On the subcompleteness of some Namba-type forcings

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§1 In this paper we show — under the assumption CH $+2^{\omega_1} = \omega_2$ — that two Namba-like forcings $\mathbb{N}'$ and $\mathbb{N}^*$ are subcomplete. $\mathbb{N}'$ is the set of Namba trees with a finite stem $s = \text{stem}(T)$ s.t. for all $t \in T$, either $t = s{\restriction}n$ for some $n < |s|$ or else $t \supset s$ has ω_2 many immediate successors. The salient property of $\mathbb{N}'$-generic sequences $\langle \gamma_i : i < \omega \rangle$ is that, whenever $F : \omega_2 \to \omega_2$ is a function in the ground model, then:

$$\bigvee n \bigwedge m \geq n \quad F(\gamma_m) < \gamma_{m+1}.$$

$\mathbb{N}^*$ is defined like $\mathbb{N}'$ except that we impose the stronger requirement that if $t \in T$ and $t \supset \text{stem}(T)$, then:

$$\{\alpha \mid t^\frown\langle\alpha\rangle \in T\} \text{ is stationary in } \omega_2.$$

The salient property of $\mathbb{N}^*$-generic sequences is that whenever $A \subset \omega_2$ is club in the ground model, then:

$$\bigvee n \bigwedge m \geq n \quad \gamma_m \in A.$$

Both forcings add no reals, assuming CH in the ground model.

$\mathbb{N}'$ has been treated extensively in the literature, especially [PIF]. We are not aware of previous treatments of $\mathbb{N}^*$ and would be grateful for any references.

*As part of the Ronald Jensen Nachlass Project, this LaTeX file was produced by Andreas Lietz from Jensen's handwritten notes. Claude produced a preliminary version, which was proofread and corrected by Andreas Lietz.

In [DSP] we generalized Shelah's notions of "dee-complete" and "ω_1-proper" forcing to "dee-subcomplete" and "ω_1-subproper". (Unfortunately we inadvertently changed the term "dee-complete" into "dee-proper". We apologize for this and will avoid doing so in the future.) We showed that, under CH, $\mathbb{N}'$ had both properties and, therefore, can be iterated without adding reals. Quite recently we introduced the notion of "almost subcomplete forcing" and proved an iteration theorem for these forcings. Assuming $\mathrm{CH} + 2^{\omega_1} = \omega_2$ we showed that $\mathbb{N}'$ and $\mathbb{N}^*$ are both almost subcomplete. We then belatedly realized that our proof showed $\mathbb{N}'$, $\mathbb{N}^*$ to be, in fact, fully subcomplete. (Embarrassingly, this leaves us with no "real" applications for almost subcomplete forcing.)

Our proofs will deal mainly with $\mathbb{N}^*$, though we shall briefly indicate the changes to be made in proving the same results for $\mathbb{N}'$. After that we introduce the basic theory of $\mathcal{L}$-forcing in §3. (For this the reader may also consult [LF], which for present purposes is more suitable than the rather abstruse treatment in [Sing].) In §4 we prove the equivalence of $\mathbb{N}^*$ with an $\mathcal{L}$-forcing. In §5 we then prove the main result. In §6 we discuss further properties of the forcings $\mathbb{N}$, $\mathbb{N}'$, and $\mathbb{N}^*$. Throughout this paper — except in §3 — we assume CH. In §5 we also assume: $2^{\omega_1} = \omega_2$.

§2 Namba-Forcing

Definition (Namba forcing)**.** Namba forcing is the set $\mathbb{N}$ of trees $T \subset (\omega_2)^{<\omega}$ s.t.

- $t \in T \to t{\restriction}m \in T$ for $m < |t|$. (We write $|t| = \mathrm{dom}(t)$ for $t \in (\omega_2)^{<\omega}$.)
- If $t \in T$, then there are ω_2 many $s \in T$ s.t. $t \subset s$.
- Each $t \in T$ is a finite, ascending sequence of ordinals in ω_2.

$\mathbb{N}$ is partially ordered by inclusion: $T \leq T'$ in $\mathbb{N} \;\leftrightarrow\; T \subset T'$.
For $s \in T \in \mathbb{N}$ we set:

$$T_{(s)} = \{t \in T \mid t \subset s \vee s \subset t\}.$$

The *stem* of T is the largest s s.t. $T = T_{(s)}$.

Definition (Namba$'$ forcing)**.** Namba$'$ forcing is the set $\mathbb{N}' \subset \mathbb{N}$ of trees T s.t. whenever $t \in T$ and $t \supset \mathrm{stem}(T)$, then t has ω_2 immediate successors in T. $\mathbb{N}'$ has been studied at length in [PIF] and elsewhere. In [DSP] we showed that $\mathbb{N}'$ is ω_1-subproper and dee-subcomplete.

Definition (Namba* forcing)**.** Namba*-forcing is the set of $T \in \mathbb{N}'$ s.t.

$$\{\alpha \mid t^\frown\langle\alpha\rangle \in T\} \text{ is stationary in } \omega_2,$$

whenever $t \supset \mathrm{stem}(T)$, $t \in T$.

Note. *It seems likely that $\mathbb{N}^*$ has also been treated in the literature, but at present we don't know where. We would be grateful for any references.*

Definition. We say that $b \in (\omega_2)^\omega$ is a *branch through* T iff $b{\restriction}n \in T$ for $n < \omega$. If G is $\mathbb{N}^*$-generic, then $b = b_G =: \bigcup\bigcap G$ is a branch through $(\omega_2)^{<\omega}$. Conversely, G is recoverable from b by:

$$G = G_b =: \{T \in \mathbb{N}^* \mid b \text{ is a branch through } T\}.$$

(Similarly for $\mathbb{N}, \mathbb{N}'$.) We call $b = \langle\gamma_i : i < \omega\rangle$ $\mathbb{N}^*$-generic iff G_b is $\mathbb{N}^*$-generic.

$\mathbb{N}^*$ then has the following property:

Lemma 1. *Let b be $\mathbb{N}^*$-generic over V. Let $C \in V$ be club in ω_2. Then*

$$(*) \qquad \bigvee n \bigwedge i \geq n \ \gamma_i \in C,$$

where $b = \langle \gamma_i : i < \omega \rangle$.

Proof. Let $T \in \mathbb{N}^*$, put $n = |\text{stem}(T)|$. Set:

$$T' = \{t \in T \mid \bigwedge i \geq n \ t(i) \in C\}.$$

Then $T' \in \mathbb{N}^*$, $\text{stem}(T') = \text{stem}(T)$ and $T' \leq T$ in $\mathbb{N}^*$. But every branch through T' satisfies $(*)$. QED (Lemma 1)

An entirely similar proof shows:

Lemma 2. *Let b be $\mathbb{N}'$-generic over V. Let $C \in V$ be club in ω_2 and let:*

$$C(\gamma) = \min\{\beta > \gamma \mid \beta \in C\} \quad \textit{for } \gamma < \omega_2.$$

Then:

$$\bigvee n \bigwedge i \geq n C(\gamma_i) < \gamma_{i+1}, \quad \textit{where } b = \langle \gamma_i : i < \omega \rangle.$$

We also mention that the properties of being $\mathbb{N}$-generic, $\mathbb{N}'$-generic, or $\mathbb{N}^*$-generic are mutually exclusive.

$\mathbb{N}^*$ satisfies the following *weak amalgamation lemma*:

Lemma 3 (Weak Amalgamation Lemma)**.** *Let $\langle T_n \mid n < \omega \rangle$ be s.t. for $n < \omega$:*

- $T_{n+1} \leq T_n$ *in* $\mathbb{N}^*$
- $\text{stem}(T_n) = \text{stem}(T_{n+1})$
- $T_n \restriction n = T_{n+1} \restriction n$

(where $T \restriction n =: \{t \in T \mid |t| \leq n\}$.)

Then $T \in \mathbb{N}^*$ where:

$$T = \bigcap_n T_n = \bigcup_n T_n \restriction n.$$

QED (Lemma 3)

It is easily seen that Lemma 3 also holds with $\mathbb{N}'$ in place of $\mathbb{N}^*$.

The following *specialization lemma* will be used to show that $\mathbb{N}^*$ adds no reals:

Lemma 4 (Specialization Lemma)**.** *Assume* CH. *Let* $T \in \mathbb{N}^*$, $f\colon T \to \omega_1$. *There is* $T' \leq T$ *in* $\mathbb{N}^*$ *s.t.*

$$|s| = |s'| \to f(s) = f(s') \quad \textit{for all } s, s' \in T'.$$

(Hence there is $g\colon \omega \to \omega_1$ *s.t.* $f(s) = g(|s|)$ *for* $s \in T'$*.)*

Proof. For each $g\colon \omega \to \omega_1$ let G_g be the following game:

At stage i, I picks a club $C_i \subset \omega_2$ s.t. $C_i \subset \bigcap_{j<i} C_j$. If $i \leq m = |\mathrm{stem}(T)|$, then $C_i = \omega_2$. II then picks γ_i s.t. $\langle \gamma_0, \ldots, \gamma_i \rangle \in T$, $\gamma_i \in C_i$ and $f(\gamma_0, \ldots, \gamma_i) = g(i)$ if possible. If II has no move, then I wins. If I does not win at any finite stage, then II wins.

Claim. *II has a winning strategy for some* G_g.

Proof. Suppose not. Then I has a winning strategy S_g for every g. $S_g(i)$ is then a club set in ω_2 and we set:

$$S(i) = \bigcap_{g\colon \omega \to \omega_1} S_g(i).$$

Then S is a strategy which wins every game G_g. Now pick a branch $\langle \gamma_i : i < \omega \rangle$ through T s.t. $\gamma_i \in S(i)$ for $i < \omega$. Then $\langle \gamma_i : i < \omega \rangle$ is a play which defeats S. Contradiction! QED (Claim)

Now let II have a strategy S for G_g. Let T' be the set of $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in T$ s.t. for some sequence $C_0, \ldots, C_{n-1}$ of possible plays for I we have: $\nu_i = S(C \restriction i)$ $(i < n)$. It is easily seen that $T' \in \mathbb{N}^*$ and $\mathrm{stem}(T') = \mathrm{stem}(T)$. QED (Lemma 4)

Note. *Lemma 4 goes through for* $\mathbb{N}'$ *in place of* $\mathbb{N}^*$*. To prove it we alter the proof to have I pick a* $\gamma_i < \omega_2$*. The contradictory "universal" strategy* S *for I is then defined by:*

$$S(i) = \sup_{g\colon \omega \to \omega_1} S_g(i).$$

Everything goes through as before.

Corollary 5. *Assume* CH. *Then* $\mathbb{N}^*$ *adds no reals.*

Proof. Let $\Vdash \dot{f}\colon \check{\omega} \to \check{\omega}_1$.

Claim. $\dot{f}^G \in V$ *for all* $\mathbb{N}^*$*-generic* G.

Proof. Let $T \in \mathbb{N}^*$. It suffices to show:

Subclaim. *There is $T' \leq T$ s.t. $T' \Vdash \dot{f} = \check{f}$ for some $f \in V$.*

Proof. We first apply the amalgamation theorem, forming T_n $(n < \omega)$ s.t.

- $\mathrm{stem}(T_n) = \mathrm{stem}(T)$
- $T_n \restriction n = T_{n+1} \restriction n$
- $T_{n+1} \leq T_n$.

Simultaneously we assign

$$g_n \colon T_n \restriction n \to \omega_1^{<\omega}$$

as follows: Let $s = \mathrm{stem}(T)$. For $n \leq |s|$, set:

$$T_n = T, \qquad g_n(t) = \emptyset.$$

Now let $n \geq |s|$. We form T_{n+1}, g_{n+1}. For each $t \in T_n$ s.t. $|t| = n+1$ look at $(T_n)_{(t)}$ and pick a maximal $p \leq n$ s.t. there exist $T'_t \leq (T_n)_{(t)}$ and $\langle \gamma_0, \ldots, \gamma_{p-1} \rangle$ s.t.:

- $\mathrm{stem}(T'_t) = t = \mathrm{stem}((T_n)_{(t)})$
- $T'_t \Vdash \dot{f}(i) = \gamma_i$ for $i < p$,

(We could have: $p = 0$, $T'_t = (T_n)_{(t)}$.)

Set: $T_{n+1} = \bigcup\limits_{\substack{t \in T_n \\ |t| = n+1}} T'_t.$

$$g_{n+1}(t) = \langle \gamma_0, \ldots, \gamma_{p-1} \rangle \quad \text{for } t \in T_{n+1},\ |t| = n+1.$$

For $t \in T_{n+1}$, $|t| \leq n$ we set $g_{n+1}(t) = g_n(t)$.

Set $T'' = \bigcap_n T_n = \bigcup_n (T_n \restriction n)$. Define $g \colon T'' \to \omega_1^{<\omega}$ by:

$$g = \bigcup_n g_n.$$

Then $T'' \in \mathbb{N}^*$, $T'' \leq T$, $\mathrm{stem}(T'') = \mathrm{stem}(T)$.

We now apply specialization. Let $T' \leq T''$ s.t. $\mathrm{stem}(T') = \mathrm{stem}(T)$ and $g(t) = g(t')$ for $|t| = |t'|$, $t, t' \in T'$. Then $g(t) = \tilde{f}(|t|)$ for a $\tilde{f} \colon \omega \to (\omega_1)^{<\omega}$. It is easily seen that $t \subset s \to g(t) \subset g(s)$. Hence $\tilde{f}(i) \subset \tilde{f}(i')$ for $i \leq i' < \omega$. Set: $f = \bigcup_i \tilde{f}(i)$. Then $\mathrm{dom}(f) \leq \omega$ and $T' \Vdash \dot{f} \restriction |\tilde{f}(i')| = \tilde{f}(i')$ for $i' < \omega$. Then it suffices to show:

Claim. $\mathrm{dom}(f) = \omega$.

Proof. Suppose not. Then for some n we have:

$$i' \geq n \rightarrow \tilde{f}(i') = f.$$

Pick $t \in T'$, $|t| \geq n$. Let $d = \mathrm{dom}(f) < \omega$. Pick $T^* \leq T'$ in $\mathbb{N}^*$ s.t.

$$T^* \Vdash \dot{f}(\check{d}) = \check{\gamma} \quad \text{for some } \gamma.$$

We assume w.l.o.g. that $|s| \geq m \geq d$ where $s = \mathrm{stem}\, T^*$. Then $s \in T''$. Hence $s \in T_{|s|-1}$ and there is $T^* \leq (T_{|s|-1})_{(s)}$ s.t. $T^* \Vdash \dot{f}(\check{d}) = \check{\gamma}$. By our construction we have $g(s) = f$. But the existence of T^* shows that $d = \mathrm{dom}(f) \in \mathrm{dom}(g(s))$.

Contradiction! QED (Corollary 5)

Note. *Exactly the same proof shows that* $\mathbb{N}'$ *adds no reals.*

§3 $\mathcal{L}$-Forcing

A major step in our proof will be to show that $\mathbb{N}^*$ is equivalent to an $\mathcal{L}$-forcing. By "$\mathcal{L}$-forcing" I refer to a body of related forcings, all of which have in common that the conditions represent certain statements in an infinitary language. The criterion for being a condition is then that the statement represented is consistent. Beyond that, however, I have not had much success in stating a general definition, even of those $\mathcal{L}$-forcings which do not add reals (which is the relevant case here). The definition I give here is closer to that given in [LF] rather than the more abstruse definition in [Sing]. I shall state some lemmas without proof, but it should be possible to reconstruct their proof from [LF].

We first introduce some notation. Set:

$$L_\beta^{A_1,\ldots,A_m} =: \langle M, \in, A_1 \cap M, \ldots, A_m \cap M\rangle \quad \text{where } M = L_\beta[A_1,\ldots,A_m].$$

If $M = L_\beta^{A_1,\ldots,A_m}$, we set:

$$M^{B_1,\ldots,B_m} =: L_\beta^{A_1,\ldots,A_m,B_1,\ldots,B_m}.$$

Finally we note the following trivial

Fact. *Let $\bar{M} = L_{\bar{\beta}}^{\bar{A}}$, $M = L_\beta^A$. Let $\bar{\pi}\colon \bar{M} \prec M$. Given classes $\bar{B}_1,\ldots,\bar{B}_n$ and $B_1,\ldots,B_n$, there is at most one extension $\bar{\pi}^* \supset \bar{\pi}$ s.t. $\bar{\pi}^*\colon \bar{M}^{\bar{B}_1,\ldots,\bar{B}_n} \prec M^{B_1,\ldots,B_n}$.*

Now fix $M = L_\beta^A$ where $\beta > 2^\omega$ is a cardinal. Let A be so chosen that $\mathcal{P}(\omega) \subset L_\beta^A$ and the property of being a cardinal is absolute in M for all ordinals $\alpha < \beta$.

Let $N = \langle H_{\beta^+}, \ldots\rangle$ be a structure of countable type. (In our specific example we take $\beta = \omega_2$ and $N = \langle H_{\omega_3}, \in, <\rangle$ where $<$ is a well ordering of H_{ω_3}.) (We shall usually include a well ordering of H_{β^+} among the predicates of N.)

N is then an admissible set and we can apply Barwise Theory to it. (For a brief account of Barwise Theory see [Sing].) Consider an infinitary language $\mathcal{L}$ on N with:

<u>Predicate:</u> $\dot{\in}$

<u>Constants:</u> $\dot{M}, \dot{\pi}, \dot{B}, \underline{x}$ $(x \in N)$

<u>Axioms:</u>

- ZFC$^-$ (meaning the finite axioms of ZFC$^-$)
- $\bigwedge v\,(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z\in x} v = \underline{z})$
- $\dot{M} = \langle \dot{M}_i \mid i \leq \omega_1\rangle$, $\dot{\pi} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \omega_1\rangle$
- $\dot{\pi}$ is a commutative sequence of elementary embeddings $\dot{\pi}_{ij} \colon \dot{M}_i \prec \dot{M}_j$.
- $\dot{M}_{\omega_1} = \underline{M}$; $\bigwedge i < \omega_1\,(\dot{M}_i$ is countable and transitive)
- $\dot{\alpha}_i = \mathrm{crit}(\dot{\pi}_{i,i+1})$, where $\dot{\alpha}_i = (\omega_1)^{\dot{M}_i}$
- $\dot{\beta}_i < \dot{\alpha}_{i+1}$, where $\dot{\beta}_i = \omega_1^{\dot{M}_i}$
- $\dot{B} \subset \dot{M}$
- $H_{\omega_1} = \underline{H_{\omega_1}}$.

We refer to these axioms as the *basic axioms.* $\mathcal{L}$ might, of course, contain further axioms (but no further predicates and constants). We then use $\mathcal{L}$ to define a set $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of forcing conditions. Before proceeding to that, however, we recall some definitions:

Definition (Well-founded core)**.** Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \ldots\rangle$ be a model of ZFC$^-$. By the *well founded core* of $\mathfrak{A}$ (wfc($\mathfrak{A}$)) we mean the set of $x \in \mathfrak{A}$ s.t. $\in^{\mathfrak{A}} \cap\, e(x)^2$ is well founded, where $e(x) =$: the closure of $\{x\}$ under $\in^{\mathfrak{A}}$.

Definition (Grounded)**.** $\mathfrak{A}$ is *grounded* iff wfc($\mathfrak{A}$) is transitive and $x \in y \leftrightarrow x \in^{\mathfrak{A}} y$ for $x, y \in \mathrm{wfc}(\mathfrak{A})$.

Note. *In [LF] we called such $\mathfrak{A}$ "solid".*

It is then easy to see that:

Fact. *Let $\mathfrak{A}$ be a grounded model of the basic axioms. Then:*

- $\underline{x}^{\mathfrak{A}} = x$ *for* $x \in N$
- $H^{\mathfrak{A}}_{\omega_1} = H_{\omega_1}$
- *Let* $\tilde{M} = \dot{M}^{\mathfrak{A}}$*. Then* $\tilde{M} = \langle M_i \mid i \leq \omega_1\rangle$ *where* $M_i = L^{A_i}_{\beta_i}$ *is transitive and countable and* $M_{\omega_1} = M$*.*

- *Let* $\tilde{\pi} = \dot{\pi}^{\mathfrak{A}}$. *Then* $\tilde{\pi} = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ *is a continuous, commutative sequence of elementary embeddings* $\pi_{ij} \colon M_i \prec M_j$. *Moreover* $\mathrm{crit}(\pi_{i,i+1}) = \alpha_i =: \omega_1^{M_i}$ *for* $i < \omega_1$.
- $\beta_i < \alpha_{i+1}$ *for* $i < \omega_1$
- *Let* $B = \dot{B}^{\mathfrak{A}}$, $B_i = \pi_{i,\omega_1}^{-1}$ "B. *Then* $B_i \in H_{\omega_1}$ *and* $\pi_{ij} \colon \langle M_i, B_i \rangle \to \langle M_j, B_j \rangle$ *is a structure preserving map* ($i \leq j \leq \omega_1$)

We now turn to the definition of the condition $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$. We first define a set of *preconditions* $\widetilde{\mathbb{P}}$:

Definition (Preconditions)**.** $\widetilde{\mathbb{P}}$ is the set of pairs $p = \langle P_0, P_1 \rangle$ s.t.

(a) $P_0 = \langle M^p, \pi^p, b^p \rangle \in H_{\omega_1}$, where:

- $M^p = \langle M_i^p \mid i \leq \delta \rangle$, $\pi^p = \langle \pi_{ij}^p \mid i \leq j \leq \delta \rangle$ where $\pi_{ij}^p \colon M_i^p \prec M_j^p$ is a continuous commutative system of elementary embeddings. (Set $|p| =: \delta$.)
- $b^p \subset M_\delta^p$

(b) P_1 is a countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $a \subset M$ and $\bar{a} \subset M_{|p|}^p$. (We might refer to $M_{|p|}^p$ by $\bar{M}$ if p is clear from the context)

We then define:

Definition (The formula φ_p)**.** Let $p \in \widetilde{\mathbb{P}}$. φ_p is the formula:

$$\underline{M^p} = \langle \dot{M}_i \mid i \leq |p| \rangle \ \wedge \ \underline{\pi^p} = \langle \dot{\pi}_{ij} \mid i \leq j \leq |p| \rangle$$
$$\wedge \bigwedge_{\langle a, \bar{a} \rangle \in P_1} \dot{\pi}_{|p|,\omega_1} \colon \underline{\langle M_{|p|}, \bar{a} \rangle} \prec \underline{\langle M, a \rangle}^1$$
$$\wedge \ \underline{b^p} = \dot{\pi}_{|p|,\omega_1}^{-1} \text{“}\dot{B}.$$

Definition ($\mathbb{P}_{\mathcal{L}}$)**.** $\mathbb{P} = \mathbb{P}_{\mathcal{L}} =$: the set of $p \in \widetilde{\mathbb{P}}$ s.t. $\mathcal{L}(p) =: \mathcal{L} + \varphi_p$ is consistent.

Definition. Let $p \in \mathbb{P}$. We set:

$$F^p = P_1, \qquad R^p = \mathrm{rng}(F^p), \qquad D^p = \mathrm{dom}(F^p).$$

[1] $\langle M, a \rangle$; hence $\langle M_{|p|}^p, \bar{a} \rangle$ need not be amenable!

We partially order $\mathbb{P}$ by:

Definition (Partial order)**.**

$p \leq q \leftrightarrow: \Big(|p| \geq |q| \;\wedge\; M^q = M^p\restriction(|q|+1) \;\wedge\; \pi^q = \pi^p\restriction(|q|+1)^2 \;\wedge\; R^q \subset R^p\Big).$

Lemma 0.1 $(F^p)^{-1}$ is a function.

Proof. Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$. Then $\langle a, \bar{a}\rangle \in F^p \leftrightarrow \bar{a} = (\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1})^{-1}\text{“}a$. QED (0.1)

Lemma 0.2 Let R^p be closed under set difference. Then F^p injects D^p onto R^p.

Proof. Let $\langle a, \bar{a}\rangle, \langle b, \bar{b}\rangle \in F^p$. It suffices to show:

Claim. $\bar{a} \subset \bar{b} \to a \subset b.$

Set $\bar{c} = \bar{b}\setminus\bar{a}$, $c = b\setminus a$. Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$. Let $\pi = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}$. Then:

$$(F^p)^{-1}(c) = \pi^{-1}\text{“}(b \setminus a) = \pi^{-1}\text{“}b \setminus \pi^{-1}\text{“}a = \bar{b} \setminus \bar{a} = \bar{c},$$

Hence $\bar{b} \subset \bar{a} \to \bar{c} = \emptyset \to c = \emptyset \to b \subset a$, since $\pi\colon \langle M^p_{|p|}, \bar{c}\rangle \prec \langle M, c\rangle$. QED (0.2)

Lemma 0.3 Let $p \leq q$.

(i) If $\langle a, \bar{a}\rangle \in F^q$ and $\langle a, a'\rangle \in F^p$, then $\pi^p_{|q|,|p|}\colon \langle M^q_{|q|}, \bar{a}\rangle \prec \langle M^p_{|p|}, a'\rangle$.

(ii) $B^q = (\pi^p_{|q|,|p|})^{-1}\text{“}B^p$.

Proof. Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$. Let $\bar{\pi} = \dot{\pi}^{\mathfrak{A}}_{|q|,\omega_1}$, $\pi = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}$. Then $\pi^p_{|q|,|p|} = \pi^{-1} \circ \bar{\pi}$. QED (0.3)

Definition. $\pi^p_i = \pi^p_{i,\omega_1} =\colon F^p \circ \pi^p_{i,|p|}$ for $i \leq |p|$.

We obviously have:

Lemma 0.4 π^p_i is a partial injection of M^p_i into M. Moreover, if $p \leq q$, then $\pi^q_i \subset \pi^p_i$ for $i \leq |q|$.

Definition (Compatibility)**.** Let $p, q \in \mathbb{P}$.

$$p \| q \leftrightarrow: p, q \text{ are compatible in } \mathbb{P}$$

$$p \perp q \leftrightarrow: \neg(p \| q).$$

We then get:

Lemma 1.1 $p \| q \leftrightarrow \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

Proof. $(\rightarrow)$ Let $r \le p, q$. Then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$.

$(\leftarrow)$ Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p) \cup \mathcal{L}(q)$. Then ω_1 is regular in $\mathfrak{A}$ and $R^p \cup R^q$ is countable in $\mathfrak{A}$. There is then an $\alpha < \omega_1$ s.t. $\alpha \ge |p|, |q|$ and for all $a \in R^p \cup R^q$ we have:

$$\dot{\pi}^{\mathfrak{A}}_{\alpha,\omega_1} : \langle \dot{M}^{\mathfrak{A}}_{\alpha}, a^* \rangle \prec \langle M, a \rangle$$

where $a^* = (\dot{\pi}^{\mathfrak{A}}_{\alpha,\omega_1})^{-1} ``a$. (Hence $a^* \in H_{\omega_1}$.) This clearly exists, since $\mathfrak{A}$ models ZFC^- and H_{ω_1} is absolute in $\mathfrak{A}$.

Define $r \in \widetilde{\mathbb{P}}$ by:

$$M^r = \langle \dot{M}^{\mathfrak{A}}_i \mid i \le \alpha \rangle, \qquad \pi^r = \langle \dot{\pi}^{\mathfrak{A}}_{ij} \mid i \le j \le \alpha \rangle,$$

$$b^r = (\dot{\pi}^{\mathfrak{A}}_{\alpha,\omega_1})^{-1} ``\dot{B}^{\mathfrak{A}}$$

$$F^r = \{ \langle a, a^* \rangle \mid a \in R^p \cup R^q \}.$$

Then $\mathfrak{A} \models \mathcal{L}(r)$. Hence $r \in \mathbb{P}$. But $r \le p, q$ in $\mathbb{P}$. QED (Lemma 1.1)

The proof of $(\leftarrow)$ in Lemma 1.1 is a model for many similar proofs. Using it, we get the *extension lemma*:

Lemma 1.2 Let $p \in \mathbb{P}$. Let u be any countable collection of subsets of M. Then there is $q \le p$ s.t. $u \subset R^q$.

(The proof turns on the fact that if $\mathfrak{A}$ is a grounded model of $\mathcal{L}(p)$, then u is countable in $\mathfrak{A}$, since $u \in N = H_{\omega_3} \subset \mathfrak{A}$.)

Corollary 1.3 Let $p \in \mathbb{P}$, $u \subset M$ be countable. Then there is $q \le p$ s.t. $u \subset \mathrm{rng}(\pi^q_{|q|})$.

Lemma 1.4 Let $p \in \mathbb{P}$. Let $u \subset M^p_i$ be finite. There is $q \le p$ s.t. $u \subset \mathrm{dom}(\pi^q_i)$.

Lemma 1.5 Let $p \in \mathbb{P}$, $|p| \le \alpha < \omega_1$. There is $q \le p$ s.t. $|q| \ge \alpha$.

Fuller proofs of these lemmas can be found in [LF]. Using these lemmas we get:

Lemma 2 Let G be $\mathbb{P}$-generic over V. Define

$$M^G = \langle M_i^G \mid i \leq \omega_1 \rangle, \quad \pi^G = \langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle, \quad b^G = \langle b_i^G \mid i \leq \omega_1 \rangle$$

as follows: (Let $\omega_1 =: \omega_1^V$)

$$M_i^G =: M_i^p, \quad \pi_{ij}^G =: \pi_{ij}^p, \quad b_i^G =: b_i^p \quad \text{for } p \in G,\ i \leq |p|,$$

$$\pi_{i,\omega_1}^G =: \bigcup_{\substack{p \in G \\ i \leq |p|}} \pi_{i,\omega_1}^p, \qquad B^G = b_{\omega_1}^G =: \bigcup_{i < \omega_1} \pi_{i,\omega_1}^G \text{``} b_i^G.$$

$$M_{\omega_1}^G =: M, \qquad \pi_{\omega_1,\omega_1}^G = \mathrm{id} \restriction M.$$

Then:

(A) M^G, π^G is a continuous commutative sequence of elementary embeddings.

(B) $\omega_1 \in \mathrm{rng}(\pi_{i,\omega_1}^G)$. Moreover:

$$\alpha_i^G = \mathrm{crit}(\pi_{i,\omega_1}^G) \text{ where } \alpha_i^G = (\pi_{i,\omega_1}^G)^{-1}(\omega_1).$$

(C) $\beta_i^G < \alpha_{i+1}^G$ where $\beta_i^G = \mathrm{On} \cap M_i^G$.

(D) $b_i^p = (\pi_{i,\omega_1}^G)^{-1} \text{``} B^G$ for all $p \in G$

(E) If $p \in G$, $\alpha = \omega_1^V$, and $\langle a, \bar{a} \rangle \in F^p$, then $\pi_{|p|,\alpha}^G \colon \langle M_{|p|}^p, \bar{a} \rangle \prec \langle M, a \rangle$

Note. *G is recoverable from M^G, π^G, B^G by:*

$$p \in G \leftrightarrow \bigvee i < \omega_1 \Big(p_0 = \langle M^G \restriction (i+1), \pi^G \restriction (i+1)^2, (\pi_{i,\omega_1}^G)^{-1} \text{``} B^G \rangle$$

$$\wedge \bigwedge \langle a, \bar{a} \rangle \in p_1 \ \ \pi_{i,\omega_1}^G \colon \langle M_i^G, \bar{a} \rangle \prec \langle M, a \rangle \Big).$$

We now introduce an important concept.

Definition. Let $W = \langle H_\gamma, M, <, \ldots \rangle$ be a model of countable or finite type, where $\gamma \geq \beta^+$ is a cardinal and $<$ well orders H_γ.

p conforms to W iff $p \in \mathbb{P}$, W is as above, and whenever $a_1, \ldots, a_n \in R^p$ and $b \subset M$ is W-definable in $a_1, \ldots, a_n$, then $b \in R^p$.

(This holds for $n = 0$ as well, meaning that any W-definable $b \subset M$ is in R^p. By Lemma 0.2 we then have: F^p bijects D^p onto R^p.)

By the extension lemma it follows easily that $\{p \mid p \text{ conforms to } W\}$ is dense in $\mathbb{P}$.

In [LF] we prove the following deep lemma:

Lemma 3.1 Let p conform to W. There is a unique $\bar{W}$ s.t.

(i) $\bar{W}$ is transitive and of the same type as W.

(ii) If $a_1, \ldots, a_n \in R^p$ and $b \subset M$ is W-definable from $a_1, \ldots, a_n$, then $\bar{a}_1^p, \ldots, \bar{a}_n^p \in \bar{W}$ and $\bar{b}^p$ is $\bar{W}$-definable from $\bar{a}_1^p, \ldots, \bar{a}_n^p$ by the same definition.

(iii) Each $x \in \bar{W}$ is $\bar{W}$-definable from parameters in $M^p_{|p|} \cup D^p$.

Definition (Pulldown)**.** We call $\bar{W}$ the *pulldown* of W by p and denote it by $\mathrm{PD}(p, W)$.

Lemma 3.2 Let $\bar{W} = \mathrm{PD}(p, W)$. If $\mathfrak{A}$ is any grounded model of $\mathcal{L}(p)$, then $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} \cup F^p$ extends uniquely to a $\pi\colon \bar{W} \prec W$.

Lemma 3.3 Let $\bar{W} = \mathrm{PD}(p, W)$. Let G be $\mathbb{P}$-generic over V, s.t. $p \in G$. Then $\pi^G_{|p|,\omega_1} \cup F^p$ extends uniquely to $\pi\colon \bar{W} \prec W$.

We note that by Lemma 2:

Lemma 4.1 Let G be $\mathbb{P}$-generic over V. Then $p \in G \leftrightarrow V[G] \models \varphi_p$.

Lemma 4.2 Let G be $\mathbb{P}$-generic over V. If $V[G]$ has no new reals, then $V[G] \models \Psi$ for all basic axioms Ψ.

Of course $\mathcal{L}$ may contain further axioms. Since, however, we design $\mathcal{L}$ with a view to defining the set of conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$, we are unlikely to adopt an axiom which we do not expect to be forced by $\mathbb{P}$. Thus, in the best case $\mathcal{L}$ will be good in the following sense:

Definition (Good)**.** $\mathcal{L}$ is *good* iff whenever G is $\mathbb{P}$-generic over V, then $V[G] \models \Psi$ for all axioms Ψ.

In general, the axioms of $\mathcal{L}$ will be so chosen, that goodness will follow as soon as we have shown that the forcing $\mathbb{P}$ adds no reals, the verification of the other axioms being trivial.

Obviously we have:

Lemma 4.3 Let $\mathcal{L}$ be good. If $p \in G$, G is $\mathbb{P}$-generic over V, then

$$\vdash_{\mathcal{L}} (\varphi_p \to \Psi) \ \longrightarrow \ V[G] \models \Psi.$$

§4 An $\mathcal{L}$-Forcing Equivalent to $\mathbb{N}^*$

From now on assume CH. We apply the theory in §3, taking $\beta = \omega_2$, $N = H_{(2^{\omega_1})^+}$, and $M = L^A_{\omega_2} = \langle L_{\omega_2}[A], \in, A\rangle$, where $A \subset \omega_2$ is s.t. $L_{\omega_1}[A] = H_{\omega_1}$ and ω_1 is the largest cardinal in $L_{\omega_2}[A]$. $\mathcal{L}$ is the infinitary language on N which, in addition to the usual predicate, constants and basic axioms, has the further axioms:

- $\dot{B} = \langle \dot{\gamma}_i \mid i < \omega\rangle$ is monotone and cofinal in $\underline{\omega_2}$
- $\bigvee m \bigwedge n \geq m\ \dot{\gamma}_n \in \underline{C}$, for every club $C \subset \omega_2$
- $\text{rng}(\dot{\pi}_{i,\underline{\omega_1}}) =$ the smallest $X \prec \underline{M}$ s.t. $\dot{\alpha}_0 \cup \{\dot{\alpha}_j \mid j < i\} \cup \{\dot{\gamma}_n \mid n < \underline{\omega}\} \subset X$, for $i \leq \omega_1$.

Lemma 1 $\mathcal{L}$ is consistent.

Proof. Let $\langle \gamma_i \mid i < \omega\rangle$ be $\mathbb{N}^*$-generic over V. Set: $B = \langle \gamma_i \mid i < \omega\rangle$. Define M_i, π_{ij} as follows: Define $\langle \alpha_i \mid i \leq \omega_1\rangle$, $\langle X_i \mid i \leq \omega_1\rangle$ by:

$$X_i =: \text{the smallest } X \prec M \text{ s.t. } \{\alpha_j \mid j < i\} \cup \{\gamma_n \mid n < \omega\} \subset X$$

$$\alpha_i =: \omega_1 \cap X_i.$$

Let $\pi_{i,\omega_1} : M_i \overset{\sim}{\longleftrightarrow} M \restriction X_i$, where M_i is transitive. Set: $\pi_{ij} = \pi^{-1}_{j,\omega_1} \circ \pi_{i,\omega_1}$ for $i \leq j \leq \omega_1$.

Then

$$\langle H^V_{\omega_4}[B], \langle M_i \mid i \leq \omega_1\rangle, \langle \pi_{ij} \mid i \leq j \leq \omega_1\rangle, B\rangle$$

models $\mathcal{L}$. QED (Lemma 1)

Note. *There is a more elementary proof of Lemma 1 which makes no mention of* $\mathbb{N}^*$.

Set: $\mathbb{P} =: \mathbb{P}_{\mathcal{L}}$. If G is $\mathbb{P}$-generic, then it adds $M^G = \langle M^G_i \mid i \leq \omega_1\rangle$, $\pi^G = \langle \pi^G_{ij} \mid i \leq j \leq \omega_1\rangle$ and $B^G = \langle \gamma^G_i \mid i < \omega\rangle$ as defined in §3.

We call $B = \langle \gamma_i \mid i < \omega\rangle$ a $\mathbb{P}$-*generic sequence* iff $B = B^G$ for a $\mathbb{P}$-generic G. We shall show:

Theorem. *$B = \langle \gamma_i \mid i < \omega\rangle$ is $\mathbb{P}$-generic iff it is $\mathbb{N}^*$-generic.*

The proof stretches over many lemmata. We shall, in fact, define a forcing $\mathbb{N}^{**}$ in which $\mathbb{N}^*$ lies dense. We then prove the theorem with $\mathbb{N}^{**}$ in place of $\mathbb{N}^*$.

We first note that we may have to deal with structures $\langle M, A\rangle$, where $A \subset M$ lies in V but $\langle M, A\rangle$ is not amenable. A number of familiar arguments do not work. However, we do have:

Lemma 2 Let $\langle a, \bar{a}\rangle \in F^p$, $\bar{M} = M^p_{|p|}$. Assume (in any extension of V) that $\bar{\pi}\colon \langle \bar{M}, \bar{a}\rangle \longrightarrow_{\Sigma_0} \langle M, a\rangle$ cofinally. Then $\bar{\pi}\colon \langle \bar{M}, \bar{a}\rangle \prec \langle M, a\rangle$.

Proof. We first note that for any Σ_n (Π_n) formula φ there is a canonical Σ_n (Π_n) formula $\bar{\varphi}$ s.t.

$$(1) \qquad \bigvee x \in u\, \varphi(x, \vec{y}) \leftrightarrow \bar{\varphi}(u, \vec{y}),$$

holds in $\langle M, a\rangle$. This follows by iterated application of:

$$(2) \qquad \bigwedge x \in u \bigvee z\, \psi(x, z, \vec{y}) \;\leftrightarrow\; \bigvee w \bigwedge x \in u \bigvee z \in w\, \psi(x, z, \vec{y})$$

and:

$$(3) \qquad \bigvee x \in u \bigvee z\, \psi(x, z, \vec{y}) \;\leftrightarrow\; \bigvee w \bigvee x \in u \bigvee z \in w\, \psi(x, z, \vec{y}).$$

(3) is trivial, as in the direction ($\leftarrow$) of (2). To prove ($\rightarrow$), note that, by the regularity of ω_2, if the premise holds, then

$$\bigwedge x \in u \bigvee z \in L_\gamma[A]\, \psi(x, z, \vec{y})$$

for a $\gamma < \omega_2$.

(1) holds uniformly in $\langle \bar{M}, \bar{a}\rangle$:

Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$. Then $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}\colon \langle \bar{M}, \bar{a}\rangle \prec \langle M, a\rangle$.

We now prove by induction on φ that if φ is Σ_n, then:

$$(5) \quad \langle M, a\rangle \models \varphi[\pi(\vec{x})] \leftrightarrow \langle \bar{M}, \bar{a}\rangle \models \varphi[\vec{x}].$$

We proceed by induction on n. The case $n = 0$ is trivial. Now let $n = m+1$, $\varphi = \bigvee z\, \psi(z, \vec{x})$, where ψ is Π_m. It suffices to show:

Claim. $\langle M, a\rangle \models \varphi[\pi(\vec{x})] \rightarrow \langle \bar{M}, \bar{a}\rangle \models \varphi[\vec{x}]$.

Assume $\langle M, a\rangle \models \varphi[\pi(\vec{x})]$. Pick $\gamma < \beta =: \beta_{|p|}$ big enough that:

$$\langle M, a\rangle \models \bigvee z \in \pi(u)\, \psi[z, \pi(\vec{x})]$$

where $u = L_\gamma^{\bar{A}}$. Then by (1):

$$\langle M, a\rangle \models \bar{\psi}[\pi(u), \pi(\vec{x})]$$

where $\bar{\psi}$ is Π_m. Hence:

$$\langle \bar{M}, \bar{a}\rangle \models \bar{\psi}[u, \vec{x}]$$

Hence $\langle \bar{M}, \bar{a}\rangle \models \bigvee z \in u\, \psi[z, \vec{x}]$. QED (Lemma 2)

We now define:

Definition. $\mathbb{N}^{**} =:$ the set of subtrees $T \subset (\omega_2)^{<\omega}$ s.t. for every $t \in T$ there is a subtree $T' \subset T$ s.t. $t \subset \mathrm{stem}(T')$ and $T' \in \mathbb{N}^*$.

Since $\mathbb{N}^*$ is dense in $\mathbb{N}^{**}$ we have:

$$\mathrm{BA}(\mathbb{N}^*) \simeq \mathrm{BA}(\mathbb{N}^{**}).$$

Definition. Let $p \in \mathbb{P}$. $T_p =: \{s \in (\omega_2)^{<\omega} \mid \mathrm{Con}(\mathcal{L} + \varphi_p + \bigwedge\!\!\!\bigwedge_{i<|s|} \dot{\gamma}_i = \underline{s(i)})\}$.

Lemma 3 $T_p \in \mathbb{N}^{**}$.

Proof. Let $s \in T_p$. For each $m \geq |s|$ define a game G_m as follows. On the i-th move:

I plays C_i where $C_i \subset \omega_2$ is club and $C_i = \omega_2$ if $i < m$.
II plays $\gamma_i \in C_i$ s.t. $\langle \gamma_0, \ldots, \gamma_i\rangle \in T_p$ and $\gamma_i = s(i)$ if $i < |s|$.
I then wins at i iff II has no play. II wins iff I does not win at any i.

Claim. *II has a winning strategy for some game G_m.*

Proof. Suppose not. Then I has a winning strategy S^m for each $m \in [|s|, \omega)$.
Set $C = \{\lambda < \omega_2 \mid \bigwedge m < \omega \bigwedge t \in \lambda^{<\omega}\ \lambda \in S^m(t)\}$.
Then C is club in ω_2.
Hence there are $p' \leq p$, $m < \omega$ s.t.

$$p' \Vdash \bigwedge i \geq m\, \dot{\gamma}_i \in \underline{C} \ \wedge \bigwedge i < |s|\, \dot{\gamma}_i = \underline{s(i)}.$$

Successively choose γ_i $(i < \omega)$ s.t. $\mathrm{Con}(\mathcal{L}(p') + \bigwedge\!\!\!\bigwedge_{h<i} \dot{\gamma}_h = \underline{\gamma_h})$. Then $\langle \gamma_i \mid i < \omega\rangle$ is a play by II in G_m which defeats S^m. QED (Claim)

Now let II have a winning strategy S for G_m. Let T' be the result of applying S to all possible plays of I. Then $T' \in \mathbb{N}^*$, $s \subset \mathrm{stem}(T')$, $m =: |\mathrm{stem}(T')|$ and $T' \leq T_p$. QED (Lemma 3)

Lemma 4 Let $\langle \gamma_0, \ldots, \gamma_i \rangle = b_p \restriction (i+1)$. Let $\langle \gamma_0, \ldots, \gamma_i \rangle \in T_p$. There is a unique $\pi^i \colon J^{A^p}_{\bar{\gamma}_i} \prec J^A_{\gamma_i}$ defined by:

Let f be M-least s.t. $f \colon \omega_1 \xrightarrow{\text{onto}} \gamma_i$. Let $\bar{f}$ be $M^p_{|p|}$-least s.t. $\bar{f} \colon \alpha^p_i \xrightarrow{\text{onto}} \bar{\gamma}_i$. Then $\pi^i(\bar{f}(\zeta)) = f(\zeta)$ for $\zeta < \alpha^p_i$.

Moreover:

- $\pi^h = \pi^i \restriction J^{A^p}_{\bar{\gamma}_h}$ for $h < i$ $\quad (M^p_{|p|} = J^{A^p}_{\beta_p})$
- If $\langle a, \bar{a} \rangle \in F^p$, then

$$\pi^i \colon \langle J^{A^p}_{\bar{\gamma}_i}, \bar{a} \cap J^{A^p}_{\bar{\gamma}_i} \rangle \prec \langle J^A_{\gamma_i}, a \cap J^A_{\gamma_i} \rangle.$$

Proof. It is clear that π^i, if it exists, is uniquely defined by: $\pi^i \circ \bar{f} = f$. To show existence, let $\mathfrak{A}$ be a grounded model of $\mathcal{L} + \varphi_p$ s.t. $\dot{\pi}^{\mathfrak{A}}_{i,\omega_1}(\bar{\gamma}_h) = \gamma_h$ for $h \leq i$. Then $\dot{\pi}^{\mathfrak{A}}_{i,\omega_1}(\bar{f}) = f$. Hence $\pi^i = \dot{\pi}^{\mathfrak{A}}_{i,\omega_1} \restriction J^{A^p}_{\bar{\gamma}_i}$ has all of the above properties. QED (Lemma 4)

Lemma 5 Let $T' \leq T_p$ in $\mathbb{N}^{**}$. There is $q \leq p$ in $\mathbb{P}$ s.t. $q \Vdash \langle \dot{\gamma}_i \mid i < \omega \rangle$ is a branch in T'.

Proof. Let G be $\mathbb{N}^{**}$-generic, $T' \in G$. Let $\langle \gamma_i \mid i < \omega \rangle$ be the generic sequence given by G. Since $\langle \gamma_i \mid i < \omega \rangle$ is a branch in $T' \subset T_p$, there is for each $i < \omega$ the unique $\pi^i \colon J^{A^p}_{\bar{\gamma}_i} \prec J^A_{\gamma_i}$ defined as in Lemma 4. Then:

(1) $\pi^i \subset \pi^j$ for $i \leq j < \omega$.

Let f_i be M-least s.t. $f_i \colon \omega_1 \xrightarrow{\text{onto}} \gamma_i$. Let $\bar{f}_i$ be M-least s.t. $\bar{f}_i \colon \alpha^p_i \xrightarrow{\text{onto}} \bar{\gamma}_i$. Clearly, if $\bar{f}_i \in L^{A^p}_{\bar{\gamma}_j}$, then $f_i \in L^A_{\gamma_j}$, since by the definition of T_p there is (via a sufficient generic collapse) a grounded model $\mathfrak{A}$ of $\mathcal{L}(p) + \bigwedge\!\!\!\bigwedge_{n \leq j} \dot{\pi}^{\mathfrak{A}}_{j,\omega_1}(\bar{\gamma}_n) = \gamma_n$. But then, letting $\pi = \dot{\pi}^{\mathfrak{A}}_{j,\omega_1}$, we have $\pi(\bar{f}_i) = f_i$. Hence $\bar{\pi} = \pi \restriction L^{A^p}_{\bar{\gamma}_i}$ and $\pi^j = \pi \restriction L^A_{\gamma_j} \supset \pi^i$. QED (Lemma 5)

Set: $\pi = \bigcup_i \pi^i$. Then $\pi \colon \langle \bar{M}, \bar{a} \rangle \longrightarrow_{\Sigma_0} \langle M, a \rangle$ whenever $\bar{M} = M^p_{|p|}$ and $\langle a, \bar{a} \rangle \in F^p$. Hence $\pi \colon \langle \bar{M}, \bar{a} \rangle \prec \langle M, a \rangle$ by Lemma 2. QED (1)

Let $\alpha = \alpha_p$. Clearly $\mathrm{rng}(\pi)$ is the smallest $X \prec M$ s.t. $\alpha \cup \{\gamma_i \mid i < \omega\} \subset X$. (We set: $M^p_i = L^{A^p_i}_{\alpha^p_i}$, $\alpha_p = \alpha^p_{|p|}$, $A^p = A^p_{|p|}$.)

Define $\langle \alpha_i \mid i \le \omega_1 \rangle$, $\langle X_i \mid i \le \omega_1 \rangle$ by:

$$X_i =: \text{the smallest } X \prec M \text{ s.t. } \alpha \cup \{\alpha_j \mid j < i\} \cup \{\gamma_n \mid n < \omega\} \subset X$$

$$\alpha_i =: \omega_1 \cap X_i.$$

Clearly $X_i = \text{rng}(\pi \circ \pi^p_{i,|p|})$ for $i \le |p|$. Define $\langle M_i \mid i \le \omega_1 \rangle$, $\langle \pi_{ij} \mid i \le j \le \omega_1 \rangle$ by: $\pi_{i,\omega_1} \colon M_i \overset{\sim}{\longleftrightarrow} \langle X_i, \in, A \cap X_i \rangle$ when M_i is transitive,

$$\pi_{ij} = \pi^{-1}_{j,\omega_1} \circ \pi_{i,\omega_1} \quad \text{for } i \le j \le \omega_1.$$

Then $\pi = \pi_{|p|,\omega_1}$ and:

$$M^p = \langle M_i \mid i \le |p| \rangle, \qquad \pi^p = \langle \pi_{ij} \mid i \le j \le |p| \rangle.$$

For $a \in R^p$ set: $a_i = (\pi_{i,\omega_1})^{-1}``a$ $(i \le \omega_1)$. Then $a_{|p|} = \bar{a}$, where $\langle a, \bar{a} \rangle \in F^p$.

Clearly $T' \subset M$. For $i \le \omega_1$ set: $T'_i = (\pi_{i,\omega_1})^{-1}``T'$. There is a limit $\lambda \ge |p|$ s.t. $\pi_{\lambda,\omega_1} \colon \langle M_\lambda, a_\lambda \rangle \prec \langle M, a \rangle$ for all $a \in R^p$ and $\pi_{\lambda,\omega_1} \colon \langle M_\lambda, T'_\lambda \rangle \prec \langle M, T' \rangle$.

Set

$$q_0 = \langle M{\restriction}(\lambda+1), \pi{\restriction}(\lambda+1)^2, \langle \gamma_n \mid n < \omega \rangle \rangle$$
$$q_1 = \{\langle a, a_\lambda \rangle \mid a \in R^p\} \cup \{\langle T', T'_\lambda \rangle\}.$$

Then $q \in \mathbb{P}$, since:

$$\langle H_\theta, \langle M_i \mid i \le \omega_1 \rangle, \langle \pi_{ij} \mid i \le j \le \omega_1 \rangle, \langle \gamma_n \mid n < \omega \rangle \rangle$$

is a model of $\mathcal{L}(q)$ for sufficiently large regular θ in $V[G]$. Clearly, then, $q \le p$ in $\mathbb{P}$.

Claim. *Let $G' \ni q$ be $\mathbb{P}$-generic. Set $\gamma'_i = \gamma_i^{G'}$ $(i \le \omega)$. Then $\langle \gamma'_i \mid i < \omega \rangle$ is a branch through T'.*

Proof. Since $\pi_{\lambda,\omega_1} \colon \langle M_\lambda, T'_\lambda \rangle \prec \langle M, T' \rangle$ and $\pi_{\lambda,\omega_1}(\gamma_i^\lambda) = \gamma_i$ for $i < \omega$, where $\langle \gamma_i \mid i < \omega \rangle$ is a branch through T', we know that $\langle \gamma_i^\lambda \mid i < \omega \rangle$ is a branch through T'_λ — i.e. $\gamma^\lambda{\restriction}n \in T'_\lambda$ for $n < \omega$. Since $\pi^G_{\lambda,\omega_1} \colon \langle M_\lambda, T'_\lambda \rangle \prec \langle M, T' \rangle$, it follows that:

$$\gamma'{\restriction}n = \pi^G_{\lambda,\omega_1}(\gamma^\lambda{\restriction}n) \in T'$$

for $n < \omega$. QED (Lemma 5)

Note that if q, T' are as in Lemma 5, then $T_q \leq T'$. (Otherwise there is $s \in T_q$ s.t. $s \notin T'$. But then there is $q' \leq q$ s.t. $q' \Vdash \langle \dot{\gamma}_i \mid i \leq n \rangle = s$, where $n = |s|$. Let $G \ni q'$ be $\mathbb{P}$-generic. Then $q \in G$ and $\langle \gamma_i^G \mid i < \omega \rangle$ is not a branch in T'. Contradiction!)

Corollary 6 Let Δ be strongly dense in $\mathbb{N}^{**}$. Then $\{p \mid T_p \in \Delta\}$ is dense in $\mathbb{P}$.

Proof. Let $T' \leq T_p$, $T' \in \Delta$. Pick $q \leq p$ s.t. $T_q \leq T'$. Then $T_q \in \Delta$.
QED (Corollary 6)

But then:

Corollary 7 Let $B = \langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{P}$-generic. Then B is $\mathbb{N}^{**}$-generic.

Proof. Let $B = B^G$ where G is $\mathbb{P}$-generic. Then

$$G' = \{T \in \mathbb{N}^{**} \mid \bigvee p \in G\ T_p \leq T\}$$

is $\mathbb{N}^{**}$-generic. But B is a branch in every $T \in G'$. Hence B is the $\mathbb{N}^{**}$-generic sequence given by G'.
QED (Corollary 7)

It remains only to prove the converse. By a virtual repetition of the proof of Lemma 5 we have:

Lemma 8 Let $T \in \mathbb{N}^{**}$. There is $p \in \mathbb{P}$ s.t. $T_p \leq T$.

Proof. Let $G \ni T$ be $\mathbb{N}^{**}$-generic. Let $\langle \gamma_i \mid i < \omega \rangle$ be the generic sequence given by G. Define $\alpha_i, X_i, M_i, \pi_{ij}$ $(i \leq j \leq \omega_1)$ exactly as before but without reference to a previously chosen $p \in \mathbb{P}$. (I.e. we set:

$$X_i = \text{the smallest } X \prec M \text{ s.t. } \{\alpha_j \mid j < i\} \cup \{\gamma_n \mid n < \omega\} \subset X.)$$

As before there is $\lambda < \omega_1$ s.t. $\pi_{\lambda,\omega_1} : \langle M_\lambda, T_\lambda \rangle \prec \langle M, T \rangle$, where $T_\lambda = \pi^{-1}_{\lambda,\omega_1} {}``T$. Define

$$p = \langle \langle M_i \mid i \leq \lambda \rangle, \langle \pi_{ij} \mid i \leq j \leq \lambda \rangle, B_\lambda \rangle$$

where $B_\lambda = \pi^{-1}_{\lambda,\omega_1} {}``\{\gamma_n \mid n < \omega\}$

$$P_1 = \{\langle T, T_\lambda \rangle\}.$$

The rest of the proof is as before. QED (Lemma 8)

Lemma 9 Let $B = \langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{N}^{**}$-generic. Then $\langle \gamma_i \mid i < \omega \rangle$ is $\mathbb{P}$-generic.

Proof. Suppose not. Then there is $T \in \mathbb{N}^{**}$ s.t. $T \Vdash \dot{B}$ is not $\mathbb{P}$-generic. Let $p \in \mathbb{P}$ s.t. $T_p \leq T$. Let $G \ni p$ be $\mathbb{P}$-generic. Let $B = B^G$. Then B is $\mathbb{N}^{**}$-generic and, in fact, $B = B^{G'}$ where $G' = \{T \mid \bigvee p \in G\ T_p \leq T\}$ is $\mathbb{N}^{**}$-generic. But $T \in G'$ and B is $\mathbb{P}$-generic. Contradiction! QED (Lemma 9)

This proves the Theorem.

Note. *Carrying this further, we could show that* $\mathrm{BA}(\mathbb{N}^*) = \mathrm{BA}(\mathbb{P})$.

Note. *If $\mathcal{L}'$ is like $\mathcal{L}$ except that the axiom:*

$$\bigvee m \bigwedge n \geq m\ \dot{\gamma}_n \in \underline{C} \quad \textit{for all club } C \subset \omega_2$$

is replaced by:

$$\bigvee m \bigwedge n \geq m\ \dot{\gamma}_{n+1} > F(\dot{\gamma}_n) \quad \textit{for all}$$
$$F : \omega_2 \to \omega_2,$$

then a virtual repetition of our proof shows:

Theorem. $B' = \langle \gamma_i \mid i < \omega \rangle$ *is* $\mathbb{P}_{\mathcal{L}'}$*-generic iff it is* $\mathbb{N}'$*-generic.*

§5 The Conclusion

From now on assume $\mathrm{CH} + 2^{\omega_1} = \omega_2$. We now prove that $\mathbb{N}^*$ is subcomplete. The definition of subcompleteness can be found e.g. in [Sing]. The notion of subcompleteness involves the notion of *fullness*, which we first define:

Definition (Grounded)**.** Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \ldots \rangle$ model the axiom of extensionality. $\mathfrak{A}$ is *grounded* iff its well founded core $A = \mathrm{wfc}(\mathfrak{A})$ is transitive and $\in \cap A^2 = \in^{\mathfrak{A}} \cap A^2$.

Definition (Full)**.** Let M be a transitive ZFC^- model. M is *full* iff there is a grounded ZFC^- model $\mathfrak{A}$ s.t.

- $M \in \mathrm{wfc}(\mathfrak{A})$
- If $B \in \mathfrak{A}$ s.t. $B \subset M$, then $\langle M, B \rangle$ is a ZFC^- model. (In other words, M looks like a 2nd order ZFC^- model in $\mathfrak{A}$.)

We then say that $\mathfrak{A}$ *witnesses the fullness* of M.

Note. *This notion is sometimes called "almost full" with the word "full" being reserved for the case that* $\mathfrak{A}$ *is transitive.*

We note the following facts:

Fact 1 If $\mathfrak{A}$ witnesses the fullness of M and ρ is the least ordinal s.t. $L_\rho(M)$ is admissible, then $L_\rho(M) \subset \mathrm{wfc}(\mathfrak{A})$.

Fact 2 If $\bar{M}$ is full and $f\colon \bar{M} \prec M$ cofinally, then M is full. (In fact, if $\bar{\mathfrak{A}}$ witnesses the fullness of $\bar{M}$, then $\mathfrak{A}$ witnesses the fullness of M, where $f^*\colon \bar{\mathfrak{A}} \prec \mathfrak{A}$ is the liftup of f.)

Combining these facts, it is easy to show:

Fact 3 Let $\pi\colon \bar{M} \prec M$ cofinally, where $\bar{M}$ is full. Let $\bar{\rho}$ be least s.t. $L_{\bar{\rho}}(\bar{M})$ is admissible and ρ be least s.t. $L_\rho(M)$ is admissible. Let:

$$L_{\bar{\rho}}(\bar{M}) \models \Psi[\bar{M}, x_1, \ldots, x_n], \quad \text{where } x_1, \ldots, x_n \in \bar{M}.$$

Then

$$L_\rho(M) \models \Psi[M, \pi(x_1), \ldots, \pi(x_n)].$$

We also define:

Definition. Let $N = L_\tau^{A_1,\dots,A_m} = \langle L_\tau[A_1,\dots,A_m], \in, A_1,\dots,A_m\rangle$ be a ZFC$^-$ model. Let $X \subset N$ and let $\delta \in N$ be a cardinal in N.

$$C_\delta^N(X) = \text{the smallest } Y \prec N \text{ s.t. } (\delta+1)\cup X \subset Y.$$

Theorem. $\mathbb{N}^*$ *is subcomplete.*

At the end of the proof we shall note the rather small changes needed to prove that $\mathbb{N}'$ is subcomplete.

Fix $\Omega > 2^{\overline{\overline{\mathbb{N}^*}}}$. Let $W = L_\tau^A = \langle L_\tau[A], \in, A\rangle$ be a transitive ZFC$^-$ model s.t. $H_\Omega \in W$. Let $\pi\colon \bar{W} \prec W$, where $\bar{W}$ is countable, transitive, and full. Assume that $\pi(\bar{\Omega}) = \Omega$ and $\pi(\bar{\mathbb{N}}^*) = \mathbb{N}^*$. In order to prove subcompleteness, it suffices to show that, given any pair $\langle \bar{s}, s\rangle$ with $\pi(\bar{s}) = s$ and any $\bar{G}$ which is $\bar{\mathbb{N}}^*$-generic over $\bar{W}$, there is a $T \in \mathbb{N}^*$ which forces the following statement:

If $G \ni T$ is $\mathbb{N}^*$-generic over V, there is $\sigma \in V[G]$ s.t.

(i) $\sigma\colon \bar{W} \prec W$

(ii) $\sigma(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{s}) = \Omega, \mathbb{N}^*, s$

(iii) $\sigma``\bar{G} \subset G$

(iv) $C_{\omega_2}^W(\operatorname{rng}\sigma) = C_{\omega_2}^W(\operatorname{rng}\pi)$.

The proof will stretch over many sublemmas.

Before proceeding further, we list some definitions and facts developed in [Sing].

Definition (Cofinal)**.** Let M be transitive and $\mathfrak{A}$ a grounded model. We call an embedding $\pi\colon M \to \mathfrak{A}$ *cofinal* iff for each $x \in \mathfrak{A}$ there is $u \in M$ s.t. $\mathfrak{A} \models x \in \pi(u)$.

Fact 4 Let M be a transitive ZFC$^-$ model and $\pi\colon M \longrightarrow_{\Sigma_0} \mathfrak{A}$ cofinally. Then $\pi\colon M \prec \mathfrak{A}$.

Fact 5 Let $\mathfrak{A}$ be a ZFC$^-$ model, M transitive and $\pi\colon M \longrightarrow_{\Sigma_0} \mathfrak{A}$ cofinally. Then $\pi\colon M \prec \mathfrak{A}$.

Definition. Let M be a transitive ZFC$^-$ model and $\tau \in M$ a regular cardinal in M. $\pi\colon M \prec \mathfrak{A}$ *τ-cofinally* iff for each $x \in \mathfrak{A}$ there is $u \in M$ s.t. $\overline{\overline{u}} < \tau$ in M and $\mathfrak{A} \models x \in \pi(u)$.

Definition (Liftup)**.** Let $\bar{M}$ be a grounded ZFC$^-$ model and $\bar{\tau} \in \mathrm{wfc}(\bar{M})$ regular in $\bar{M}$. Let $\bar{H} = H_{\bar{\tau}}^{\bar{M}}$ and let $\bar{\pi}\colon \bar{H} \prec H$ cofinally, where H is transitive. We say that $\langle \mathfrak{A}, \pi\rangle$ is a *liftup* of $\langle \bar{M}, \bar{\pi}\rangle$ iff $\pi \supset \bar{\pi}$ and $\pi\colon \bar{M} \prec \mathfrak{A}$ $\bar{\tau}$-cofinally. (In this case we also say that π is a liftup of $\bar{\pi}$.)

Fact 6 The liftup $\langle \mathfrak{A}, \pi\rangle$ of $\langle \bar{M}, \bar{\pi}\rangle$ always exists and is determined up to isomorphism by $\langle \bar{M}, \bar{\pi}\rangle$. The liftup is unique if $\mathfrak{A}$ is transitive.

Fact 7 ("Interpolation Lemma") Let $\pi'\colon \bar{M} \prec M'$ and $\bar{\pi} = \pi' \restriction \bar{H}\colon \bar{H} \prec H$ cofinally, where $\bar{H} = H_{\bar{\tau}}^{\bar{M}}$ is as above. Then the transitive liftup $\langle M, \pi\rangle$ of $\langle \bar{M}, \bar{\pi}\rangle$ exists. Moreover, there is a unique $\sigma\colon M \prec M'$ s.t. $\sigma\pi = \pi'$ and $\sigma \restriction H = \mathrm{id}$.

Another well known fact is:

Fact 8 If $\pi\colon \bar{M} \prec M$ τ-cofinally, then $\pi(\tau) = \sup \pi``\tau$.

Note. *To prove the theorem it will suffice to show that if $\bar{G}, \bar{s}, s$ are as stated, then in the generic collapse of a sufficient cardinal there exist G, g s.t. G is $\mathbb{N}^*$-generic over V, $g \in V[G]$ and (i)–(iv) hold. There will then be a $T \in G$ which forces this.*

We shall also make use of an auxiliary forcing $\mathbb{C}$ which adds a club $C \subset \omega_2$ which is almost contained in every club set in the ground model:

$$\bigvee \alpha < \omega_2\ C \setminus \alpha \subset A \quad \text{whenever } A \in V \text{ is club in } \omega_2.$$

The conditions are pairs $\langle \alpha, A\rangle$ s.t. A is club in ω_2 and $\alpha < \omega_2$. They are partially ordered by:

$$\langle \alpha', A'\rangle \leq \langle \alpha, A\rangle \leftrightarrow: \big(\alpha' \geq \alpha \wedge A' \subset A \wedge A' \cap \alpha = A \cap \alpha\big).$$

These conditions are closed under ω_1-chains and hence do not add new subsets of the ground model of size ω_1. For this reason it is also *complete* in Shelah's sense. Moreover $\mathbb{C}$ satisfies the ω_3-chain condition, since whenever $\langle \alpha, A\rangle, \langle \beta, B\rangle$ are incompatible, then $\langle \alpha, A \cap \alpha\rangle \neq \langle \beta, B \cap \beta\rangle$. (Otherwise $\langle \alpha, A \cap B\rangle \leq \langle \alpha, A\rangle, \langle \beta, B\rangle$.) Hence $\mathbb{C}$ does not change cardinals or cofinalities. If G is $\mathbb{C}$-generic, we set:

$$C = C_G =: \bigcap \{A \mid \bigvee \langle \alpha, A\rangle \in G\} = \bigcup \{A \cap \alpha \mid \langle \alpha, A\rangle \in G\}.$$

C then has the above property. We call it the *generic club* added by G. G is then recoverable from C by:

$$G = \{\langle \alpha, A\rangle \in \mathbb{C} \mid C \subset A \wedge A \cap \alpha = C \cap \alpha\}.$$

Since $\mathbb{C}$ is definable in W, there is $\bar{\mathbb{C}} \in \bar{W}$ s.t. $\pi(\bar{\mathbb{C}}) = \mathbb{C}$.

Definition (Good pair)**.** $\langle G, G'\rangle$ is a *good pair* for $\bar{W}$ iff

- G is $\bar{\mathbb{N}}^*$-generic, adding the generic sequence $\langle \bar{\gamma}_n \mid n < \omega\rangle$
- G' is $\bar{\mathbb{C}}$-generic, adding the generic club $\bar{C}$
- $\bigvee n \bigwedge m \geq n\ \bar{\gamma}_m \in \bar{C}$.

Lemma 1 Let $T \in \bar{\mathbb{N}}^*$. There is a good pair $\langle G, G'\rangle$ s.t. $T \in G$.

Proof. Let $\langle \Delta_i \mid i < \omega\rangle$ enumerate the dense subsets of $\bar{\mathbb{N}}^*$ and $\langle \Delta'_i \mid i < \omega\rangle$ enumerate the dense subsets of $\bar{\mathbb{C}}$. We define:

$$T_i \in \bar{\mathbb{N}}^*, \quad \langle \alpha_i, A_i\rangle \in \bar{\mathbb{C}}$$

by induction on $i < \omega$ as follows.

$$T_0 = T, \qquad \langle \alpha_0, A_0\rangle = \langle 0, \omega_2\rangle;$$

Now let $T_i, \langle \alpha_i, A_i\rangle$ be given. Let $s_i = \mathrm{stem}(T_i)$. Set:

$$T'_i = \{t \in T_i \mid \bigwedge j \geq |s_i|\ t(j) \in A_i \setminus \alpha_i\}.$$

Then $T'_i \leq T_i$ in $\bar{\mathbb{N}}^*$. Pick $T_{i+1} \leq T'_i$ s.t. $T_{i+1} \in \Delta_i$. Pick $\alpha'_i < \omega_2$ s.t. $\alpha'_i > s_{i+1}(j)$ for all $j < |s_{i+1}|$. Then $\langle \alpha'_i, A_i\rangle \leq \langle \alpha_i, A_i\rangle$ in $\bar{\mathbb{C}}$. Pick

$$\langle \alpha_{i+1}, A_{i+1}\rangle \leq \langle \alpha'_i, A_i\rangle \text{ s.t. } \langle \alpha_{i+1}, A_{i+1}\rangle \in \Delta'_i.$$

$C = \bigcup_i (\alpha_i \cap A_i)$ is then $\bar{\mathbb{C}}$-generic and $b = \langle \gamma_n \mid n < \omega\rangle =: \bigcup \bigcap_{i<\omega} T_i$ is an $\bar{\mathbb{N}}^*$-generic sequence.

Claim. $\bigwedge i \geq |s_0|\ \gamma_i \in C$.

Let $i < |s_{j+1}|$. Then $\gamma_i = s_{j+1}(i) < \alpha'_j$. Hence $\gamma_i \in \alpha'_j \cap A_j = \alpha'_j \cap C$. QED (Lemma 1)

But then:

Corollary 2 Let G be $\bar{\mathbb{N}}^*$-generic over $\bar{W}$. There is G' s.t. $\langle G, G'\rangle$ is a good pair.

Proof. Let $\rho =$ the least ρ s.t. $L_\rho(\bar{W})$ is admissible. Since $\bar{W}$ is full, we know that $\mathcal{P}(\bar{\mathbb{N}}^*) \cap L_\rho(\bar{W}) \subset \bar{W}$. Hence G is $\bar{\mathbb{N}}^*$-generic over $L_\rho(\bar{W})$ and $L_\rho(\bar{W}[G])$ is admissible. Let $\mathcal{L}$ be the infinitary language over $L_\rho(\bar{W}[G])$ with:

<u>Predicate:</u> $\dot{\in}$

<u>Constants:</u> $\underline{x}$ $(x \in L_\rho(\bar{W}[G]))$, $\dot{G}$, $\dot{C}$

<u>Axioms:</u> ZFC$^-$; $\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z\in x} v = \underline{z})$; $\dot{G}$ is $\underline{\mathbb{C}}$-generic over $\underline{\bar{W}}$; giving a generic club set $\dot{C}$; $\bigvee n \bigwedge m \geq n\ \underline{\gamma}(m) \in \dot{C}$, where $\underline{\gamma} = \langle \gamma(i) \mid i < \omega\rangle$ is the generic sequence given by $\underline{G}$.

It suffices to show:

Claim. *$\mathcal{L}$ is consistent, since if $\mathfrak{A}$ is a grounded model of $\mathcal{L}$, then $\langle G, \dot{G}^{\mathfrak{A}}\rangle$ is a good pair.*

Suppose not. The consistency of $\mathcal{L}$ is a $\Pi_1(L_\rho[\bar{W}[G]])$ statement, so there is $T \in G$ which forces it to be false. Let $\langle G_0, G_1\rangle$ be a good pair with $T \in G_1$. Then the corresponding language $\mathcal{L}'$ on $L_\rho[\bar{W}[G_0]]$ is inconsistent. But it is consistent, since $\langle H_{\omega_1}, G_1\rangle$ is a model. Contradiction! QED (Corollary 2)

Now let $\bar{G}$ be $\bar{\mathbb{N}}^*$-generic over $\bar{W}$ with generic sequence $\langle \bar{\gamma}_i \mid i < \omega\rangle$. Let $\langle \bar{G}, \bar{G}'\rangle$ be a good pair and let $\bar{C}$ be the generic club given by $\bar{G}'$. Since $\mathbb{C}$ is a complete forcing, there is G' which is $\mathbb{C}$-generic over V s.t. $\pi``\bar{G}' \subset G'$. Let $C$ be the generic club given by $G'$. Then $\pi``\bar{C} \subset C$. But then π extends uniquely to a $\pi' \supset \pi$ s.t. $\pi' \colon \bar{W}[\bar{C}] \prec W[C]$, $\pi'(\bar{C}) = C$. We now prove:

Lemma 3 Assume the collapse of a sufficient cardinal to ω. Then there is σ s.t.

(i) $\sigma \colon \bar{W} \prec W$

(ii) σ takes $\omega_2^{\bar{W}}$ cofinally to $\omega_2^W = \omega_2$

(iii) $\sigma``\bar{C} \subset C$

(iv) $C^W_{\omega_2}(\operatorname{rng}\pi) = C^W_{\omega_2}(\operatorname{rng}\sigma)$

Before proving Lemma 3 we note a consequence. Let $\gamma_i = \sigma(\bar\gamma_i)$. Then $\langle \gamma_i \mid i < \omega\rangle$ is almost contained in C. But C is almost contained in every club $A \subset \omega_2$ in the ground model. Hence $\langle \gamma_i \mid i < \omega\rangle$ is almost contained in every such A. If we knew that $\langle \gamma_i \mid i < \omega\rangle$ were an $\mathbb{N}^*$-generic sequence given by G and that $\sigma \in V[G]$, we would be done. However, we do not know that and shall have to do a further argument, using Löwenheim–Skolem to apply Lemma 3 to a countable model instead of $V[C]$.

Proof of Lemma 3. Let $\bar H_0 = (H_{\omega_2})^{\bar W[\bar C]}$ and let $\langle W_0[C_0], \pi_0'\rangle$ be the liftup of $\langle \bar W[\bar C], \pi' \restriction \bar H_0\rangle$ (noting that $\bar W$ is definable in $\bar W[\bar C]$ from the predicates in $\bar W$). Set $\pi_0 = \pi_0' \restriction \bar W$. Then $\pi_0 \colon \bar W \prec W_0$ (noting that $\bar W$ is uniformly $\bar W[\bar C]$-definable from the predicates in $\bar W$).

(1) $\pi_0 \colon \bar W \prec W_0$, $\omega_2^{\bar W}$-cofinally.

Proof. Let $x \in W_0$. Then $x \in \pi_0'(u)$, where $u \in \bar W[\bar C]$, $u \subset \bar W$ and $\overline{\overline{u}} < \omega_2$ in $\bar W$. But the forcing $\bar{\mathbb{C}}$ adds no new subsets of $\bar W$ of size $< \omega_2$. Hence $u \in \bar W$ and $\pi_0'(u) = \pi_0(u)$. QED (1)

By the same argument we have: $\bar H_0 = (H_{\omega_2})^{\bar W}$. Thus:

(2) $\langle W_0, \pi_0\rangle$ is the liftup of $\langle \bar W, \pi \restriction \bar H_0\rangle$.

But if $x \in \pi(u)$, $u \in \bar W$, $\overline{\overline{u}} < \omega_2$ in $\bar W$, we have $x = \pi(f)(\nu)$ for a $\nu < \omega_1$, where $f \in \bar W$ maps ω_1 onto u. Then

(3) $W_0 = C_{\omega_1}^{W_0}(\operatorname{rng} \pi_0)$.

Now let $\rho_0 =$ the least ρ s.t. $L_\rho(W_0)$ is admissible. Then W_0 is a 2nd order ZFC$^-$ model in $L_{\rho_0}(W_0)$, by the fullness of W_0.

It follows that C_0 is $\mathbb{C}_0$-generic over $L_{\rho_0}(W_0)$, where $\pi_0(\bar{\mathbb{C}}) = \mathbb{C}_0$ and $\pi_0'(\bar C) = C_0$. Thus ρ_0 is minimal s.t. $L_{\rho_0}(W_0[C_0])$ is admissible. Let $\mathcal{L}_0$ be the following language on $L_{\rho_0}(W_0)$:

<u>Predicate:</u> $\dot\in$

<u>Constants:</u> $\underline{x}$ $(x \in L_{\rho_0}(W_0))$, $\dot\sigma_0'$, $\dot\sigma_0$

<u>Axioms:</u> ZFC$^-$; $\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$;

$$\dot\sigma_0' \colon \underline{\bar W[\bar C]} \prec \underline{W_0[C_0]};$$

$$\dot\sigma_0'(\underline{\bar\Omega}, \underline{\bar{\mathbb{N}}^*}, \underline{\bar{\mathbb{C}}}, \underline{\bar C}) = \underline{\Omega_0}, \underline{\mathbb{N}_0^*}, \underline{\mathbb{C}_0}, \underline{C_0}$$

where $\pi_0'(\bar\Omega, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}, \bar C) = \Omega_0, \mathbb{N}_0^*, \mathbb{C}_0, C_0$;

$$\dot\sigma_0 = \dot\sigma_0' \restriction \underline{W_0};$$

$$\dot{\sigma}_0 : \underline{\bar{W}} \prec \underline{W_0},\ \omega_2^{\bar{W}}\text{-cofinally.}$$

Then $\mathcal{L}_0$ is consistent, since $\langle H_\theta, \pi', \pi\rangle$ is a model for sufficiently regular θ. The statement that $\mathcal{L}_0$ is consistent is a $\Pi_1(L_{\rho_0}(W_0[C_0]))$ statement about $\bar{W}, \bar{\Omega}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}, \bar{C}$ and $\Omega_0, \mathbb{N}_0^*, \mathbb{C}_0, C_0$.

Now let $\bar{H}_1 = (H_{\omega_3})^{\bar{W}[\bar{C}]}$. Let $\langle W_1[C_1], \pi_1'\rangle$ be the liftup of $\langle \bar{W}[\bar{C}], \pi' \restriction \bar{H}_1\rangle$. Let ρ_1 be the least ρ s.t. $L_\rho(W_1[C_1])$ is admissible. Note that $C_1 = C$, since $\bar{C} \in \bar{H}_1$ and $\pi' \restriction \bar{H}_1 = \pi_1' \restriction \bar{H}_1$. Let $\mathcal{L}_1$ be the language on $L_{\rho_1}(W_1[C_1])$ defined from $W_1, C_1, \Omega_1, \mathbb{N}_1^*, \mathbb{C}_1, \bar{W}, \bar{C}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}, \bar{C}$ as $\mathcal{L}_0$ was defined on $L_{\rho_0}(W_0[C_0])$ from $W_0, C_0, \Omega_0, \ldots$, etc. (where $\Omega_1, \mathbb{N}_1^*, \mathbb{C}_1 = \pi_1'(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}})$). By the interpolation lemma there is $\mu : W_0[C_0] \prec W_1[C_1]$ s.t. $\mu \restriction H_0 = \mathrm{id}$ and $\mu\pi_0' = \pi_1'$. Hence $\mu(C_0, \Omega_0, \mathbb{N}_0^*, \ldots) = C_1, \Omega_1, \mathbb{N}_1^*, \ldots$. Applying Fact 3 we then conclude:

(1) $\mathcal{L}_1$ is consistent.

Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}_1$. Let $\sigma_1 = \dot{\sigma}^{\mathfrak{A}}$. Then:

(1) $\sigma_1 : \bar{W} \prec W_1$, $\omega_2^{\bar{W}}$-cofinally.

- $\sigma_1(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}) = \Omega_1, \mathbb{N}_1^*, \mathbb{C}_1$
- $\sigma_1``\bar{C} \subset C$

On the other hand:

(2) $\pi_1 : \bar{W} \prec W_1$, $\omega_3^{\bar{W}}$-cofinally.

Proof. Let $x \in W_1$. Then $x \in \pi_0'(u)$ where $u \in \bar{W}[\bar{C}]$, $\overline{\overline{u}} \leq \omega_1$ in $\bar{W}[\bar{C}]$ and $u \subset \bar{W}$. But $\bar{\mathbb{C}}$ satisfies the ω_3-chain condition in $\bar{W}$, which implies that $u \subset v$ for a $v \in \bar{W}$ s.t. $\overline{\overline{v}} \leq \omega_2$ in $\bar{W}$. Thus $x \in \pi_0(v)$. QED (2)

But then:

$$(3)\ C_\tau^{W_1}(\mathrm{rng}\, \sigma_1) = C_\tau^{W_1}(\mathrm{rng}\, \pi_1) = W_1, \quad \text{where } \tau = \omega_2^{W_1}.$$

By (2) we also have:

(4) $\langle W_1, \pi_1\rangle$ is the liftup of $\langle \bar{W}, \pi \restriction \bar{H}_1\rangle$, where $\bar{H}_1 = (H_{\omega_3})^{\bar{W}}$.

But then there is $\delta : W_1 \prec W$ s.t. $\delta \restriction (H_{\omega_3})^{W_1} = \mathrm{id}$ and $\delta\pi_1 = \pi$. Set: $\sigma = \delta \circ \sigma_1$. Then:

- $\sigma : \bar{W} \prec W$
- $\sigma``\bar{C} \subset C$, since $\delta \restriction C = \mathrm{id}$
- $C_\tau^W(\mathrm{rng}\, \sigma) = \delta``C_\tau^{W_1}(\mathrm{rng}\, \sigma_1) = \delta``W_1$

- $C^W_\tau(\operatorname{rng}\pi) = \delta``C^{W_1}_\tau(\operatorname{rng}\pi_1) = \delta``W_1$

for $\tau = \omega_2^{W_1} = \omega_2$, since $\delta\restriction\tau = \mathrm{id}$. Trivially we also have:

$$\sigma(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}) = \pi(\bar{\Omega}, \bar{\mathbb{N}}^*, \bar{\mathbb{C}}) = \Omega, \mathbb{N}^*, \mathbb{C}.$$

QED (Lemma 3)

Now let $\langle \delta_n \mid n < \omega\rangle$ be a Namba-generic sequence over $V[C]$. Work in $V[C][\delta]$. Let θ be regular in V s.t. $W \in H_\theta$. Set:

$$H = \langle H_\theta, \in, <, W, \Omega, \mathbb{N}^*, \bar{W}, \pi\rangle,$$

where $<$ is a well ordering of H in V. Then cardinals are not collapsed in $H[C]$ and $\pi'\colon \bar{W}[\bar{C}] \prec W[C]$ is $H[C]$-definable in C. Set:

$$X =: \text{the smallest } X \prec H[C] \text{ s.t. } \{C\} \cup \{\delta_n \mid n < \omega\} \subset X.$$

Let $\mu'\colon \tilde{H}[\tilde{C}] \overset{\sim}{\longleftrightarrow} H[C]\restriction X$ s.t. $\mu'(\tilde{C}) = C$ and $\tilde{H}$ is transitive. Set $\mu = \mu' \restriction \tilde{H}$. Then $\mu\colon \tilde{H} \overset{\sim}{\longleftrightarrow} H\restriction X$. Let $\tilde{H} = \langle |\tilde{H}|, \in, \tilde{<}, \tilde{W}, \ldots, \bar{W}, \tilde{\pi}\rangle$. Let $\tilde{\pi}'$ be defined from $\tilde{C}$ in $\tilde{H}[\tilde{C}]$ as π was defined in $H[C]$. Then:

$$\tilde{\pi}'\colon \bar{W}[\bar{C}] \prec \tilde{W}[\tilde{C}], \qquad \tilde{\pi}'(\bar{C}) = \tilde{C},$$

where $\tilde{\pi} = \tilde{\pi}'\restriction\tilde{W}$. Note that μ takes $\tilde{\omega}_2 = \omega_2^{\tilde{H}}$ cofinally to ω_2.

Lemma 4 Every $x \in \tilde{H}$ is $\tilde{H}$-definable from parameters in $\omega_1^{\tilde{H}} \cup \{\tilde{\delta}_n \mid n < \omega\}$, where $\mu(\tilde{\delta}_n) = \delta_n$.

Proof of Lemma 4. Let $x \in \tilde{H}$. x is $\tilde{H}[\tilde{C}]$-definable from $\tilde{C}$ and a parameter $p = \langle \tilde{\delta}_{i_1}, \ldots, \tilde{\delta}_{i_m}\rangle$. Let $x = t^{\tilde{H}[\tilde{C}]}(p, \tilde{C})$. Let u be the set of $z \in \tilde{H}$ s.t. $q \Vdash t(\check{p}, \dot{C}) = \check{z}$ for some $q \in \mathbb{C}$. Then $u \in \tilde{H}$. Moreover, $\overline{\overline{u}} \leq \omega_2$ in $\tilde{H}$, since $\mathbb{C}$ satisfies the ω_3-chain condition. Clearly u is $\tilde{H}$-definable in p. Set $f =$ the $\tilde{H}$-least $f\colon \omega_2 \xrightarrow{\text{onto}} u$. Then f is $\tilde{H}$-definable in p and $x = f(\xi)$ for a $\xi < \omega_2$ in $\tilde{H}$.

Let $\xi < \tilde{\delta}_m$. Let g be $\tilde{H}$-least s.t. $g\colon \omega_1 \xrightarrow{\text{onto}} \tilde{\delta}_m$. Then $f \circ g$ is $\tilde{H}$-definable in $p, \tilde{\delta}_m$ and $x = f \circ g(\nu)$ for a $\nu < \omega_1^{\tilde{H}}$. QED (Lemma 4)

Applying the argument of Lemma 3 to $\tilde{H}[\tilde{C}]$ instead of $V[C]$, we see that:

Lemma 5 There is $\tilde{\sigma} \in H_{\omega_1}$ s.t.

(i) $\tilde{\sigma}\colon \bar{W} \prec \tilde{W}$

(ii) $\tilde{\sigma}$ takes $\omega_2^{\bar{W}}$ cofinally to $\omega_2^{\tilde{W}} = \tilde{\omega}_2$ (where $\tilde{\omega}_2 = \omega_2^{\tilde{H}}$).

(iii) $\tilde{\sigma}``\bar{C} \subset \tilde{C}$

(iv) $C_{\tilde{\omega}_2}^{\tilde{W}}(\operatorname{rng}\tilde{\sigma}) = C_{\tilde{\omega}_2}^{\tilde{W}}(\operatorname{rng}\tilde{\pi})$.

We now use our construction to define a precondition in the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ defined in §4. We define $p = \langle P_0, P_1\rangle$ by:

Definition. $P_0 = \langle M^p, \pi^p, b^p\rangle$ where:

- $\operatorname{dom}(M^p) = \operatorname{dom}(\pi^p) = \{\alpha_0\}$, where $\alpha_0 = \omega_1^{\tilde{H}}$
- $M_0^p = \tilde{\pi}(\bar{M}) = \mu^{-1}(M)$; $\pi_{00}^p = \mathrm{id}\restriction M_0^p$
- $b^p = \tilde{\pi}(\bar{B})$, where $\bar{B} = B^{\bar{G}}$, $\bar{G}$ is the $\bar{\mathbb{N}}^*$-generic set over $\bar{W}$ fixed at the outset.

To define P_1 we let $\langle \varphi_i(v_1, \ldots, v_{n_i}) \mid i < \omega\rangle$ enumerate the H-formulae and set:

$$A_i = \{\langle x_1, \ldots, x_{n_i}\rangle \in M^{n_i} \mid H \models \varphi_i[x_1, \ldots, x_{n_i}]\}$$
$$\tilde{A}_i = \{\langle x_1, \ldots, x_{n_i}\rangle \in (M_0^p)^{n_i} \mid \tilde{H} \models \varphi_i[x_1, \ldots, x_{n_i}]\}.$$

Then $\mu(\tilde{A}_i) = A_i$ for $i < \omega$. We set:

$$P_1 = \{\langle A_i, \tilde{A}_i\rangle \mid i < \omega\}.$$

p is then a putative condition, i.e. $p \in \widetilde{\mathbb{P}}$.

Lemma 6 $p \in \mathbb{P}$.

Proof. We show that $\mathcal{L}(p)$ is consistent by constructing a model for it in $V[C][\delta]$, using the fact that ω_1 is not collapsed. We construct $\langle \alpha_i \mid i \leq \omega_1\rangle$, $\langle X_i \mid i \leq \omega_1\rangle$ as follows:

$$X_0 = \mu``M_0^p, \qquad \alpha_0 = \alpha_0^p = \omega_1^{M_0^p}.$$

Set $B = \langle \gamma_n \mid n < \omega\rangle$, where:

$$\gamma_n = \mu(\tilde{\gamma}_n) = \mu(\gamma_n^p) \quad \text{for } n < \omega.$$

It is easily seen that X_0 is the smallest $X \prec M$ s.t. $\alpha_0 \cup \{\gamma_n \mid n < \omega\} \subset X$. For $j > 0$ set:

$$X_j = \text{the smallest } X \prec M \text{ s.t. } \bigcup_{i<j} X_i \cup \{\alpha_i\} \subset X,$$

$$\alpha_j = \omega_1 \cap X_j.$$

Set: $\pi_{i,\omega_1} \colon M_i \overset{\sim}{\longleftrightarrow} M{\restriction}X_i$, $\pi_{ij} = \pi_{j,\omega_1}^{-1} \circ \pi_{i,\omega_1}$ for $i \le j \le \omega_1$. Note that since $\bar{\gamma}_m \in \bar{C}$ for sufficiently large m, we have: $\tilde{\gamma}_m = \tilde{\sigma}(\bar{\gamma}_m) \in \tilde{C}$ for sufficient m, since $\tilde{\sigma}``\bar{C} \subset \tilde{C}$. But $\pi_{0,\omega_1} = \mu{\restriction}M_0$, where $\mu``\tilde{C} \subset C$. Hence $\gamma_m \in C$ for sufficiently large m. But whenever $A \subset \omega_2$, $A \in V$ is club in ω_2, then $\bigvee \alpha < \omega_2\ C \setminus \alpha \subset A$. It follows that $\bigvee m \bigwedge n \ge m\ \gamma_n \in A$ for all such A. Hence, letting η be a sufficiently large regular cardinal in $V[C][\delta]$, we see that

$$\mathfrak{A} = \langle H_\eta, \langle M_i \mid i \le \omega_1\rangle, \langle \pi_{ij} \mid i \le j \le \omega_1\rangle, B\rangle$$

models $\mathcal{L}$ (where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as in §4). But $\pi_{0,\omega_1} = \mu{\restriction}M_0^p$ and $\mu(\tilde{A}_i) = A_i$ for $i < \omega$. Hence:

$$\pi_{0,\omega_1} \colon \langle M_0^p, \tilde{A}_i\rangle \prec \langle M, A_i\rangle \quad \text{for } i < \omega.$$

Thus $\mathfrak{A}$ models $\mathcal{L}(p)$. QED (Lemma 6)

Now let G' be $\mathbb{P}$-generic with $p \in G'$. Let $B' = B^{G'}$, $\pi' = \pi^{G'}$, $M' = M^{G'}$. Since p conforms to H, we know that π'_{0,ω_1} extends to $\pi^* \colon \tilde{H} \prec H'$. Set: $\sigma' = \pi^* \circ \tilde{\sigma}$. Then:

(i) $\sigma' \colon \bar{W} \prec W$

(ii) σ' takes $\omega_2^{\bar{W}}$ cofinally to ω_2

Since $\pi^*(\tilde{\pi}) = \pi^* \circ \tilde{\pi} = \pi$, and

$$C_{\tilde{\omega}_2}^{\tilde{W}}(\operatorname{rng}\tilde{\sigma}) = C_{\tilde{\omega}_2}^{\tilde{W}}(\operatorname{rng}\tilde{\pi}),$$

we conclude:

(iii) $C_{\omega_2}^{W}(\operatorname{rng}\sigma') = C_{\omega_2}^{W}(\operatorname{rng}\pi)$.

Now let $G =$ the set of $T \in \mathbb{N}^*$ s.t. $\bigvee p \in G'\ T_p \le T$. Then G is $\mathbb{N}^*$-generic and B' is the sequence given by G. If $\bar{T} \in \bar{G}$, then $\bar{B}_i = \langle \bar{\gamma}_n \mid n < \omega\rangle$ is a branch in $\bar{T}$. Hence $B' = \langle \sigma'(\bar{\gamma}_n) \mid n < \omega\rangle$ is a branch in $\sigma'(\bar{T})$. Hence $\sigma'(\bar{T}) \in G$. Thus:

(iv) $\sigma'``\bar{G} \subset G$.

QED (Theorem)

We now sketch the changes that must be made in order to prove the subcompleteness of $\mathbb{N}'$. If A is club in ω_2, let:

$$F_A(\alpha) =: \text{the least } \beta \in A \text{ s.t. } \alpha < \beta.$$

Then if C is $\mathbb{C}$-generic, the function F_C eventually majorizes every $F\colon \omega_2 \to \omega_2$ in the ground model.

We change the definition of good pair, requiring only

$$\bigvee n \bigwedge m \geq n\ F_C(\gamma_m) \leq \gamma_{m+1}$$

instead of $\bigvee n \bigwedge m \geq n\ \gamma_m \in C$.

In the proof of Lemma 1 we alter the definition of T_i' to read:

$$T_i' = \{t \in T_i \mid \bigwedge j \geq |s_i|\ (\alpha_i < t_j \wedge F_{A_i}(t_j) \leq t_{j+1})\}.$$

The proofs of Lemma 1, Corollary 2 go through as before. The remaining proofs then go through with cosmetic changes.

§6 Some Further Properties of $\mathbb{N}, \mathbb{N}', \mathbb{N}^*$

Set: $\mathbb{P}_0 = \mathbb{N}$, $\mathbb{P}_1 = \mathbb{N}'$, $\mathbb{P}_2 = \mathbb{N}^*$ and $\mathbb{P}_3 = \mathbb{C} * \dot{\mathbb{N}}$ (where $\dot{\mathbb{N}}$ is a name for Namba forcing in $V^{\mathbb{C}}$).

We say that $\gamma = \langle \gamma_i \mid i < \omega \rangle$ is $\mathbb{P}_h$-*conforming* ($h \leq 3$) iff

- γ is monotone and cofinal in ω_2^V
- $\gamma \in V[G]$ for a G which is $\mathbb{P}_h$-generic.

We prove: (assuming CH)

Theorem (assuming a sufficient collapse to ω)**.** *If γ is $\mathbb{P}_h$-conforming, it is not $\mathbb{P}_j$-conforming for any $j \neq h$.*

But this implies:

Corollary Let $\mathbb{B}_i = \mathrm{BA}(\mathbb{P}_i)$ be the canonical complete Boolean algebra over $\mathbb{P}_i$. Then $\mathbb{B}_i, \mathbb{B}_j$ are *not* isomorphic for $i \neq j$.

Definition (Magical)**.** Working in any extension of V, define:
$\delta = \langle \delta_i \mid i < \omega \rangle$ is *magical* iff δ is monotone, cofinal in $\omega_2 =: \omega_2^V$ and whenever $A \in V$ is club in ω_2, then $\bigvee n \bigwedge i \geq n \; \delta_i \in A$.

Lemma 1 Let $\gamma = \langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{N}'$-generic. Then γ is not magical. In fact, there is $A \in V$ club in ω_2 s.t. $\gamma_i \notin A$ for $i < \omega$.

Proof. We show that for every $T \in \mathbb{N}'$ there is $T' \leq T$, $A \in V$ s.t. A is club and $\bigwedge t \in T' \; \mathrm{rng}(t) \cap A = \emptyset$.

We first find T' as follows. Let $s = \mathrm{stem}(T)$. We construct T_n s.t. $T = T_0 \geq T_1 \geq \cdots \geq T_n \geq \cdots$ and $T_m{\restriction}n = T_n{\restriction}n$ for $m \geq n$, (where $T{\restriction}n = \{t \in T \mid |t| \leq n\}$).

For $i \leq |s|$ set $T_i = T$. Now let $m \geq 1$. For each $t \in T_i$, $|t| = i$, select A_t club in ω_2 s.t. $\{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\} \setminus A_t$ is unbounded in ω_2. Set

$$T_{i+1} = \{t' \in T_i \mid t'(i) \notin A_{t'{\restriction}i}\} = \bigcup \{(T_i)_{(t^\frown\langle\alpha\rangle)} \mid t^\frown\langle\alpha\rangle \in T_i, |t| = i, \alpha \notin A_t\}.$$

Then $T' = \bigcup_i T_i{\restriction}i = \bigcap_i T_i$ has the property that:

$$t \in T', |t| = i \;\rightarrow\; \{\alpha \mid t^\frown\langle\alpha\rangle \in T\} \cap A_t = \emptyset.$$

For each $n \geq |s|$ set:

$$A_n = \{\lambda \mid \bigwedge t \in \lambda^n \, (t \in T' \rightarrow \lambda \in A_t)\}.$$

Then A_n is club in ω_2 and

$$(t \in T',\ |t| = n,\ t^\frown\langle\alpha\rangle \in T') \to \alpha \notin A_n.$$

Set $A = \bigcap_n A_n$. Then:

$$t \in T' \to \bigwedge i\ t(i) \notin A.$$

But we have shown that the set of such T' is dense in $\mathbb{N}'$. Hence if G is $\mathbb{N}'$-generic, there is such T', A with $T' \in G$. Hence $\bigwedge i\ \gamma_i \notin A$.

QED (Lemma 1)

Lemma 2 Let W be an extension of V with $H^W_{\omega_1} = H^V_{\omega_1}$. Let $\gamma = \langle \gamma_i \mid i < \omega\rangle$, $\gamma' = \langle \gamma'_i \mid i < \omega\rangle \in W$ be monotone and cofinal in ω_2. Then $\gamma' \in V[\gamma]$.

Proof. Let $f_i \in V$ s.t. $f_i \colon \omega_1 \xrightarrow{\text{onto}} \gamma_i$. Set $X_\alpha = \bigcup_i f_i``\alpha$ for $\alpha \leq \omega_1$. Then $\langle X_\alpha \mid \alpha \leq \omega_1\rangle \in V[\gamma]$ and for $i < \omega$ let δ_i be least s.t. $\gamma'_i \in X_{\delta_i}$. Then $\alpha = \sup_i \delta_i < \omega_1$. But there is $g \in V[\gamma]$ s.t. $g \colon \omega \xrightarrow{\text{onto}} X_\alpha$. Hence $\{\gamma'_i \mid i < \omega\} = g``a \in V[\gamma]$ for an $a \subset \omega$. QED (Lemma 2)

Lemma 3 There is no magical sequence in $V[\gamma]$ where γ is $\mathbb{N}'$-generic.

Proof. Let $\delta = \langle \delta_i \mid i < \omega\rangle$ be a counterexample. Let $A \in V$ be a club in ω_2 s.t. $\gamma_i \notin A$ for $i < \omega$. Let $m < \omega$ s.t. $\delta_i \in A$ for $i \geq m$. Then $\langle \delta_i \mid i \geq m\rangle$ is a counterexample which is disjoint from γ. Hence we may w.l.o.g. assume $\{\delta_i \mid i < \omega\} \cap \{\gamma_i \mid i < \omega\} = \emptyset$. By thinning the sequence δ we may also assume that there is a monotone sequence $\langle m_i \mid i < \omega\rangle$ in ω s.t. $\gamma_{m_i} < \delta_i < \gamma_{m_i+1}$ for $i < \omega$. Let $B \in V$, $B \subset \omega_2$ s.t. $H_{\omega_1} = L_{\omega_1}[B]$ and $\omega_1 =$ the largest cardinal in $L_{\omega_2}[B]$. Then there is $\alpha < \omega_1$ s.t. $\{\delta_i \mid i < \omega\} \subset X$, where $X =$ the smallest $X \prec \langle L_{\omega_2}[B], B\rangle$ s.t. $\alpha \cup \{\gamma_n \mid n < \omega\} \subset X$. We can assume w.l.o.g. that $\alpha = \omega_1 \cap X$. Let $\pi \colon \langle L_\beta[\bar{B}], \bar{B}\rangle \overset{\sim}{\longleftrightarrow} X$. Let $\pi(\bar{\gamma}_m, \bar{\delta}_i) = \gamma_m, \delta_i$. Then $\bar{\gamma}_{m_i} < \bar{\delta}_i < \bar{\gamma}_{m_i+1}$ for $i < \omega$.

Let $T \in G$ (where $\gamma = \bigcup\bigcap G$ and G is $\mathbb{N}'$-generic) s.t.

$$T \Vdash \Big(\dot{\pi} \colon \langle L_\beta[\check{\bar{B}}], \check{\bar{B}}\rangle \prec \langle L_{\omega_2}[\check{B}], \check{B}\rangle\ \wedge\ \dot{\pi}(\check{\bar{\gamma}}_m) = \dot{\gamma}_m \text{ for } m < \omega \\ \wedge\ \dot{\pi}(\check{\bar{\delta}}_n) = \dot{\delta}_n \text{ for } n < \omega\Big)$$

where $\dot{\gamma}$ is the canonical name for γ and $\dot{\delta}^G = \delta$, and $\dot{\pi}$ is defined from $\check{\alpha}, \dot{\gamma}$ as above.

Then:

(1) If $t \in T$, $|t| = m_i + 2$, then there is $\eta < t(m_i+1)$ s.t. $T_{(t)} \Vdash \check{\eta} = \dot{\delta}_i$.

Proof. Let $\bar{f}_n =$ the $\langle L_\beta[\bar{B}], \bar{B}\rangle$-least f s.t. $f\colon \alpha \xrightarrow{\text{onto}} \bar{\gamma}_n$ $(n < \omega)$. Let $f_t =$ the $\langle L_{\omega_2}[B], B\rangle$-least f s.t. $f\colon \omega_1 \xrightarrow{\text{onto}} t(n)$ for $t \in T$, $|t| = n+1$. Whenever $G \ni T$ is $\mathbb{N}'$-generic, then $\dot{\pi}^G(\bar{f}_n(\nu)) = f_{\langle\gamma_0,\ldots,\gamma_n\rangle}(\nu)$ for $\nu \leq \alpha$.

Thus, setting $\pi_t = f_t \circ \bar{f}_n^{-1}$ for $|t| = n+1$, $t \in T$, we have: $T_{(t)} \Vdash \dot{\pi}\restriction\check{\bar{\gamma}}_n = \check{\pi}_t$. In particular: if $|t| = m_i+2$, $t \in T$ and $\eta = \bar{\pi}_t(\bar{\delta}_i)$, then $T_{(t)} \Vdash \check{\eta} = \dot{\delta}_i$.QED (1)

We now thin T as follows. For $i < \omega$ we define T_i s.t. $T_{i+1} \leq T_i$, $T_i\restriction(m_i + 1) = T_j\restriction(m_i+1)$ and $\mathrm{stem}(T_i) = \mathrm{stem}(T_j)$ for $i \leq j$. We then set: $T' = \bigcap_i T_i = \bigcup_i T_i\restriction(m_i+1)$. Then $T' \in \mathbb{N}'$. Define T_i by: $T_i = T$ if $m_i + 1 < |s|$ where $s = \mathrm{stem}(T)$. If $m_i + 1 \geq |s|$, proceed as follows. We pick for each $t \in T_i$ s.t. $|t| = m_i + 1$ a set $A_i = A_i^t \subset \{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\}$ as follows: Set: $\eta_\alpha = \eta_\alpha^t = \pi_{t^\frown\langle\alpha\rangle}(\bar{\delta}_i)$. (Hence $t(m_i) < \eta_\alpha < \alpha$.)

<u>Case 1.</u> For all $\mu < \omega_1$ there is α s.t. $\eta_\alpha > \mu$. Successively pick α_i s.t. $\eta_{\alpha_i} > \sup_{i'<i} \alpha_{i'}$. Set: $A = \{\alpha_i \mid i < \omega_2\}$.

<u>Case 2.</u> Case 1 fails. Then there is η s.t. $\mathrm{Card}(\{\alpha \mid \eta_\alpha = \eta\}) = \omega_2$. Set $A = \{\alpha \mid \eta_\alpha = \eta\}$.

We then set:

$$\begin{aligned} T_{i+1} &= \{t \mid t \in T_i \text{ and } t(m_i+1) \in A_i^t \text{ if } |t| \geq m_i + 2\} \\ &= \bigcup \{T_{t^\frown\langle\alpha\rangle} \mid |t| = m_i + 1, t \in T_i, \alpha \in A_i^t\}. \end{aligned}$$

If $t \in T'$ and $|t| = m_i + 1$, then we have for $t^\frown\langle\alpha\rangle \in T'$: either $\eta_\alpha^t < \min\{\alpha \mid t^\frown\langle\alpha\rangle \in T'\}$ or $\eta_\alpha^t > \sup\{\beta \mid t^\frown\langle\beta\rangle \in T' \wedge \beta < \alpha\}$.

If we let $B_t =$ the closure of $\{\alpha \mid t^\frown\alpha \in T'\}$ in ω_2, then $\eta_\alpha^t \notin B_t$. Set:

$$B_i = \{\lambda \mid \bigwedge t \in \lambda^{m_i+1}\, (t \in T' \to \lambda \in B_t)\}.$$

Then $\eta_\alpha^t \notin B_i$, since otherwise $\eta_\alpha^t \in B_t$, since $t(h) < \eta_\alpha^t$ for $h \leq m_i$. If we set: $B = \bigcap_{i<\omega} B_i$, it follows that $\eta_\alpha^t \notin B$ whenever $t^\frown\langle\alpha\rangle \in T'$ and $|t| = m_i + 1$.

This means that if $G \ni T'$ is $\mathbb{N}'$-generic, given the generic sequence $\langle\gamma_n \mid n < \omega\rangle$, then $\delta_i \notin B$ for $i < \omega$ (where $\delta_i = \eta_{\gamma_{m_i+1}}^{\langle\gamma_0,\ldots,\gamma_{m_i}\rangle}$). Thus $\langle\delta_i \mid i < \omega\rangle$ is not magical. Contradiction! QED (Lemma 3)

Lemma 4 Let γ be $\mathbb{N}^*$-generic. Then γ is a *maximal* magical sequence in the following sense: let $\delta = \langle\delta_i \mid i < \omega\rangle$, where $\delta \in V[\gamma]$ is magical. Then

$$\bigvee n \bigwedge m \geq n\ \delta_m \in \{\gamma_j \mid j < \omega\}.$$

Proof. Suppose not. Let δ be a counterexample. Then there is a monotone function $\langle m_i \mid i < \omega \rangle$ s.t. $\bigvee j \ \delta_j \in (\gamma_{m_i}, \gamma_{m_i+1})$ for $i < \omega$. Pick $\delta_{m_i} \in (\gamma_{m_i}, \gamma_{m_i+1})$. Then $\langle \delta_{m_i} \mid i < \omega \rangle$ is magical. Hence we may assume w.l.o.g. that $\delta_i \in (\gamma_{m_i}, \gamma_{m_i+1})$ for $i < \omega$, where $\langle m_i \mid i < \omega \rangle$ is monotone. We again let $B \subset \omega_2$ s.t. $L_{\omega_1}[B] = H_{\omega_1}$ and ω_1 is the largest cardinal in $L_{\omega_2}[B]$ (where $B \in V$). Define $\alpha, X, \pi \colon \langle L_\beta[\bar{B}], \bar{B} \rangle \overset{\sim}{\longleftrightarrow} X$ exactly as in Lemma 3. Let $\pi(\bar{\gamma}_m, \bar{\delta}_i) = \gamma_m, \delta_i$. Then $\bar{\gamma}_{m_i} < \bar{\delta}_i < \bar{\gamma}_{m_i+1}$ as before. Let G be the $\mathbb{N}^*$-generic set giving the sequence γ.

There is $T \in G$ s.t.

$$T \Vdash \Big(\dot{\pi} \colon \langle L_\beta[\check{\bar{B}}], \check{\bar{B}} \rangle \prec \langle L_{\omega_2}[\check{B}], \check{B} \rangle \ \wedge \ \dot{\pi}(\check{\bar{\gamma}}_n) = \dot{\gamma}_n \text{ for } n < \omega \\ \wedge \ \dot{\pi}(\check{\bar{\delta}}_n) = \dot{\delta}_n \text{ for } n < \omega \Big)$$

where $\dot{\gamma}^G = \gamma$ and $\dot{\delta}^G = \delta$ and $\dot{\pi}$ is defined from $\check{\alpha}, \dot{\gamma}$ as above.

Then just as before we get:

(1) If $t \in T$, $|t| = m_i + 2$, there is $\eta < t(m_i + 1)$ s.t. $T_{(t)} \Vdash \check{\eta} = \dot{\delta}_i$.

For $t \in T$, $|t| = m_i + 1$, $t^\frown\langle\alpha\rangle \in T$, let $\eta_\alpha = \eta^t_\alpha =$: that η s.t. $T_{t^\frown\langle\alpha\rangle} \Vdash \check{\eta} = \dot{\delta}_i$.

Imitating the proof in Lemma 3 we then define successive thinnings T_i of T s.t.: $T' = \bigcap_i T_i$. If $m_i + 1 < |s|$, $s = \text{stem}(T)$, we set $T_i = T$. Now let $m_i + 1 \geq |s|$ and let T_i be given. For $t \in T_i$, $|t| = m_i + 1$, we define: $A = A_t = \{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\}$ as follows. Since $\eta_\alpha < \alpha$ for $t^\frown\langle\alpha\rangle \in T_i$, there must be η s.t. $\{\alpha \mid t^\frown\alpha \in T_i \wedge \eta_\alpha = \eta\}$ is stationary. Set $A = \{\alpha \mid t^\frown\alpha \in T_i \wedge \eta_\alpha = \eta\}$.

Set $T_{i+1} = \{t \in T_i \mid t(m_i + 1) \in A_{t \restriction (m_i+1)} \text{ if } |t| > m_i + 1\}$. It follows as before that if $B = B_t =$: the closure of $\{\alpha \mid t^\frown\alpha \in T'\}$ for $|t| = m_i + 1$, then $\eta_{t^\frown\alpha} \notin B_t$ for $t^\frown\alpha \in T'$. Setting

$$B_i = \{\lambda \mid \bigwedge t \in \lambda^{m_i+1} \, (t \in T' \to \lambda \in B_t)\}$$

we again have: $\eta_{t^\frown\alpha} \notin B_i$. But then $\eta_{t^\frown\alpha} \notin B$ where $B = \bigcap_i B_i$. Hence if $T' \in G$ and G is $\mathbb{N}^*$-generic, we conclude that $\delta_i \notin B$ for $i < \omega$. Contradiction! QED (Lemma 4)

Definition (Quasi-magical)**.** Define $\delta = \langle \delta_i \mid i < \omega \rangle$ to be *quasi-magical* iff whenever $F \in V$ and $F \colon \omega_2 \to \omega_2$, then $\bigvee n \bigwedge m \geq n \ \delta_{m+1} \geq F(\delta_m)$.

We recall the following known theorem about $\mathbb{N}$:

Lemma 5 Let G be $\mathbb{N}$-generic. Then:

(a) If $\delta = \langle \delta_i \mid i < \omega \rangle \in V[G]$ is monotone and cofinal in ω_2, then δ is an $\mathbb{N}$-generic sequence.

(b) No $\mathbb{N}$-generic sequence is quasi-magical.

Finally we note:

Lemma 6 Let G be $\mathbb{C} * \dot{\mathbb{N}}$-generic. Let $\gamma = \langle \gamma_i \mid i < \omega \rangle \in V[G]$ be magical. Then there is a magical $\gamma' = \langle \gamma'_i \mid i < \omega \rangle$ s.t. $\{\gamma_i \mid i < \omega\} \cap \{\gamma'_i \mid i < \omega\} = \emptyset$.

Proof. Let $V[G] = V[C][\gamma]$ where C is $\mathbb{C}$-generic and γ is Namba-generic over $V[C]$. Then there is a monotone sequence $\langle m_i \mid i < \omega \rangle$ s.t. $(\gamma_{m_i}, \gamma_{m_i+1}) \cap C \neq \emptyset$. This follows from the $\mathbb{C}$-genericity of C. Pick $\gamma'_i \in (\gamma_{m_i}, \gamma_{m_i+1}) \cap C$. Then $\gamma' = \langle \gamma'_i \mid i < \omega \rangle$ has the desired property. QED (Lemma 6)

Now let: $\mathbb{P}_0 = \mathbb{N}$, $\mathbb{P}_1 = \mathbb{N}'$, $\mathbb{P}_2 = \mathbb{N}^*$, $\mathbb{P}_3 = \mathbb{C} * \dot{\mathbb{N}}$.

Definition. $\gamma = \langle \gamma_i \mid i < \omega \rangle$ is $\mathbb{P}_n$-conforming iff γ is monotone and cofinal in ω_2 and $\gamma \in V[G]$ for $\mathbb{P}_n$-generic G.

Theorem. *If γ is $\mathbb{P}_n$-conforming ($n \leq 3$), then it is not $\mathbb{P}_m$-conforming for any $m \neq n$.*

Proof. (1) Let γ be $\mathbb{P}_0$-conforming. Then γ is not $\mathbb{P}_n$-conforming for $n > 0$.

Proof. If $\gamma \in V[G]$ and G is $\mathbb{P}_n$-generic, then there is a quasi-magical $\delta \in V[G]$. Hence $\delta \in V[\gamma]$ by Lemma 2, contradicting Lemma 5.

(It follows, of course, that no $\mathbb{P}_n$-conforming sequence is $\mathbb{P}_0$-conforming if $n > 0$.)

Using Lemma 1, we then get:

(2) Let γ be $\mathbb{P}_1$-conforming. Then γ is not $\mathbb{P}_n$-conforming for $n > 1$.

Using Lemma 4 and Lemma 6 we then get:

(3) Let γ be $\mathbb{P}_2$-conforming. Then γ is not $\mathbb{P}_3$-conforming. QED

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