(\omega, {<}\kappa) = \bigcup_{\alpha<\kappa} \mathbb{C}_\alpha$ so that each $\mathbb{C}_\alpha$ is isomorphic to $\mathrm{Col}(\omega, \alpha)$ and $\Delta(\vec{\mathbb{C}}) = \kappa$. Now there must be a club C of $\beta < \kappa$ such that π restricts to an isomorphism

$$\pi \upharpoonright \bigcup_{\alpha<\beta} \mathbb{B}_\alpha : \bigcup_{\alpha<\beta} \mathbb{B}_\alpha \to \bigcup_{\alpha<\beta} \mathbb{C}_\alpha$$

Now we have $\bigcup_{\alpha<\beta} \mathbb{C}_\alpha <_{\mathrm{reg}} \mathrm{Col}(\omega, {<}\kappa)$. But as π is an iso, this must mean that $\bigcup_{\alpha<\beta} \mathbb{B}_\alpha <_{\mathrm{reg}} \mathbb{B}$ as well, so that:

$$C \subseteq \Delta(\vec{\mathbb{B}})$$

" $\Rightarrow$ " : As remarked before, we can assume wlog that all $\mathbb{B}_\alpha$ are complete subalgebras of $\mathbb{B}$. Since $\Delta(\vec{\mathbb{B}})$ is a club, we may as well rename $\vec{\mathbb{B}}$ so that $\Delta(\vec{\mathbb{B}}) = \kappa$. To ease notation, we will define

$$\mathbb{B}_{<\beta} = \bigcup_{\alpha<\beta} \mathbb{B}_\alpha$$

for any $\beta < \kappa$. I claim that we may assume that wlog for all $\alpha < \kappa$, $\mathbb{B}_\alpha \cong \mathrm{Col}(\omega, \gamma_\alpha)$ for some $\gamma_\alpha < \kappa$. To see this, we show that there are unboundedly many $\beta < \kappa$ so that the completion of $\mathbb{B}_{<\beta}$ is $\mathrm{Col}(\omega, \gamma)$ for some γ. So let $\alpha_0 < \kappa$ be given. Define $\alpha_{n+1} > \alpha_n$ so that $|\mathbb{B}_{\alpha_n}|$ is countable in $V^{\mathbb{B}_{\alpha_{n+1}}}$. Let $\beta = \sup_{n<\omega} \alpha_n$. Then $\mathbb{B}_{<\beta}$ has size $|\beta|$ and collapses $|\beta|$ and hence its completion is isomorphic to $\mathrm{Col}(\omega, |\beta|)$. As $\mathbb{B}_{<\beta} <_{\mathrm{reg}} \mathbb{B}$, its completion is a complete subalgebra of $\mathbb{B}$.
The above also shows that we may as well assume that at all limit $\beta < \kappa$, $\mathbb{B}_\beta$ is just the completion of $\mathbb{B}_{<\beta}$.
Now with this out of the way, we will do a zig-zag argument using the lifting property of the collapse forcing $\mathrm{Col}(\omega, \gamma)$. As before we write $\mathrm{Col}(\omega, {<}\kappa) = \bigcup_{\alpha<\kappa} \mathbb{C}_\alpha$.

Claim 3. *For any complete embedding $\pi : \mathbb{B}_\beta \to \mathrm{Col}(\omega, {<}\kappa)$ and $\delta < \kappa$, there is a $\beta < \beta' < \kappa$ and a complete embedding*

$$\pi^+ : \mathbb{B}_{\beta'} \to \mathrm{Col}(\omega, {<}\kappa)$$

that extends π so that $\mathbb{C}_\delta \subseteq \mathrm{ran}(\pi^+)$.

Proof. We can find $\delta \leqslant \gamma < \kappa$ so that $\operatorname{ran}(\pi \upharpoonright \mathbb{B}_\beta) \subseteq \mathbb{C}_\gamma$. Furthermore we can find some $\beta < \beta' < \kappa$ of size larger than $|\mathbb{B}_\beta|$. Now π^{-1} is a complete embedding of a complete subalgebra of $\mathbb{C}_{\gamma_\beta}$ into $\mathbb{B}_{\beta'}$. As $\mathbb{B}_{\beta'}$ is isomorphic to a collapse forcing, by the lifting property of collapses (Lemma 26.9 in [Jec03]), there is a complete embedding

$$\eta : \mathbb{C}_\gamma \to \mathbb{B}_{\beta'}$$

that extends π^{-1}. Essentially the same argument shows that we can find a complete embedding

$$\pi^+ : \mathbb{B}_{\beta'} \to \operatorname{Col}(\omega, {<}\kappa)$$

that extends η^{-1} (and thus π). □

Using this, we inductively construct a club sequence $\langle \beta_\gamma \mid \gamma < \kappa \rangle$ and complete embeddings

$$\pi_\gamma : \mathbb{B}_{\beta_\gamma} \to \operatorname{Col}(\omega, {<}\kappa)$$

such that $\operatorname{ran}(\pi_\gamma) \supseteq \mathbb{C}_\gamma$. The successor step is handled by the above claim. At limit steps, let $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$. we already have

$$\bigcup_{\alpha<\gamma} \pi_\alpha : \mathbb{B}_{<\gamma} \to \operatorname{Col}(\omega, {<}\kappa)$$

and this is a complete complete embedding as the construction gives that for some $\delta < \kappa$ we have $\bigcup_{\alpha<\gamma} \operatorname{ran}(\pi_\alpha) = \mathbb{C}_{<\delta} <_{\text{reg}} \operatorname{Col}(\omega, {<}\kappa)$. As $\mathbb{B}_\gamma$ is just the completion of $\mathbb{B}_{<\gamma}$, (so that consequently $\mathbb{B}_{<\gamma}$ is dense in $\mathbb{B}_\gamma$) there is a unique extension of $\bigcup_{\alpha<\gamma} \pi_\alpha$ to a complete embedding

$$\pi_\gamma : \mathbb{B}_\gamma \to \operatorname{Col}(\omega, {<}\kappa)$$

Thus finally,

$$\pi := \bigcup_{\gamma<\kappa} \pi_\gamma : \mathbb{B} \cong \operatorname{Col}(\omega, {<}\kappa)$$

is an isomorphism.

□

The proof of Theorem 34.6 can be saved though. With the notation from there, Exercise 34.5 is used to infer that

$$V^{P*Q} \cong V^{\operatorname{Col}(\omega,<\kappa)}$$

which is not quite true. If we let $\mathbb{B} = \mathbb{B}(P * Q)$, then $\mathbb{B}$ satisfies (34.6), but there is no hope that $\Delta(\vec{\mathbb{B}})$ contains a club. Anyhow, κ is weakly compact and as

$$\text{“}\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha \text{ is } \kappa\text{-cc”}$$

is a Π^1_1-statement over $(V_\kappa, \in, \mathbb{B}, \vec{\mathbb{B}})$, the weak compactness of κ gives that $\Delta(\vec{\mathbb{B}})$ is stationary. Thus we can force a club through there in V^{P*Q}, where κ is ω_1; call that forcing R. If $H = H_0 * H_1 * H_2$ is $P * Q * R$-generic, then $V[H]$ has the same reals as $V[H_0 * H_1]$ and we can check that in $V[H]$ there is an isomorphism of $\mathbb{B}$ and $\mathrm{Col}(\omega, {<}\kappa)^V$ and thus a filter G that is V-generic filter for the latter forcing such that $V[G]$ has all the reals of $V[H_0 * H_1]$. Thus after all there is still an elementary embedding

$$j : L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$$

A further example where $\vec{\mathbb{B}}$ witnesses that $\mathbb{B}$ satisfies (34.6), yet is not isomorphic to $\mathrm{Col}(\omega, {<}\kappa)$ is

$$\mathbb{B} = \mathrm{Col}(\omega, \lambda) \oplus \mathrm{Col}(\lambda, {<}\kappa)$$

with

$$\mathbb{B}_\alpha = \mathrm{Col}(\omega, \lambda) \oplus \mathrm{Col}(\lambda, {<}\alpha)$$

Here $\lambda < \kappa$ is a regular uncountable cardinal. Then

$$\Delta(\vec{\mathbb{B}}) \cap \mathrm{Lim} = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) \geqslant \lambda\}$$

does not contain a club. Hence there is no dense embedding from $\mathbb{B}$ into $\mathrm{Col}(\omega, {<}\kappa)$. One can check that if $\mathbb{B}$ and $\mathrm{Col}(\omega, {<}\kappa)$ were forcing equivalent, there would be such a dense embedding and hence they are not. This results in a headache in some lifting arguments.

References

[Jec03] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.