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Forcing "NS_{ω_1} Is ω_1-Dense" From Large Cardinals

A Journey Guided by the Stars

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Abstract

We answer a question of Woodin by showing that "NS_{ω_1} is ω_1-dense" holds in a stationary set preserving extension of any universe with a cardinal κ which is a limit of $<\kappa$-supercompact cardinals. We do so by introducing a forcing axiom we call Q-Maximum, prove it consistent from large cardinals and show that it implies the version of Woodin's $(*)$-axiom for $\mathbb{Q}_{\max}$. Along the way we produce a number of other new instances of Asperó-Schindler's result $\mathrm{MM}^{++} \Rightarrow (*)$.
In the second part, we show that the κ-mantle, i.e. the intersection of all grounds which extend to V via forcing of size $<\kappa$, may fail to be a model of AC for various instances of κ. This answers a question of Usuba.

Für meine Familie.

Preface

Kurt Gödel's incompleteness theorems shattered David Hilbert's dream of a framework of mathematics which is complete, consistent and powerful all at once. The axiom system ZFC is the leading compromise on these three factors. It is powerful enough so that (almost) all arguments of the mathematical practice can be carried out within it and it is believed to be consistent by most mathematicians. Unfortunately, it turns out to come short in terms of completeness. The holy grail of set theory is finding a canonical extension of ZFC which is complete "for all intents and purposes", that is it should answer at least all natural questions of mathematics. This is, of course, a vague endeavour, nonetheless significant progress has been made toward this goal. The interesting undecidable statements come in two flavours:

(i) Statements about the height of the universe, typical examples are the existence of large cardinals.

(ii) Questions on the width of the universe, e.g. CH or Suslin's hypothesis[1].

The statements of the second type are typically the ones that can be attacked by Paul Cohen's method of forcing and leave set theorists with control over them. In fact, inner model theory has produced a number of theories deciding all interesting statements of this latter type[2], for example $\mathrm{ZFC} + V = L$. Unfortunately, models of these theories, at least of the ones we have a good understanding of, are too small to contain large enough large cardinals and are deemed unsatisfactory on these grounds. For example, there are no measurable cardinals in L. Another approach is building canonical theories "level by level" along the H-hierarchy. A good indicator of canonicity for a sentence is resistance against a change of truth by forcing. For H_ω, there is not much to argue about, all transitive models of ZFC have the same H_ω. For H_{ω_1}, the situation is already more complicated since H_{ω_1} and its theory can be changed significantly by forcing over "small" models of ZFC. The picture changes when large cardinals are involved:

Theorem (Woodin) *Assume there is a proper class of Woodin cardinals. Then the theory of H_{ω_1} (in fact that of $L(\mathbb{R})$), even with ground model reals as parameters, cannot be changed by forcing.*

This is strong evidence for the canonicity of the theory of H_{ω_1}, indeed of $L(\mathbb{R})$, assuming enough large cardinals exist.

[1]Suslin's hypothesis states that every complete dense c.c.c. linear order without endpoints is isomorphic to $\mathbb{R}$.

[2]This is only an empirical observation rather than a mathematical fact.

When moving higher up to H_{ω_2}, one cannot expect an exact analog of Woodin's result above since CH can be expressed by a Σ_2-formula over H_{ω_2} and can be changed at will[3]. Instead, one might ask for less: Is it possible that the Σ_1-theory of H_{ω_2} with ground model parameters cannot be changed by forcing? This is also quickly refuted as ω_1 can be collapsed and moreover, stationary sets can be killed by forcing and doing so introduces a new Σ_1 fact over H_{ω_2}. This suggests to only consider stationary set preserving extensions and to be more generous and look at the structure

$$\left(H_{\omega_2};\in,\mathrm{NS}_{\omega_1}\right)$$

with an additional relation for the nonstationary ideal on ω_1 instead. Demanding Σ_1-absoluteness for this structure to stationary set preserving extensions is exactly the forcing axiom BMM^{++} and is consistent modulo large cardinals. Two almost disjoint sets of tools have been developed in an effort to establish independence of such axioms and many others regarding the structure $(H_{\omega_2};\in,\mathrm{NS}_{\omega_1})$.

The first set begins starts with finite support iterations of c.c.c. forcings and then flourished with Shelah's theory of proper and semiproper forcing. This world is governed by forcing iteration theorems of the form "if $\mathbb{P}$ is a countable support(-style) iteration of forcings (forced to be) in a class Γ then $\mathbb{P}$ is in Γ". These are used in practice to make sure that an iteration of forcings one is interested in at the moment hopefully preserves basic structure, namely

$$\left(H_{\omega_2};\in,\mathrm{NS}_{\omega_1}\right)^V \prec_{\Sigma_0} \left(H_{\omega_2};\in,\mathrm{NS}_{\omega_1}\right)^{V^{\mathbb{P}}}.$$

We will call this area the "iterated forcing world".

The second set of tools has its roots in Steel-Van Wesep's proof of consistency of "NS_{ω_1} is saturated" from strong determinacy assumptions and was then developed much further by Woodin into the theory of $\mathbb{P}_{\max}$ and its variations resulting in the book [Woo10]. Here, ground models are models of determinacy, and the building blocks of the forcings are countable transitive models M together with an ideal I on ω_1^M, which are generically iterable. That means that taking an ultrapower of M by an M-ultrafilter which "generically" completes the dual filter associated to I results in a wellfounded model and further, this procedure can be iterated transfinitely and only produces wellfounded models. The power of AD is used to (and is necessary to) produce many "well-behaved" such structures. We will refer to this set of techniques as the"generic iterations world".

It has been observed empirically that if a problem can be solved using the tools from one of these worlds then often it is possible to give a solution using tools of the other world instead. Examples include the problems of finding models of ZFC in which e.g.

[3]CH is really the main perpetrator by Woodin's Σ^2_1-absoluteness theorem.

(*i*) NS_{ω_1} is saturated,

(*ii*) Cichoń's diagram takes a specific set of values,

(*iii*) Borel's conjecture[4] holds,

(*iv*) $\square_{\omega_2}$ fails or

(*v*) Club Bounding[5] holds.

In fact, (*i*), (*iv*) and (*v*) are all consequences of the canonical maximal forcing axiom MM^{++} of the iterated forcing world, but also hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, assuming $\mathrm{AD}^{L(\mathbb{R})}$. Recently, Asperó-Schindler have build a rigorous bridge between these two worlds: They proved $\mathrm{MM}^{++} \Rightarrow (*)$, the latter being the natural maximality axiom of the generic iterations world[6].

The goal of this thesis is to add details to this bridge and present more ways in which these worlds are connected. A number of consistency proofs within the world of generic iterations were not yet replicated in the world of iterated forcing. A prominent offender here was the statement "NS_{ω_1} is ω_1-dense"[7]. Using a variation on the $\mathbb{P}_{\max}$-method, Woodin has produced generic extensions of canonical models of determinacy in which this principle holds. We will show that this can be forced from large cardinals over otherwise arbitrary models of ZFC as well. Our argument will naturally prove a new implication of the form $\mathrm{MM}^{++} \Rightarrow (*)$.

Translating arguments from one world into the other is valuable for a number of reasons. First of all, it leads to the development of new methods which can be useful for further research. In this case, our methods But also, each world comes with its own restrictions. For example, the $\mathbb{P}_{\max}$-method is tied to well-behaved models of determinacy and usually does not provide a way to extend, in principle, any model of ZFC (with sufficiently many large cardinals) into a model of the theory in question. Moreover, usually the large cardinal assumptions seemingly necessary to carry out the arguments in the two worlds respectively differ, so one might provide a more optimal solution consistency-wise than the other. For example, we will make use of an inaccessible limit of supercompact cardinals to force "NS_{ω_1} is ω_1-dense", while Woodin's method makes use of the optimal assumption of ZF + AD. To give an example of the flip side of this, Paul Larson answered a question of Shelah-Zapletal using a variation of $\mathbb{P}_{\max}$ under the assumption ZF + AD and we will provide an analogous argument in the iterate forcing world making use of the weaker and optimal assumption of ZFC + "there is a Woodin

[4]This states that all strong measure zero sets are countable.

[5]That is, every map $f\colon \omega_1 \to \omega_1$ is bounded by a canonical function on a club.

[6]They even proved an equivalence of $(*)$ with a fragment of MM^{++} assuming a proper class of Woodin cardinals exists.

[7]An ideal I on ω_1 is ω_1-dense if $(\mathcal{P}(\omega_1)/I)^+$ has a dense subset of size ω_1.

cardinal".
Along the way to force "NS_{ω_1} is ω_1-dense" from large cardinals, we will frequently take time to explore the surrounding area. Among other things, we will prove multiple new instances of $\mathrm{MM}^{++} \Rightarrow (*)$ with a unified approach. For example we will introduce and prove consistent from large cardinals a forcing axiom which implies the $(*)$-axiom for Woodin's $\mathbb{S}_{\max}$. This gives another translation between the two worlds: $\mathbb{S}_{\max}$ forces the statement Ψ_S^+ over $L(\mathbb{R})$, assuming $\mathrm{AD}^{L(\mathbb{R})}$ and to the best of the authors knowledge, this has not been replicated before in the world of iterated forcing.
Another byproduct of our approach is the construction of a model in which Todorčević's Strong Reflection Principle holds, but Moore's Mapping Reflection Principle fails.

The second part of this thesis presents some contributions toward Set-Theoretic Geology. This area of set theory reverses the usual perspective on forcing: In practice, forcing is used as a tool to bring certain structure into existence which may not have been there before, so there is a clear underlying upward direction. Instead, Set-Theoretic Geology shifts this and looks down instead and analyzes the structure of the collection of all forcing grounds, i.e. models of ZFC which extend to V via forcing. One of the central objects here is that of the *mantle* $\mathbb{M}$, the intersection of all grounds. Toshimichi Usuba [Usu17] showed that $\mathbb{M}$ is always a model of ZFC, solving the most important open problem of Set-Theoretic Geology at that time. There is a number of ways to define variations of the mantle by intersecting only over a subcollection of all grounds. Perhaps the most natural variation on the mantle is the following: Let κ be a cardinal and define the κ-mantle $\mathbb{M}_\kappa$ as the intersection of all grounds which extend to V via a forcing of size $<\kappa$. The κ-mantle has its origins in Usuba's proof of the Bedrock Axiom[8] from an extendible cardinal. It also appears naturally in the computation of the mantle in certain canonical inner models, which lead to the *Varsovian models* which have been studied by Grigor Sargsyan, Ralf Schindler and later Farmer Schlutzenberg, see [SS18], [SSS21] and [Sch22b].
Usuba has shown that assuming κ to be merely a strong limit cardinal implies that $\mathbb{M}_\kappa \models \mathrm{ZF}$.

Question (Usuba,[Usu18])**.** Is $\mathbb{M}_\kappa$ always a model of ZFC?

We will answer this question negatively by constructing models of ZFC in which $\mathbb{M}_\kappa$ fails to satisfy the axiom of choice for three different types of cardinals κ.

[8]The Bedrock Axiom asserts the existence of a minimal ground w.r.t. inclusion.

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Contents

Part I

Forcing "NS_{ω_1} Is ω_1-Dense" From Large Cardinals

0 Introduction

0.1 History of "NS_{ω_1} is ω_1-dense"

In 1930, Stanislaw Ulam published an influential paper [Ula30] dealing with a question of Stefan Banach generalizing the measure problem of Lebesgue. He proved the following theorem:

Theorem 1 (Ulam). *Suppose κ is an uncountable cardinal and there is a σ-additive real-valued measure on κ which*

(i) measures all subsets of κ and

(ii) vanishes on points.

Then there is a weakly inaccessible cardinal $\leqslant \kappa$.

Ulam noticed that he could strengthen his conclusion if he replaces *real-valued* by *0-1-valued*. In more modern terminology, his second result reads:

Theorem 2 (Ulam). *Suppose κ is an uncountable cardinal and there is a nonprincipal σ-complete ultrafilter on κ. Then there is a (strongly) inaccessible cardinal $\leqslant \kappa$.*

These theorems gave birth to what are now known as real-valued measurable cardinals and measurable cardinals respectively. In the interest of having all subsets of some cardinal κ be measured in some sense, instead of increasing the size of κ, it is also possible to increase the number of allowed filters that measure. Henceforth Ulam considered the following question:

Question 3. Suppose κ is an uncountable cardinal below the least inaccessible. What is the smallest possible size of a family $\mathcal{F}$ of σ-closed nonprincipal filters on κ so that every subset of κ is measured by some filter in $\mathcal{F}$?

Let us call the cardinal in question the Ulam number of κ, $\mathrm{Ulam}(\kappa)$. Ulam's second theorem above can be rephrased as "$\mathrm{Ulam}(\kappa) > 1$". Indeed, Ulam proved in unpublished work that $\mathrm{Ulam}(\kappa) \geqslant \omega$. At some point, Ulam proposed this question to Paul Erdős, who, together with Leonidas Alaoglu, improved Ulam's result to "$\mathrm{Ulam}(\kappa) \geqslant \omega_1$" [Erd50]. The problem, this time in the special case $\kappa = \omega_1$, was apparently revitalized by appearing in the 1971 collection of unsolved problems in set theory popularized by Erdős and Hajnal [EH71]: Shortly after, Karel Prikry [Pri72] produced a model in which $\mathrm{Ulam}(\omega_1) = 2^{\omega_1} = \omega_2$, and did the same again with a different method in [Pri76].
A critical step towards a model in which $\mathrm{Ulam}(\omega_1) = \omega_1$ was taken by Alan D. Taylor: Building on earlier work of Baumgartner-Hajnal-Maté [BHM75], Taylor provided [Tay79] an impressive amount of statements equivalent to a natural strengthening of "$\mathrm{Ulam}(\omega_1) = \omega_1$", here is a shortened list.

Theorem 4 (Taylor)**.** *The following are equivalent:*

(*i*) *There is a family of normal filters witnessing* $\mathrm{Ulam}(\omega_1) = \omega_1$*.*

(*ii*) *There is a* σ*-closed uniform* ω_1*-dense ideal on* ω_1*.*

(*iii*) *There is a normal uniform* ω_1*-dense ideal on* ω_1*.*

The formulation (*iii*) is much better suited for set-theoretical arguments. We also mention that Taylor proved that all the above statements fail under MA_{ω_1}.
Thus what remains of Ulam's original question was reduced to: Is the existence of a normal uniform ω_1-dense ideal on ω_1 consistent with ZFC? This was answered positively by W. Hugh Woodin in three different ways. The first was by forcing over a model of $\mathrm{AD}_{\mathbb{R}}$+"Θ is regular", already in the fall of 1978. (unpublished). At that time, this theory was not yet known to be consistent relative to large cardinals. Naturally, somewhat later he did so from large cardinals:

Theorem 5 (Woodin, unpublished[9])**.** *Assume there is an almost-huge cardinal* κ*. Then there is a forcing extension in which there is a normal uniform* ω_1*-dense ideal on* $\omega_1 = \kappa$*.*

This finally resolved the question relative to large cardinals. But can the canonical normal uniform ideal, namely NS_{ω_1}, have this property? It is known that NS_{ω_1} behaves a little different in this context.

Theorem 6 (Shelah, [She86])**.** *If* NS_{ω_1} *is* ω_1*-dense then* $2^\omega = 2^{\omega_1}$*. In particular* CH *fails.*

This is not true for other normal uniform ideals on ω_1, for example CH holds in the model Woodin constructs from an almost huge cardinal. One can also ask about the exact consistency strength of the existence of such a normal uniform ω_1-dense ideal on ω_1. Both these questions were answered in subsequent work by Woodin, building on his $\mathbb{P}_{\max}$-technique.

Theorem 7 (Woodin, [Woo10, Corollary 6.150])**.** *The following theories are equiconsistent:*

(*i*) ZFC + "*There are infinitely many Woodin cardinals.*"

(*ii*) ZFC + "NS_{ω_1} *is* ω_1*-dense.*"

(*iii*) ZFC + "*There is a normal uniform* ω_1*-dense ideal on* ω_1*.*"

[9] A proof can be found in Foreman's handbook article [For10].

The direction $(iii) \Rightarrow (i)$ makes use of Woodin's core model induction technique, the argument is unpublished. We refer the interested reader to [RS14] where part of this is proven. Woodin's method for $(i) \Rightarrow (ii)$ is by forcing over $L(\mathbb{R})$, assuming AD there, with the $\mathbb{P}_{\max}$-variation $\mathbb{Q}_{\max}$. This approach has one downside: It is a forcing construction over a canonical determinacy model. $L(\mathbb{R})$ can be replaced by larger determinacy models, but $\mathbb{Q}_{\max}$ relies on a good understanding of the model in question. In practice, this is akin to an anti large cardinal assumption and leaves open questions along the lines of: Is "NS_{ω_1} is ω_1-dense" consistent together with all natural large cardinals, e.g. supercompact cardinals? Is it consistent with powerful combinatorial principles, for example SRP?
Woodin's original motivation for these results was in fact the question of generic large cardinal properties of ω_1: For example ω_1 is not measurable by Ulam's theorem, but there can be a generic extension of V with an elementary embedding $j : V \to M$ with transitive M and critical point ω_1^V. This leads to precipitous ideals on ω_1.

Definition 8. A uniform ideal I on ω_1 is precipitous if, whenever G is generic for $(\mathcal{P}(\omega_1)/I)^+$ then $\mathrm{Ult}(V, U_G)$ is wellfounded[10].

The existence of an ω_1-dense ideal is a much stronger assumption than the existence of a precipitous ideal. There is a natural well-studied intermediate principle.

Definition 9. A uniform ideal I on ω_1 is saturated if $(\mathcal{P}(\omega_1)/I)^+$ is ω_2-c.c..

Here is a short history of similar result for these principles:

(i) Mitchell forces a precipitous ideal on ω_1 from a measurable in the mid 70s, see [JMMP80].

(ii) Magidor forces "NS_{ω_1} is precipitous" from a measurable, published in [JMMP80].

(iii) Kunen [Kun78] forces a saturated ideal on ω_1 from a huge cardinal, which he invented for this purpose.

(iv) Steel-Van Wesep [SVW82] force "NS_{ω_1} is saturated" over a model of[11] $\mathrm{AD} + \mathrm{AC}_{\mathbb{R}}$.

(v) Foreman-Magidor-Shelah [FMS88] force "NS_{ω_1} is saturated" from a supercompact with semiproper forcing. Later reduced to one Woodin cardinal by Shelah[12].

[10] U_G denotes the V-ultrafilter induced by G.
[11] Woodin [Woo83] subsequently reduced the assumption to just AD .
[12] The main ideas for the argument are in [She98, XVI], a write-up by Schindler can be found in [Sch11].

Woodin's results continue this line of research for ω_1-dense ideals. But the analog of the step from (iv) to (v) for ω_1-dense ideals was missing. Accordingly, Woodin posed the following question:

Question 10 (Woodin, [Woo99, Chapter 11 Question 18 b)])**.** Assuming the existence of some large cardinal: Must there exist some semiproper partial order $\mathbb{P}$ such that

$$V^{\mathbb{P}} \models \text{“NS}_{\omega_1} \text{ is } \omega_1\text{-dense” ?}$$

We will answer this positively in this thesis.

Theorem 11. *Assume there is an inaccessible cardinal κ which is the limit of cardinals which are $<\kappa$-supercompact. Then there is a stationary set preserving forcing $\mathbb{P}$ so that*

$$V^{\mathbb{P}} \models \text{“NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

If there is an additional supercompact cardinal below κ, we can find such $\mathbb{P}$ that is semiproper.

On a different note, there has been significant interest recently into the possible $\mathbf{\Delta}_1$-definability of NS_{ω_1} (with parameters), in particular in the presence of forcing axioms. Note that NS_{ω_1} is trivially $\Sigma_1(\omega_1)$-definable, but it is independent of ZFC whether NS_{ω_1} is $\mathbf{\Pi}_1$-definable. Hoffelner-Larson-Schindler-Wu [HLSW22] show:

- (i) If BMM holds and there is a Woodin cardinal then NS_{ω_1} is not $\mathbf{\Delta}_1$-definable.
- (ii) If $(*)$ holds then NS_{ω_1} is not $\mathbf{\Delta}_1$-definable.
- (iii) Thus by Asperó-Schindler [AS21], if MM^{++} holds, NS_{ω_1} is not $\mathbf{\Delta}_1$-definable.
- (iv) It is consistent relative to large cardinals that BPFA holds and NS_{ω_1} is $\mathbf{\Delta}_1$-definable.

There is also a forthcoming paper by Ralf Schindler and Xiuyuan Sun [SS22] showing that in (iii), MM^{++} can be relaxed to MM.
If NS_{ω_1} is ω_1-dense then NS_{ω_1} is automatically $\mathbf{\Delta}_1$-definable: If $\mathcal{S}$ is a set of ω_1-many stationary sets witnessing the density, then $T \subseteq \omega_1$ is stationary iff

$$\exists C \subseteq \omega_1 \text{ a club, } \exists S \in \mathcal{S}\ C \cap S \subseteq T.$$

This was first observed by Friedman-Wu-Zdomskyy [FWZ15]. In this context, two interesting points arise from our results here: First, we isolate for the first time a forcing axiom which *implies* “NS_{ω_1} is $\mathbf{\Delta}_1$-definable”. Second, it is well known that many of the structural consequences of MM follow already from SRP, for example "NS_{ω_1} is saturated", $2^\omega = \omega_2$, SCH, etc. In contrast, in the result of Schindler-Sun, MM cannot be replaced by SRP: If appropriate large cardinals are consistent, then so is SRP together with "NS_{ω_1} is $\mathbf{\Delta}_1$-definable".

0.2 Overview

In Section 2, we introduce a combinatorial principle central to our approach, called $\diamondsuit(\mathbb{B})$. It is a certain diamond-style principle that guesses filters for a forcing $\mathbb{B}$ of size ω_1.

In Section 3, we present the theory of $\diamondsuit$-forcing which generalizes the notions of complete, proper and semiproper forcing respectively to their natural variants which preserve a distinguished witness f of $\diamondsuit(\mathbb{B})$. We prove iteration theorems for these classes which naturally generalize the iteration theorems for complete, proper and semiproper[13] forcings respectively. This allows us to formulate and prove consistent from a supercompact cardinal the principles $\mathrm{MM}(f)$ and $\mathrm{MM}^{++}(f)$ for a witness f of $\diamondsuit(\mathbb{B})$ in Section 3.7. These are the variants of MM and MM^{++} with stationary sets replaced by f-stationary sets, a natural version of stationary sets in the present context. We will show that SRP is a consequence of $\mathrm{MM}(f)$, so in particular $2^\omega = \omega_2$ and "NS_{ω_1} is saturated" follow.

In Section 4, we work with general abstract $\mathbb{P}_{\max}$-variations and modifying the methods of Asperó-Schindler [AS21], ultimately arriving at two results we call "Blueprint Theorems" that allow us to prove a variety of implications analogous to "$\mathrm{MM}^{++} \Rightarrow (*)$" later. A key tool we introduce here is that of a $\diamondsuit$-iteration.

In Section 5, we apply the theory developed so far for different instances of the forcing $\mathbb{B}$ and find an instance of $\mathrm{MM}^{++} \Rightarrow (*)$ in each case. For $\mathbb{B} = \{\mathbb{1}\}$, i.e. the trivial forcing, we indeed exactly recover the $\mathrm{MM}^{++} \Rightarrow (*)$ picture as well as the iteration theorem for semiproper forcings, but with nice supports. In the case of $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$, we end up with models in which NS_{ω_1} is saturated and there is a complete embedding $\mathrm{Col}(\omega, \omega_1) \to (P(\omega_1)/\mathrm{NS}_{\omega_1})^+$, partial progress towards "NS_{ω_1} is ω_1-dense". In case $\mathbb{B}$ is Cohen forcing, we recover the notion of weakly Lusin sequences introduced by Shelah-Zapletal in [SZ99]. We also reduce the consistency assumption used in an answer of Paul Larson to a question of the aforementioned paper by translating Larson's argument which makes use of a $\mathbb{P}_{\max}$-variation into a forcing iteration. Finally, in case $\mathbb{B}$ is a Suslin tree, we obtain iteration theorems by Miyamoto for proper forcings [Miy93] and semiproper forcings [Miy02] preserving a distinguished Suslin tree respectively. Moreover, we

[13] Though we will make use of Miyamoto's nice support instead of RCS support in the case of f-semiproper forcings.

show that if T is a strongly homogeneous Suslin tree then $\mathrm{MM}^{++}(T)$ implies the variant of the $(*)$-axiom for $\mathbb{S}^T_{\max}$, a $\mathbb{P}_{\max}$- (or $\mathbb{S}_{\max}$-) variation for a single Suslin tree due to Paul Larson.

Section 6 deals with a new MM^{++}-style forcing axiom Suslin's Minimum^{++}. Roughly speaking, it is MM^{++} restricted to forcings which also preserve all Suslin trees and the partially generic filter that is postulated to exist additionally evaluates ω_1-many names for Suslin trees to Suslin trees in V. We prove this axiom consistent from a supercompact cardinal and argue that it implies the version of the $(*)$-axiom for Woodin's $\mathbb{P}_{\max}$-variation $\mathbb{S}_{\max}$.

We are finally ready to answer Question 10 in Section 7. We introduce yet another forcing axiom QM and prove that it implies "NS_{ω_1} is ω_1-dense". We then prove a new iteration theorem for so called Q-iterations. This allows us to force QM from a supercompact limit of supercompact cardinals via semiproper forcing. We can lower the large cardinal assumption a bit if we weaken the conclusion of forcing QM to "NS_{ω_1} is ω_1-dense". We also show that QM implies the version of $(*)$ for Woodin's $\mathbb{Q}_{\max}$.

Section 8 briefly introduces a quite general $\mathbb{P}_{\max}$-variation we denote by $\mathbb{F}_{\max}$ and analyzes the extension of $L(\mathbb{R})$ by $\mathbb{F}_{\max}$ assuming $\mathrm{AD}^{L(\mathbb{R})}$. This approach treats the $\mathbb{P}_{\max}$-variations associated with the forcing axioms $\mathrm{MM}^{++}(f)$ uniformly in the same way $\mathrm{MM}^{++}(f)$ treats a number of concrete forcing axioms uniformly.

In Section 9, we investigate maximal models of $\mathfrak{d} = \aleph_1$. Shelah-Zapletal introduced a $\mathbb{P}_{\max}$-variation $P_{\mathfrak{d}=\aleph_1}$ and showed that the extension of $L(\mathbb{R})$ by $P_{\mathfrak{d}=\aleph_1}$ is a canonical model of $\mathfrak{d} = \aleph_1$. We use yet another iteration theorem by Miyamoto to prove consistent from a supercompact the axiom $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$, which is MM^{++} conditioned to the existence of a dominating family of size $\aleph_1$. We once again prove that $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ implies the $(*)$-axiom associated to $P_{\mathfrak{d}=\aleph_1}$. Section 10 achieves the same picture, but for $\mathfrak{b} = \aleph_1$.

In the appendix, Section 11, we provide some more related results. Among them, we show that MRP is not a consequence of SRP. We go on to prove some structural consequences of PFA from $\mathrm{PFA}(f)$, the natural variant of PFA for a witness f of $\diamondsuit(\mathbb{B})$, instead. This is hampered by the fact that both MRP and the P-ideal dichotomy may fail. We also characterize exactly when Namba forcing is f-semiproper and doing so generalize a result of Shelah [She98, XII] in the classical semiproper case.

A reader mainly interested in forcing "NS_{ω_1} is ω_1-dense" from large cardinals may choose to only read Sections 2-4, optionally Subsection 5.2, but definitely Section 7.

1 Notation

First, we fix some notation. We will extensively deal with countable elementary substructures $X \prec H_\theta$ for large regular θ. We will make frequent use of the following notation:

Definition 1.1. Suppose X is any extensional set.

(*i*) M_X denotes the transitive isomorph of X.

(*ii*) $\pi_X \colon M_X \to X$ denotes the inverse collapse.

(*iii*) $\delta^X := \omega_1 \cap X$.

In almost all cases, we will apply this definition to a countable elementary substructure $X \prec H_\theta$ for some uncountable cardinal θ. In some cases, the X we care about lives in a generic extension of V, even though it is a substructure of H_θ^V. In that case, δ^X will always mean $X \cap \omega_1^V$.

We will also sometimes make use of the following convention in order to "unclutter" arguments.

Convention 1.2. If $X \prec H_\theta$ is an elementary substructure and some object a has been defined before and $a \in X$ then we denote $\pi_X^{-1}(a)$ by $\bar{a}$.

We will make use of this notation only if it is unambiguous.

Definition 1.3. If X, Y are sets then $X \sqsubseteq Y$ holds just in case

(*i*) $X \subseteq Y$ and

(*ii*) $\delta^X = \delta^Y$.

We use the following notions of clubs and stationarity on $[H_\theta]^\omega$:

Definition 1.4. Suppose A is an uncountable set.

(*i*) $[A]^\omega$ is the set of countable subsets of A.

(*ii*) $\mathcal{C} \subseteq [A]^\omega$ is a club in $[A]^\omega$ if

  a) for any $X \in [A]^\omega$ there is a $Y \in \mathcal{C}$ with $X \subseteq Y$ and

  b) if $\langle Y_n \mid n < \omega \rangle$ is a $\subseteq$-increasing sequence of sets in $\mathcal{C}$ then $\bigcup_{n<\omega} Y_n \in \mathcal{C}$.

(*iii*) $\mathcal{S} \subseteq [A]^\omega$ is stationary in $[A]^\omega$ if $\mathcal{S} \cap \mathcal{C} \neq \emptyset$ for any club $\mathcal{C}$ in $[A]^\omega$.

Next, we explain our notation for forcing iterations.

Definition 1.5. Suppose $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is an iteration and $\beta \leqslant \gamma$. We consider elements of $\mathbb{P}$ as functions of domain (or length) γ.

(*i*) If $p \in \mathbb{P}_\beta$ then $\mathrm{lh}(p) = \beta$.

(*ii*) If G is $\mathbb{P}$-generic then G_β denotes the restriction of G to $\mathbb{P}_\beta$, i.e.

$$G_\beta = \{p \upharpoonright \beta \mid p \in G\}.$$

Moreover, $\dot{G}_\beta$ is the canonical $\mathbb{P}$-name for G_β.

(*iii*) If G_β is $\mathbb{P}_\beta$-generic then $\mathbb{P}_{\beta,\gamma}$ denotes (by slight abuse of notation) the remainder of the iteration, that is

$$\mathbb{P}_{\beta,\gamma} = \{p \in \mathbb{P}_\gamma \mid p \upharpoonright \beta \in G_\beta\}.$$

$\dot{\mathbb{P}}_{\beta,\gamma}$ denotes a name for $\mathbb{P}_{\beta,\gamma}$ in V.

(*iv*) If G is $\mathbb{P}$-generic and $\alpha < \beta$ then $G_{\alpha,\beta}$ denotes the projection of G onto $\mathbb{P}_{\alpha,\beta}$.

There will be a number of instances were we need a structure to satsify a sufficiently large fragment of ZFC. For completeness, we make this precise.

Definition 1.6. *Sufficiently much of* ZFC is the fragment $\mathrm{ZFC}^- +$ "ω_1 exists". Here, ZFC^- is ZFC without the powerset axiom and with the collection scheme instead of the replacement scheme.

We will frequently deal with forcing axioms. Our notations for the classical forcing axioms is as follows:

Definition 1.7. Suppose Γ is a class of forcings.

(*i*) $\mathrm{FA}(\Gamma)$ holds if for any for any $\mathbb{P} \in \Gamma$ and any collection $\mathcal{D}$ of ω_1-many dense subsets of $\mathbb{P}$, there is a filter $g \subseteq \mathbb{P}$ with $g \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

(*ii*) $\mathrm{FA}^{++}(\Gamma)$ holds if for any $\mathbb{P} \in \Gamma$ and any collections

- $\mathcal{D}$ of ω_1-many dense subsets of $\mathbb{P}$ and
- $\mathcal{S}$ of ω_1-many $\mathbb{P}$-names for stationary subsets of ω_1

there is a filter $g \subseteq \mathbb{P}$ with

(*g.i*) $g \cap D \neq \emptyset$ for all $D \in \mathcal{D}$ and

(*g.ii*) $\dot{S}^g = \{\alpha < \omega_1 \mid \exists p \in g\ p \Vdash \check{\alpha} \in \dot{S}\}$ is stationary for all $\dot{S} \in \mathcal{S}$.

Now assume additionally that $X \subseteq \mathbb{R}$.

(*iii*) X-$\mathrm{BFA}(\Gamma)$ holds if X is ∞-universally Baire and for any $\mathbb{P} \in \Gamma$ we have

$$(H_{\omega_2}; \in, X)^V \prec_{\Sigma_1} (H_{\omega_2}; \in, X^*)^{V^{\mathbb{P}}}$$

where X^* denotes the reinterpretation of X in $V^{\mathbb{P}}$.

(*iv*) X-$\mathrm{BFA}^{++}(\Gamma)$ holds if X is ∞-universally Baire and for any $\mathbb{P} \in \Gamma$ we have

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, X)^V \prec_{\Sigma_1} (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, X^*)^{V^{\mathbb{P}}}.$$

$\mathrm{BFA}(\Gamma)$ (resp. $\mathrm{BFA}^{++}(\Gamma)$) is short for $\varnothing$-$\mathrm{BFA}(\Gamma)$ (resp. $\varnothing$-$\mathrm{BFA}^{++}(\Gamma)$). Also, if $\Delta \subseteq \mathcal{P}(\mathbb{R})$ is nonempty then Δ-$\mathrm{BFA}(\Gamma)$ means

$$\forall X \in \Delta\ X\text{-}\mathrm{BFA}(\Gamma)$$

and Δ-$\mathrm{BFA}^{++}(\Gamma)$ is defined analogously.

2 $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$

We introduce the central combinatorial principles. Their relevancy is motivated by the following observation: If NS_{ω_1} is ω_1-dense, then there is a dense embedding

$$\eta\colon \mathrm{Col}(\omega, \omega_1) \to (P(\omega_1)/\mathrm{NS}_{\omega_1})^+.$$

We aim to force a forcing axiom that implies this. As usual, the forcing achieving this is an iteration $\mathbb{P}$ of some large cardinal length κ which preserves ω_1 and iterates forcings of size $<\kappa$ with countable support-style supports. $\mathbb{P}$ will thus be κ-c.c. and this means that some "representation"

$$\eta_0\colon \mathrm{Col}(\omega, \omega_1) \to \mathrm{NS}^+_{\omega_1}$$

of η exists already in an intermediate extension. By "representation" we mean that in $V^{\mathbb{P}}$,

$$[\eta_0(p)]_{\mathrm{NS}_{\omega_1}} = \eta(p)$$

for all $p \in \mathrm{Col}(\omega, \omega_1)$[14]. With this in mind, one should isolate the relevant Π_1-properties which η_0 possesses in $V^{\mathbb{P}}$. Consequently, η_0 satisfies these properties in the intermediate extension. It is hopefully easier to first force an object with this Π_1-fragment and we should subsequently only force with partial orders that preserve this property. This is exactly what we will do. The relevant combinatorial properties are $\diamondsuit(\omega_1^{<\omega})$ and $\diamondsuit^+(\omega_1^{<\omega})$ and were already isolated by Woodin in his study of $\mathbb{Q}_{\max}$ [Woo10, Section 6.2]. For us, these principles will be useful in a more general context. In fact, we will replace the role of $\mathrm{Col}(\omega, \omega_1)$ by an arbitrary forcing of size $\leqslant\omega_1$. Moreover, we also strengthen Woodin's principle in a technical way that turns out to be convenient for our purposes. Most results in this Section are essentially due to Woodin and proven in [Woo10, Section 6.2].

Definition 2.1. Suppose $\mathbb{B} \subseteq \omega_1$ is a forcing.

[14] For $S \subseteq \omega_1$ and I an ideal on ω_1, $[S]_I$ denotes the equivalence class of S induced by the equivalence relation $T \sim T' \Leftrightarrow T \triangle T' \in I$.

(*i*) We say that f *guesses $\mathbb{B}$-filters* if f is a function

$$f\colon \omega_1 \to H_{\omega_1}$$

and for all $\alpha < \omega_1$, $f(\alpha)$ is a $\mathbb{B} \cap \alpha$-filter[15].

(*ii*) Suppose $\theta \geqslant \omega_2$ is regular and $X \prec H_\theta$ is an elementary substructure. We say X is *f-slim*[16] if

(*X.i*) X is countable,

(*X.ii*) $f, \mathbb{B} \in X$ and

(*X.iii*) $f(\delta^X)$ is $\mathbb{B} \cap \delta^X$-generic over M_X.

Definition 2.2. Let $\mathbb{B} \subseteq \omega_1$ be a forcing. $\diamondsuit(\mathbb{B})$ states that there is a function f so that

(*i*) f guesses $\mathbb{B}$-filters and

(*ii*) for any $b \in \mathbb{B}$ and regular $\theta \geqslant \omega_2$

$$\{X \prec H_\theta \mid X \text{ is } f\text{-slim} \wedge b \in f(\delta^X)\}$$

is stationary in $[H_\theta]^\omega$.

$\diamondsuit^+(\mathbb{B})$ is the strengthening of $\diamondsuit(\mathbb{B})$ where (*ii*) is replaced by:

$(ii)^+$ For any regular $\theta \geqslant \omega_2$

$$\{X \prec H_\theta \mid X \text{ is } f\text{-slim}\}$$

contains a club of $[H_\theta]^\omega$. Moreover, for any $b \in \mathbb{B}$

$$\{\alpha < \omega_1 \mid b \in f(\alpha)\}$$

is stationary.

We say that f *witnesses* $\diamondsuit(\mathbb{B})$, $\diamondsuit^+(\mathbb{B})$ respectively.

Remark 2.3. Observe that if f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{B}$ is separative then $\mathbb{B}$ can be "read off" from f: We have $\mathbb{B} = \bigcup_{\alpha<\omega_1} f(\alpha)$ and for $b, c \in \mathbb{B}$, $b \leqslant_{\mathbb{B}} c$ iff whenever $b \in f(\alpha)$ then $c \in f(\alpha)$ as well. Thus, it is usually not necessary to mention $\mathbb{B}$.

We introduce some convenient shorthand notation.

[15] We consider the empty set to be a filter.

[16] We use the adjective "slim" for the following reason: An f-slim $X \prec H_\theta$ cannot be too fat compared to its height below ω_1, i.e. δ^X. If $X \sqsubseteq Y \prec H_\theta$ and Y is f-slim then X is f-slim as well, but the converse can fail.

Definition 2.4. If $\mathbb{B} \subseteq \omega_1$ is a forcing, f guesses $\mathbb{B}$-filters and $b \in \mathbb{B}$ then

$$S_b^f := \{\alpha < \omega_1 \mid b \in f(\alpha)\}.$$

If f is clear from context we will sometimes omit the superscript f.

Note that if f witnesses $\diamondsuit(\mathbb{B})$, then S_b^f is stationary for all $b \in \mathbb{B}$. This is made explicit for $\diamondsuit^+(\mathbb{B})$. This is exactly the technical strengthening over Woodin's definition of $\diamondsuit(\omega_1^{<\omega}), \diamondsuit^+(\omega_1^{<\omega})$. Lemma 2.12 shows that this strengthening is natural. Moreover, this implies

$$\diamondsuit(\mathbb{B} \oplus \mathbb{C}) \Rightarrow \diamondsuit(\mathbb{B}) \wedge \diamondsuit(\mathbb{C})$$

whenever $\mathbb{B}, \mathbb{C} \subseteq \omega_1$ are forcings and $\mathbb{B} \oplus \mathbb{C}$ is the disjoint union of $\mathbb{B}$ and $\mathbb{C}$ coded into a subset of ω_1. This becomes relevant in Subsection 5.3. Nonetheless, the basic theory of these principles is not changed by a lot.

Definition 2.5. If f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is a forcing, we say that $\mathbb{P}$ *preserves* f if whenever G is $\mathbb{P}$-generic then f witnesses $\diamondsuit(\mathbb{B})$ in $V[G]$.

We remark that if f witnesses $\diamondsuit^+(\mathbb{B})$ then "$\mathbb{P}$ preserves f" still only means that f witnesses $\diamondsuit(\mathbb{B})$ in $V^{\mathbb{P}}$.

Next, we define a variant of stationary sets related to a witness of $\diamondsuit(\mathbb{B})$. Suppose $\theta \geqslant \omega_2$ is regular. Then $S \subseteq \omega_1$ is stationary iff for any club $\mathcal{C} \subseteq [H_\theta]^\omega$, there is some $X \in \mathcal{C}$ with $\delta^X \in S$. f-stationarity results from restricting to f-slim $X \prec H_\theta$ only.

Definition 2.6. Suppose f guesses $\mathbb{B}$-filters.

(i) A subset $S \subseteq \omega_1$ is *f-stationary* iff whenever $\theta \geqslant \omega_2$ is regular and $\mathcal{C} \subseteq [H_\theta]^\omega$ is club then there is some f-slim $X \in \mathcal{C}$ with $\delta^X \in S$.

(ii) A forcing $\mathbb{P}$ *preserves f-stationary sets* iff any f-stationary set is still f-stationary in $V^{\mathbb{P}}$.

We make use of f-stationarity only when f witnesses $\diamondsuit(\mathbb{B})$. However, with the above definition it makes sense to talk about f-stationarity in a forcing extension before we know that f has been preserved. Note that all f-stationary sets are stationary, but the converse might fail, see Proposition 5.15. We will later see that f-stationary sets are the correct replacement of stationary set in our context. Most prominently this notion will be used in the definition of the MM^{++}-variant $\mathrm{MM}^{++}(f)$ we introduce in Subsection 3.7. It will be useful to have an equivalent formulation of f-stationarity at hand.

Proposition 2.7. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent for any set $S \subseteq \omega_1$:*

(*i*) *S is f-stationary.*

(*ii*) *Whenever $\langle D_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{B}$, the set*

$$\{\alpha \in S \mid \forall\beta < \alpha\ f(\alpha) \cap D_\beta \neq \varnothing\}$$

is stationary.

Proof. (*i*)⇒(*ii*) can be seen by finding an f-slim $X \prec H_\theta$ with

$$\langle D_\alpha \mid \alpha < \omega_1\rangle \in X$$

and $\delta^X \in S$. So let us prove (*ii*)⇒(*i*): Let $\theta \geqslant \omega_2$ be regular and $\mathcal{C} \subseteq [H_\theta]^\omega$ be club. Let $\langle X_i \mid i < \omega_1\rangle$ a continuous increasing chain of elementary substructures of H_θ with all $X_i \in \mathcal{C}$ and $f \in X_0$. Now enumerate the dense subsets of $\mathbb{B}$ appearing along the chain $\vec{X}$ as $\langle D_\alpha \mid \alpha < \omega_1\rangle$. Let $C \subseteq \omega_1$ be club with

(*C.i*) $\alpha = \delta^{X_\alpha}$ and

(*C.ii*) $\forall\beta < \omega_1\ D_\beta \in X_\alpha \Leftrightarrow \beta < \alpha$

for all $\alpha \in C$. By (*ii*), there is some $\alpha \in S \cap C$ so that $f(\alpha) \cap D_\beta \neq \varnothing$ whenever $\beta < \alpha$. Clearly we then have X_α is f-slim and $\delta^{X_\alpha} = \alpha \in S$. □

Proposition 2.8. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent:*

(*i*) *f witnesses $\diamondsuit(\mathbb{B})$.*

(*ii*) *S^f_b is f-stationary for all $b \in \mathbb{B}$.*

(*iii*) *For any $b \in \mathbb{B}$ and sequence $\langle D_\alpha \mid \alpha < \omega_1\rangle$ of dense subsets of $\mathbb{B}$,*

$$\{\alpha \in S^f_b \mid \forall\beta < \alpha\ f(\alpha) \cap D_\beta \neq \varnothing\}$$

is stationary.

Proof. The equivalence of (*i*) and (*ii*) follows from the definitions. (*ii*) and (*iii*) are equivalent by the equivalent formulation of f-stationarity provided by Proposition 2.7. □

We mention a handy corollary.

Corollary 2.9. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Any forcing preserving f-stationary sets preserves f.*

Proposition 2.10. *Suppose f guesses $\mathbb{B}$-filters. The following are equivalent:*

(*i*) *f witnesses $\diamondsuit^+(\mathbb{B})$.*

(*ii*) *For any* $b \in \mathbb{B}$, S_b^f *is stationary and all stationary sets are* f*-stationary.*

(*iii*) *If* D *is dense in* $\mathbb{B}$ *then*

$$\{\alpha < \omega_1 \mid \ f(\alpha) \cap D \neq \varnothing\}$$

contains a club and for all $b \in \mathbb{B}$, S_b^f *is stationary.*

(*iv*) *All countable* $X \prec H_\theta$ *with* $f \in X$ *and* $\theta \geqslant \omega_2$ *regular are* f*-slim and moreover for all* $b \in \mathbb{B}$, S_b^f *is stationary.*

Proof. (*i*)⇒(*ii*) is trivial.
(*ii*)⇒(*iii*): Let $D \subseteq \mathbb{B}$ be dense and suppose

$$S := \{\alpha < \omega_1 \mid f(\alpha) \cap D \neq \varnothing\}$$

does not contain a club. Then $T = \omega_1 - S$ is stationary and hence f-stationary by (*ii*). It follows that there is some f-slim $X \prec H_{\omega_2}$ with $D \in X$ and $\delta^X \in T$. But clearly $f(\delta^X) \cap D \neq \varnothing$, contradiction.
(*iii*)⇒(*iv*): Let $\theta \geqslant \omega_2$ be regular. Let $X \prec H_\theta$ be countable with $f \in X$. Suppose $D \in X$ is dense in $\mathbb{B}$. We have

$$H_\theta \models \text{“}\exists C \subseteq \omega_1 \text{ club and for all } \alpha \in C\ f(\alpha) \cap D \neq \varnothing\text{”}.$$

Hence there is such a club C in X. But then $\delta^X \in C$ so that

$$f(\delta^X) \cap (D \cap \delta^X) = f(\delta^X) \cap D \neq \varnothing.$$

Thus $f(\delta^X)$ is generic over M_X.
(*iv*)⇒(*i*) is trivial. □

We will now give a natural equivalent formulation of $\diamondsuit^+(\mathbb{B})$. The following little observation will be handy.

Proposition 2.11. *Suppose* f *guesses* $\mathbb{B}$*-filters. Let* g *be* $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$*-generic,* U_g *the corresponding generic* V*-ultrafilter and*

$$j\colon V \to \mathrm{Ult}(V, U_g)$$

the induced ultrapower[17]*. Then*

$$j(f)(\omega_1^V) = \{b \in \mathbb{B} \mid [S_b^f]_{\mathrm{NS}_{\omega_1}} \in g\}.$$

Witnesses of $\diamondsuit^+(\mathbb{B})$ are simply codes for regular embeddings[18] of $\mathbb{B}$ into $\mathrm{NS}_{\omega_1}^+$.

[17]We assume $\mathrm{wfp}(\mathrm{Ult}(V, U_g))$ to be transitive. Note that $\omega_1^V \in \mathrm{wfp}(\mathrm{Ult}(V, U_g))$.

[18]Regular embeddings, also known as complete embeddings, are embeddings between partial orders which preserve maximal antichains.

Lemma 2.12. *The following are equivalent:*

(*i*) $\diamondsuit^+(\mathbb{B})$.

(*ii*) *There is a regular embedding* $\eta\colon \mathbb{B} \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$.

Proof. (*i*)⇒(*ii*): Let f witness $\diamondsuit^+(\mathbb{B})$ and define η via

$$\eta(b) = \left[S_b^f\right]_{\mathrm{NS}_{\omega_1}}$$

for $b \in \mathbb{B}$. It is clear that η is well-defined, preserves the order as well as incompatibility. Assume $\mathcal{A}$ is a maximal antichain in $\mathbb{B}$. Let g be generic for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$, it is our duty to show that $g \cap \eta[\mathcal{A}] \neq \varnothing$. Let $D \subseteq \mathbb{B}$ be the dense set of conditions below some element of $\mathcal{A}$. Let U_g be the generic ultrafilter associated to g and

$$j\colon V \to \mathrm{Ult}(V, U_g)$$

be the generic ultrapower. $\mathrm{Ult}(V, U_g)$ is not necessarily wellfounded, however we assume the wellfounded part $\mathrm{wfp}(\mathrm{Ult}(V, U_g))$ to be transitive and we have $\omega_1^V \in \mathrm{wfp}(\mathrm{Ult}(V, U_g))$, which suffices for our purposes. As f witnesses $\diamondsuit^+(\mathbb{B})$,

$$\{\alpha < \omega_1 \mid f(\alpha) \cap D \neq \varnothing\} \in g$$

and thus

$$\mathrm{Ult}(V, U_g) \models j(f)(\omega_1^V) \cap j(D) \neq \varnothing.$$

Now $j(D) \cap \omega_1^V = D$ and hence $j(f)(\omega_1^V) \cap \mathcal{A} \neq \varnothing$. By Proposition 2.11

$$j(f)(\omega_1^V) = \{b \in \mathbb{B} \mid \eta(b) \in g\}$$

and it follows that there is $b \in \mathcal{A}$ with $\eta(b) \in g$.

(*ii*)⇒ (*i*): Let η witness (*ii*). For $b \in \mathbb{B}$, choose $S_b \in \mathrm{NS}_{\omega_1}^+$ that represents the equivalence class $\eta(b)$ and define $f\colon \omega_1 \to H_{\omega_1}$ by

$$f(\alpha) = \{b \in \mathbb{B} \cap \alpha \mid \alpha \in S_b\}.$$

Note that we have $S_b^f = S_b \in \mathrm{NS}_{\omega_1}^+$ for all $b \in \mathbb{B}$.

Claim 2.13. $\{\alpha < \omega_1 \mid f(\alpha)$ *is a* $\mathbb{B} \cap \alpha$*-filter*$\}$ *contains a club.*

Proof. Suppose toward a contradiction that there are stationarily many $\alpha < \omega_1$ so that $f(\alpha)$ contains incompatible conditions. By normality of NS_{ω_1}, there are then incompatible $b, c \in \mathbb{B}$ so that

$$\{\alpha < \omega_1 \mid b, c \in f(\alpha)\} \in \mathrm{NS}_{\omega_1}^+.$$

But this would imply that $\eta(b), \eta(c)$ are compatible, contradiction. Similarly it follows from η being order-preserving that $f(\alpha)$ is upwards closed on a club of $\alpha < \omega_1$. □

Thus we may assume all $f(\alpha)$ are filters. Now suppose that $D \subseteq \mathbb{B}$ is dense. Pick g that is generic for $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$ and let

$$j\colon V \to \mathrm{Ult}(V, U_g)$$

be the resulting generic ultrapower. Again by Proposition 2.11, we have that

$$j(f)(\omega_1^V) = \{b \in \mathbb{B} \mid \eta(b) \in g\}$$

Thus, as η is a regular embedding, $h := j(f)(\omega_1^V)$ is generic over V for $\mathbb{B}$ and consequently meets D. Hence we have

$$[\{\alpha < \omega_1^V \mid f(\alpha) \cap D \neq \varnothing\}]_{\mathrm{NS}_{\omega_1}} \in g$$

and as g was arbitrary, we can conclude

$$\{\alpha < \omega_1^V \mid f(\alpha) \cap D = \varnothing\} \in \mathrm{NS}_{\omega_1}$$

which is what we had to show. □

The argument above suggests the following definition.

Definition 2.14. Suppose f witnesses $\diamondsuit(\mathbb{B})$. We define

$$\eta_f\colon \mathbb{B} \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$$

by $b \mapsto [S_b^f]_{\mathrm{NS}_{\omega_1}}$ and call η_f the *embedding associated to f*.

We will now show that $\diamondsuit(\mathbb{B})$ is consistent for any forcing $\mathbb{B} \subseteq \omega_1$, even simultaneously so for all such $\mathbb{B}$. We will deal with the consistency of $\diamondsuit^+(\mathbb{B})$ in the next section.

Proposition 2.15. *Assume $\diamondsuit$. Then $\diamondsuit(\mathbb{B})$ holds for any poset $\mathbb{B} \subseteq \omega_1$.*

Proof. We fix a uniform way of coding an element

$$(b, \langle D_\alpha \mid \alpha < \omega_1\rangle)$$

of $\omega_1 \times \mathcal{P}(\omega_1)^{\omega_1}$ into a subset $A \subseteq \omega_1$. We only require that on a club, $A \cap \beta$ codes $(b, \langle D_\alpha \cap \beta \mid \alpha < \beta\rangle)$. This will happen automatically for any sensible coding method. We leave the details to the reader.
Let $\vec{a} := \langle a_\beta \mid \beta < \omega_1\rangle$ witness $\diamondsuit$ and let $\mathbb{B} \subseteq \omega_1$ be a forcing. Define f as follows: Let $\beta < \omega_1$ and suppose a_β codes

$$(b, \langle D_\alpha \mid \alpha < \beta\rangle)$$

where $b \in \mathbb{B} \cap \beta$ and $\langle D_\alpha \mid \alpha < \beta\rangle$ is a sequence of dense subsets of $\mathbb{B} \cap \beta$. Then we let $f(\beta)$ be a filter in $\mathbb{B} \cap \beta$ so that

$(f.i)$ $b \in f(\beta)$ and

$(f.ii)$ for all $\alpha < \beta$, $f(\beta) \cap D_\alpha \neq \emptyset$.

Otherwise, we let $f(\beta)$ be the empty filter.
We now show that f witnesses $\diamondsuit(\mathbb{B})$: Suppose $\langle D_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{B}$ and $b \in \mathbb{B}$. Let $A \subseteq \omega_1$ code

$$(b, \langle D_\alpha \mid \alpha < \omega_1\rangle).$$

There is a club $C \subseteq \omega_1$ so that for all $\beta \in C$ both

$(C.i)$ $D_\alpha \cap \beta$ is dense in $\mathbb{B} \cap \beta$ for all $\alpha < \beta$ and

$(C.ii)$ $A \cap \beta$ codes $(b, \langle D_\alpha \cap \beta \mid \alpha < \beta\rangle)$.

Since $\vec{a}$ witnesses $\diamondsuit$,

$$S = \{\beta \in C \mid a_\beta = A \cap \beta\}$$

is stationary and by construction of f

$$S \subseteq \{\beta < \omega_1 \mid b \in f(\beta) \wedge \forall\alpha < \beta\ f(\beta) \cap D_\alpha \neq \emptyset\}$$

is stationary as well. $\square$

Corollary 2.16. *Suppose $\mathbb{B} \subseteq \omega_1$ is a forcing. Then $\diamondsuit(\mathbb{B})$ holds in $V^{\mathrm{Add}(\omega_1,1)}$.*

In a number of arguments, we will deal with f-slim $X \prec H_\theta$ that become thicker over time, i.e. at a later stage there will be some f-slim $X \sqsubseteq Y \prec H_\theta$.

Definition 2.17. In the above case of $X \sqsubseteq Y$, we denote the canonical elementary embedding from M_X to M_Y by

$$\mu_{X,Y} \colon M_X \to M_Y.$$

$\mu_{X,Y}$ is given by $\pi_Y^{-1} \circ \pi_X$.

Usually, both X and Y will be f-slim. It is then possible to lift $\mu_{X,Y}$.

Proposition 2.18. *Suppose f guesses $\mathbb{B}$-filters and $X, Y \prec H_\theta$ are both f-slim with $X \sqsubseteq Y$. Then the lift of $\mu_{X,Y}$ to*

$$\mu^+_{X,Y} \colon M_X[f(\delta^X)] \to M_Y[f(\delta^X)]$$

exists.

Proof. As $\delta^X = \delta^Y$, the critical point of $\mu_{X,Y}$ is $>\delta^X$ (if it exists). As $\pi_X^{-1}(\mathbb{B})$ is a forcing of size $\leqslant\omega_1^{M_X} = \delta^X$ and $f(\delta^X)$ is generic over both M_X and M_Y, the lift exists. $\square$

We consider the above proposition simultaneously as a definition: From now on $\mu^+_{X,Y}$ will refer to this lift if it exists.

Definition 2.19. Suppose f witnesses $\diamondsuit(\mathbb{B})$. NS_f is the ideal of f-nonstationary sets, that is

$$\mathrm{NS}_f = \{N \subseteq \omega_1 \mid N \text{ is not } f\text{-stationary}\}.$$

Lemma 2.20. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. NS_f is a normal uniform ideal.*

Proof. Clearly NS_f is an ideal and NS_f contains all bounded subsets of ω_1. To show normality, suppose that $S \in \mathrm{NS}_f^+$ and that

$$r\colon S \to \omega_1$$

is regressive. It is our duty to show that for

$$T_\alpha := r^{-1}(\{\alpha\})$$

for $\alpha < \omega_1$ we have T_α is f-stationary for some $\alpha < \omega_1$. So assume toward a contradiction that there is no such α. Then for any $\alpha < \omega_1$ we can find

(*i*) a club $C_\alpha \subseteq \omega_1$ and

(*ii*) a sequence $\vec{D}^\alpha = \langle D^\alpha_\beta \mid \beta < \omega_1\rangle$ of dense subsets of $\mathbb{B}$

so that

$$\forall\gamma \in C_\alpha \cap T_\alpha \exists\beta < \gamma\ f(\gamma) \cap D^\alpha_\beta = \emptyset.$$

Let $\langle D^*_\xi \mid \xi < \omega_1\rangle$ be an enumeration of

$$\{D^\alpha_\beta \mid \alpha, \beta < \omega_1\}.$$

There must be a club $C^* \subseteq \omega_1$ so that for all $\gamma \in C^*$:

$$\{D^*_\xi \mid \xi < \gamma\} = \{D^\alpha_\beta \mid \alpha, \beta < \gamma\}$$

By f-stationarity of S, we may find some nonzero

$$\gamma \in C^* \cap \left(\triangle_{\alpha<\omega_1} C_\alpha\right) \cap \{\alpha \in S \mid \forall\xi < \alpha\ f(\alpha) \cap D^*_\xi \neq \emptyset\}.$$

Now let $\alpha = r(\gamma) < \gamma$. Then $\gamma \in C_\alpha \cap T_\alpha$ and hence there is $\beta < \gamma$ with $f(\gamma) \cap D^\alpha_\beta = \emptyset$. But as $\gamma \in C^*$, $D^\alpha_\beta = D^*_\xi$ for some $\xi < \gamma$. However,

$$f(\gamma) \cap D^*_\xi \neq \emptyset$$

by choice of γ, contradiction.

□

3 $\diamondsuit$-Forcing

This section deals with a parameterized generalization of the usual classes of forcings associated to countable support style iterations. For a witness f of $\diamondsuit(\mathbb{B})$, we will define and develop a theory for a number of classes of forcings contained on the right hand side of the following diagram. We group these together under the label "$\diamondsuit$-forcing".

Classical	$\diamondsuit$-Forcing

$$
\begin{tikzcd}
\text{complete}(\approx \sigma\text{-closed}) \arrow[rrr, Rightarrow] \arrow[dd, Rightarrow] & & & f\text{-complete} \arrow[dd, Rightarrow] \\
& \text{c.c.c.} \arrow[dl] & f\text{-c.c.c.} \arrow[l, Rightarrow] \arrow[dr] & \\
\text{proper} \arrow[d, Rightarrow] & & & f\text{-proper} \arrow[d, Rightarrow] \\
\text{semiproper} \arrow[d, Rightarrow] & & & f\text{-semiproper} \arrow[d, Rightarrow] \\
\text{stationary set preserving} \arrow[d, Rightarrow] & & & f\text{-stationary set preserving} \arrow[d, Rightarrow] \\
\omega_1\text{-preserving} & & & f\text{-preserving} \arrow[lll, Rightarrow]
\end{tikzcd}
$$

The theory will run parallel to the classical theory of the forcing classes on the left hand side. We note that for some choices of $\mathbb{B}$ and f, no implication between these properties that does not follow from transitivity of implications presented is provable. Although, if f witnesses $\diamondsuit^+(\mathbb{B})$, then all missing horizontal reverse implications hold[19].

Usually, no implication shown here is reversible, though it is a celebrated result of Foreman-Magidor-Shelah [FMS88] that the class of stationary set preserving forcings can coincide with with the class of semiproper forcings, a critical step in the proof of consistency of Martin's Maximum. Similarly, we will see that it is possible that all f-stationary set preserving forcings are f-semiproper. A new phenomenon here is that the class of f-preserving forcings can equal the class of f-stationary set preserving forcings, which is not possible for their classical counterparts. This will happen exactly when η_f is a dense embedding. In particular, NS_{ω_1} is ω_1-dense in this case. In fact, we will construct a model in Section 7 in which all f-preserving forcings are even f-semiproper, but we are getting ahead of ourselves. We start with a closer look at f-complete forcings. Later on, the f-proper forcings make

[19] In fact, if f witnesses $\diamondsuit^+(\mathbb{B})$ and $\mathbb{B}$ is the trivial forcing with one element then we recover the Classical side as a special case of the $\diamondsuit$-forcing side.

up a natural stepping stone along the way to f-semiproper forcings. The f-c.c.c. forcings are not really relevant to our purposes, for completeness reasons we introduce them in Section 11.

3.1 f-complete forcing

We introduce the analog of σ-closed forcing in the context of $\diamondsuit$-forcing. A forcing $\mathbb{P}$ is *complete* if for any large enough regular θ and any countable $X \prec H_\theta$ with $\mathbb{P} \in X$ as well as any $g \subseteq \mathbb{P} \cap X$ which is generic over X there is some $q \in \mathbb{P}$ with $q \Vdash \dot{G} \cap \check{X} = \check{g}$, see [She98, V Definition 1.1][20]. We remark that Jensen [Jen] has shown that a forcing is complete if it is forcing equivalent to a σ-closed forcing.

Definition 3.1. Suppose f witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is f-*complete* if for all sufficiently large regular θ and all f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$ and any g that is $\bar{\mathbb{P}}$-generic over $M_X[f(\delta^X)]$ with $\bar{p} \in g$, there is $q \leqslant p$ with

$$q \Vdash \dot{G} \cap \check{X} = \pi_{\check{X}}[\check{g}].$$

Lemma 3.2. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is f-complete. Then $\mathbb{P}$ preserves f-stationary sets.*

We will prove stronger version of this Lemma later on, so we will not give a proof here.

The following results can be proven almost exactly as their analogs for complete forcings.

Lemma 3.3. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Every σ-closed forcing is f-complete.*

Lemma 3.4. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is f-complete. Then $\mathbb{P}$ does not add any countable sequences of ordinals.*

Lemma 3.5. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a countable support iteration so that for any $\alpha < \gamma$,*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is } f\textit{-complete”}.$$

Then $\mathbb{P}$ is f-complete.

We will now get to know the forcing that tries to turn a witness of $\diamondsuit(\mathbb{B})$ into a witness of $\diamondsuit^+(\mathbb{B})$. Woodin has defined this forcing in the case of $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$ in [Woo10, Section 6.2].

Definition 3.6 (Woodin)**.** Suppose f witnesses $\diamondsuit(\mathbb{B})$. $\mathbb{P}(f)$ consists of conditions (h, c) so that, for some $\alpha := \alpha^p < \omega_1$,

[20] A forcing $\mathbb{P}$ is complete in our sense if it is $\mathcal{E}$-complete for $\mathcal{E} = \{\mathcal{S}_{\aleph_0}(|\mathbb{P}|)\}$ in the sense of [She98].

(*i*) $h\colon \alpha \to \mathcal{P}(\mathbb{B})$ is a function

(*ii*) $h(\beta)$ is dense in $\mathbb{B}$ for all $\beta < \alpha$,

(*iii*) $c \subseteq \omega_1$ is closed with $\max(c) = \alpha$ and

(*iv*) for all $\beta \in c$ and $\gamma < \beta$, $f(\beta) \cap h(\gamma) \neq \varnothing$.

The order on $\mathbb{P}(f)$ is defined by

$$(h, c) \leqslant (k, d)$$

iff $h \upharpoonright \beta = k$ and $c \cap \beta + 1 = d$ where $\beta = \mathrm{dom}(k)$.
If $p \in \mathbb{P}(f)$ then we sometimes write h^p, c^p for the first and second coordinate of p, i.e. $p = (h^p, c^p)$.

The typical density arguments yield:

Proposition 3.7. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Then for any dense $D \subseteq \mathbb{B}$ there is a club $C \in V^{\mathbb{P}(f)}$ with*

$$f(\alpha) \cap D \neq \varnothing$$

for all $\alpha \in C$.

Thus to get $\diamondsuit^+(\mathbb{B})$ one simply iterates $\mathbb{P}(f)$ for a witness f of $\diamondsuit(\mathbb{B})$. For this to work, this iteration must not collapse ω_1.

Lemma 3.8. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. $\mathbb{P}(f)$ is f-complete*

Proof. Suppose θ is a sufficiently large regular cardinal and $X \prec H_\theta$ is f-slim with $\mathbb{P}(f) \in X$. Let $\bar{g}$ be generic over $M_X[f(\delta^X)]$. We can find a condition $q \in \mathbb{P}(f)$ with

$$q \leqslant \pi_X(p)$$

for all $p \in g$ as follows: For $\beta < \delta^X$ let $h(\beta) = \pi_X(h^r(\beta))$ for any/all $r \in \bar{g}$ with $\beta \in \mathrm{dom}(h^r)$ and

$$c = \bigcup_{r \in \bar{g}} c^r \cup \{\delta^X\}.$$

Claim 3.9. *$q := (h, c) \in \mathbb{P}(f)$.*

Proof. Let $\beta < \delta^X$. We have to show that $f(\delta^X) \cap h(\beta) \neq \varnothing$. But $h(\beta) \in X$ is a dense subset of $\mathbb{B}$ and hence

$$f(\delta^X) \cap h(\beta) = f(\delta^X) \cap \pi_X^{-1}(h(\beta)) \neq \varnothing$$

because $f(\delta^X)$ is generic over M_X. □

This concludes the proof. □

Lemma 3.10. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and κ is inaccessible. Let $\mathbb{P}$ be the countable support iteration of $\mathbb{P}(f)$ (as computed in the relevant extension) of length κ. Then f witnesses $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}}$.*

This result is essentially due to Woodin, see Lemma 6.44 in [Woo10]. We will later show that one can force $\diamondsuit^+(\mathbb{B})$ without the use of an inaccessible cardinal, but for now the Lemma above suits our purposes.

Proof. $\mathbb{P}$ is f-complete and preserves "f witnesses $\diamondsuit(\mathbb{B})$" along the iteration by Lemma 3.8, 3.5 and 3.2. The point is that as κ is inaccessible and as $\mathbb{P}$ is a countable support iteration with $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$, $\mathbb{P}$ satisfies the κ-chain condition. Thus any dense subset D of $\mathbb{B}$ in $V^{\mathbb{P}}$ is already in $V^{\mathbb{P}_\alpha}$ for some $\alpha < \kappa$ and hence for a club $C \in V^{\mathbb{P}_{\alpha+1}}$ of $\beta < \omega_1$ we have

$$f(\alpha) \cap D \neq \varnothing$$

by Proposition 3.7. □

As a warm-up for the later sections, we will make the nonstationary ideal on ω_1 presaturated and force $\diamondsuit^+(\mathbb{B})$ simultaneously by f-complete forcing.

Definition 3.11. The (cofinal) Strong Chang Conjecture ($\mathrm{SCC}_{\mathrm{cof}}$) holds true iff for any sufficiently large regular θ and for any countable $X \prec H_\theta$ there are cofinally many $\alpha < \omega_2$ for which there is $X \sqsubseteq Y \prec H_\theta$ with $\alpha \in Y$.

Fact 3.12 (Shelah, [She98, XIII Theorem 1.9])**.** *Suppose κ is measurable as witnessed by a normal measure U and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$ is a* RCS*-iteration so that*

($\mathbb{P}.i$) *$\mathbb{P}$ is semiproper,*

($\mathbb{P}.ii$) *for all $\alpha < \kappa$, $|\mathbb{P}_\alpha| < \kappa$,*

($\mathbb{P}.iii$) *for all $\alpha < \kappa$, $\Vdash_{\mathbb{P}_{\alpha+1}} |(2^{\omega_1})^{V^{\mathbb{P}_\alpha}}| = \aleph_1$ and*

($\mathbb{P}.iv$) *$\{\alpha < \kappa \mid \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is semiproper"$\} \in U$.*

Then the $\mathrm{SCC}_{\mathrm{cof}}$ holds in $V^{\mathbb{P}}$.

Theorem 3.13. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and δ is Woodin. There is then an extension by a f-complete forcing satisfying*

(i) *$\delta = \omega_2$,*

(ii) *f witnesses $\diamondsuit^+(\mathbb{B})$ and*

(iii) *NS_{ω_1} is presaturated.*

In general, simply forcing with $\mathrm{Col}(\omega_1, {<}\delta)$ does not work. That extension will always satisfy (i) and (iii), but (ii) may fail.

Proof. (*Sketch*) Let $\mathbb{P}$ be the countable support iteration of length δ that forces with $\mathrm{Col}(\omega_1, 2^{\omega_1})$ at inaccessible stages and with $\mathbb{P}(f)$ at all other stages. Note that $\mathbb{P}$ is f-complete by Lemma 3.8, Lemma 3.3 and Theorem 3.20. Now we have that

($\kappa.i$) $\mathbb{P}_\kappa$ is κ-c.c.,

($\kappa.ii$) f witnesses $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}_\kappa}$,

($\kappa.iii$) in $V^{\mathbb{P}_\kappa}$, the remaining part $\mathbb{P}_{\kappa,\delta}$ of the iteration $\mathbb{P}$ is complete, in particular semiproper and

($\kappa.iv$) if κ is measurable in V then $\mathrm{SCC}_{\mathrm{cof}}$ holds in $V^{\mathbb{P}_\kappa}$.

for all inaccessible $\kappa \leqslant \delta$. ($\kappa.i$) is true by a general result of Shelah [She98, XI 1.13] about revised countable support iterations[21]. ($\kappa.ii$) follows from ($\kappa.i$) as $\mathbb{P}(f)$ was used unboundedly often below κ. ($\kappa.iii$) follows from ($\kappa.ii$) and Lemma 3.2. ($\kappa.iv$) is a consequence of Fact 3.12 as by ($\kappa.iii$), the extension $V^{\mathbb{P}_\kappa}$ can be realized as required.
The proof that NS_{ω_1} is presaturated in $V^{\mathbb{P}}$ can now be carried out similarly as the one given in, e.g. [Sch11] which shows that NS_{ω_1} is presaturated in $V^{\mathrm{Col}(\omega_1,<\delta)}$. □

3.2 f-proper forcing

Suppose that f witnesses $\diamondsuit(\mathbb{B})$. We present the class of f-proper forcings, the $\diamondsuit$-forcing analog of Shelah's class of proper forcings. The main result here is that that f-proper forcings can be iterated by countable support iterations. This subsection is not necessary in the pursuit of answering Woodin's question, nonetheless it serves as a natural warm-up for dealing with f-semiproper forcing later, which we will make much more use of in practice.

Definition 3.14. Suppose $\mathbb{P}$ is a forcing and f witnesses $\diamondsuit(\mathbb{B})$.

(i) Let θ be a regular cardinal $> 2^{|\mathbb{P}|}$ and $X \prec H_\theta$ is f-slim. A condition $q \in \mathbb{P}$ is $(X, \mathbb{P}, f)$-*generic* if q is $(X, \mathbb{P})$-generic[22] and

$$q \Vdash \text{“}\check{X}[\dot{G}] \text{ is } f\text{-slim”}.$$

(ii) $\mathbb{P}$ is f-*proper* if for all sufficiently large regular θ and all f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$, for any $p \in \mathbb{P} \cap X$ there is $q \leqslant p$ that is $(X, \mathbb{P}, f)$-generic.

[21] The same result holds for countable support iterations as well by the same argument.
[22] q is $(X, \mathbb{P})$-generic if $q \Vdash$ “$\dot{G} \cap \check{X}$ is $\check{\mathbb{P}} \cap \check{X}$-generic over $\check{X}$”.

Being f-proper is neither stronger nor weaker than being proper, requiring the existence of $(X, \mathbb{P}, f)$-generic conditions is more demanding than only $(X, \mathbb{P})$-generic conditions, but we only do so on the potentially smaller class of f-slim substructures.
We remark that the product lemma for forcing gives an alternative equivalent formulation of $(X, \mathbb{P}, f)$-generic conditions.

Proposition 3.15. *Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is a forcing and $X \prec H_\theta$ is f-slim with $\mathbb{P} \in X$. The following are equivalent for $q \in \mathbb{P}$:*

(i) q is $(X, \mathbb{P}, f)$-generic.

(ii) $q \Vdash$ "$\pi_{\check{X}}^{-1}[\dot{G}]$ is generic over $M_{\check{X}}[\check{f}(\delta^{\check{X}})]$"

This observation forms the backbone of the iteration theorems we are about to prove.

Proposition 3.16. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Then any f-complete forcing is f-proper.*

Proof. Let $\mathbb{P}$ be f-complete. Let θ be sufficiently large and regular, $X \prec H_\theta$ countable and f-slim with $\mathbb{P} \in X$. Let $p \in \mathbb{P} \cap X$. Now find $\bar{g}$ with $\bar{p} \in \bar{g}$ that hits all dense subsets of $\bar{\mathbb{P}}$ in $M_X[f(\delta^X)]$. By f-completeness, there is $q \leqslant p$ so that

$$q \Vdash \dot{G} \cap \check{X} = \pi_{\check{X}}[\check{\bar{g}}].$$

By Proposition 3.15, q is $(X, \mathbb{P}, f)$-generic. □

Lemma 3.17. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is f-proper. Then $\mathbb{P}$ preserves f-stationary sets*[23].

Proof. Assume that S is f-stationary. Let $\dot{C}$ be a name for a club in ω_1 and $\langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle$ a sequence of $\mathbb{P}$-names for dense subsets of $\mathbb{B}$. Assume toward a contradiction that

$$p \Vdash \text{“}\forall \alpha \in \dot{C} \cap \check{S} \exists \beta < \alpha \ \check{f}(\alpha) \cap \dot{D}_\beta = \varnothing\text{”}. \tag{$\sharp$}$$

Let θ be sufficiently large and regular and $X \prec H_\theta$ with

$(X.i)$ X is f-slim,

$(X.ii)$ $\mathbb{P}, p, \dot{C}, \langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle \in X$ and

$(X.iii)$ $\delta^X \in S$.

X exists as S is f-stationary. Since $\mathbb{P}$ is f-proper, there is some $(X, \mathbb{P}, f)$-generic $q \leqslant p$. Let G be $\mathbb{P}$-generic with $q \in G$. Let $C = \dot{C}^G, D_\alpha = \dot{D}_\alpha^G$ for $\alpha < \omega_1$ and $\bar{G} = \pi_X^{-1}[G]$. We have

[23] In particular, f is still a witness of $\diamondsuit(\mathbb{B})$ in $V^{\mathbb{P}}$ by Corollary 2.9.

(i) $\delta^X \in C \cap S$,

(ii) $\langle D_\alpha \cap \delta^X \mid \alpha < \delta^X \rangle \in M_X[\bar{G}]$ is a sequence of dense subsets of $\mathbb{B} \cap \delta^X$ and

(iii) $f(\delta^X)$ is generic over $M_X[\bar{G}]$.

By (ii) and (iii),

$$\forall \beta < \delta^X \ f(\delta^X) \cap D_\beta \neq \emptyset.$$

But this together with (i) contradicts ($\sharp$)! $\square$

Remark 3.18. In particular, if f witnesses $\diamondsuit(\mathbb{B})$ then f-proper forcings preserve ω_1. We stress that if f witnesses $\diamondsuit(\mathbb{B})$ but not $\diamondsuit^+(\mathbb{B})$, there are f-proper forcings which are not stationary set preserving, e.g. $\mathbb{P}(f)$.

Putting together Lemma 3.3 and Lemma 3.17, we see that every σ-closed poset preserves the statement "f witnesses $\diamondsuit(\mathbb{B})$". In general though, depending on the choice of $\mathbb{B}$, the same is not true when we replace $\diamondsuit(\mathbb{B})$ by $\diamondsuit^+(\mathbb{B})$. We will see this later when we consider the case $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$.

We now go on to show that f-proper forcings can be iterated by countable support iterations. We first note that the following can be shown exactly as for proper forcing.

Proposition 3.19. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and G is generic for some f-proper forcing $\mathbb{P}$. If $X \in V[G]$ is a countable set of ordinals then there is a countable set $Y \in V$ with $X \subseteq Y$.*

This is important as it implies that the tail of a f-proper countable support iteration is (essentially) a countable support iteration from the point of view of the intermediate extension.

Theorem 3.20. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. If $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$ is a countable support iteration of f-proper forcings then $\mathbb{P}$ is f-proper.*

Proposition 3.21. *Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is f-proper and*

$$\Vdash_{\mathbb{P}} \text{“}\dot{\mathbb{Q}} \textit{ is } f\textit{-proper”}.$$

Let θ be sufficiently large and regular, $X \prec H_\theta$ f-slim with $\mathbb{P} * \dot{\mathbb{Q}} \in X$. If $(p, \dot{r}) \in \mathbb{P} * \dot{\mathbb{Q}}$ so that

(i) p is $(X, \mathbb{P}, f)$-generic and

(ii) $p \Vdash \dot{r} \in \check{X}[\dot{G}]$

then there is $\dot{q}$ such that $(p, \dot{q})$ is $(X, \mathbb{P} * \dot{\mathbb{Q}}, f)$-generic and $(p, \dot{q}) \leqslant (p, \dot{r})$. In particular, $\mathbb{P} * \dot{\mathbb{Q}}$ is f-proper.

Proof. Assume G is $\mathbb{P}$-generic with $p \in G$. Then $X[G]$ is f-slim. As $\dot{\mathbb{Q}}^G$ is f-proper, there is $s \leqslant \dot{r}^G \in \dot{\mathbb{Q}}^G$ that is $(X[G], \dot{Q}^G, f)$-generic. Thus there is a name $\dot{q}$ with

$$p \Vdash \text{“}\dot{q} \leqslant \dot{r} \wedge \dot{q} \text{ is } (\check{X}[\dot{G}], \dot{\mathbb{Q}}, \check{f})\text{-generic”}.$$

It is now easy to see that $(p, \dot{q})$ is $(X, \mathbb{P} * \dot{\mathbb{Q}}, f)$-generic and $(p, \dot{q}) \leqslant (p, \dot{r})$. $\square$

Lemma 3.22. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}_\beta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\xi \mid \alpha \leqslant \beta, \xi < \beta \rangle$ is a countable support iteration of f-proper forcings. Let $\alpha \leqslant \beta$ and $X \prec H_\theta$ be f-slim with $\mathbb{P}_\beta, \alpha \in X$. Suppose $q_0 \in \mathbb{P}_\alpha$ is $(X, \mathbb{P}_\alpha, f)$-generic and $\dot{p} \in V^{\mathbb{P}_\alpha}$ so that*

$$q_0 \Vdash_{\mathbb{P}_\alpha} \dot{p} \in (\check{\mathbb{P}}_\beta \cap \check{X}) \wedge \dot{p} \upharpoonright \alpha \in \dot{G}_\alpha.$$

Then there is a $(X, \mathbb{P}_\beta, f)$-generic condition $q \in \mathbb{P}_\beta$ with $q \upharpoonright \alpha = q_0$ and $q \Vdash_{\mathbb{P}_\beta} \dot{p} \in \dot{G}_\beta$.

Proof. We prove the Lemma by induction on β. The successor case follows from Proposition 3.21, so we may assume that β is a limit. Let

$$\langle \alpha_n \mid n < \omega \rangle$$

be an increasing cofinal sequence in $M \cap \beta$ with $\alpha_0 = \alpha$. Furthermore, let $\{\bar{D}_n \mid n < \omega\}$ be an enumeration of all dense subsets of $\pi_X^{-1}(\mathbb{P}_\beta)$ in $M_X[f(\delta^X)]$ and put

$$D_n = \pi_X[\bar{D}_n].$$

Note that D_n is not necessarily dense in $\mathbb{P}_\beta$, but it *is* dense in $\mathbb{P}_\beta \cap X$. We will construct sequences $\langle q_n \mid n < \omega \rangle$ and $\langle \dot{p}_n \mid n < \omega \rangle$ so that the following hold for all $n < \omega$:

$(A.i)$ $q_0 = q$ and $\dot{p}_0 = \dot{p}$.

$(A.ii)$ q_n is $(X, \mathbb{P}_{\alpha_n}, f)$-generic.

$(A.iii)$ $q_{n+1} \upharpoonright \alpha_n = q_n$.

$(A.iv)$ $\dot{p}_{n+1} \in V^{\mathbb{P}_{\alpha_n}}$.

$(A.v)$ q_n forces

- (a) $\dot{p}_{n+1} \in (\check{\mathbb{P}}_\beta \cap \check{X})$,
- (b) $\dot{p}_{n+1} \leqslant \dot{p}_n$,
- (c) $\dot{p}_{n+1} \in \check{D}_n$ and
- (d) $\dot{p}_{n+1} \upharpoonright \alpha_n \in \dot{G}_{\alpha_n}$.

q_0 and $\dot{p}_0$ are given by $(A.i)$. Suppose q_n and $\dot{p}_n$ have been constructed already. Let G be generic for $\mathbb{P}_{\alpha_n}$ with $q_n \in G$ and put $p_n = \dot{p}_n^G$. Then

- $p_n \in \mathbb{P}_\beta \cap X$ and
- $p_n \restriction \alpha_n \in G$.

As $q_n \in G$, $\bar{G} = \pi_X^{-1}[G]$ is generic over $M_X[f(\delta^X)]$ and thus there is $\bar{p}_{n+1} \in \bar{D}_n$ with

$$\bar{p}_{n+1} \restriction \pi_X^{-1}(\alpha_n) \in \bar{G} \text{ and } \bar{p}_{n+1} \leqslant \pi_X^{-1}(p_n).$$

Hence, letting $p_{n+1} = \pi_X(\bar{p}_{n+1})$,

$(p_{n+1}.i)$ $p_{n+1} \in \mathbb{P}_\beta \cap X$,

$(p_{n+1}.ii)$ $p_{n+1} \leqslant p_n$,

$(p_{n+1}.iii)$ $p_{n+1} \in D_n$ and

$(p_{n+1}.iv)$ $p_{n+1} \restriction \alpha_n \in G$.

By fullness, we can find a $\mathbb{P}_{\alpha_{n+1}}$-name $\dot{p}_{n+1}$ which is forced by q_n to satisfy properties $(p_{n+1}.i)$-$(p_{n+1}.iv)$. By induction, we can now find $q_{n+1} \in \mathbb{P}_{\alpha_{n+1}}$ that is $(X, \mathbb{P}_{\alpha_{n+1}}, f)$-generic with $q_{n+1} \restriction \alpha_n = q_n$ and

$$q_{n+1} \Vdash \dot{p}_{n+1} \restriction \check{\alpha}_{n+1} \in \dot{G}_{\alpha_{n+1}}.$$

This completes the construction.
Finally, let $q \in \mathbb{P}_\beta$ with $q \restriction \alpha_n = q_n$ for all $n < \omega$.

Claim 3.23. *q is $(X, \mathbb{P}_\beta, f)$-generic.*

Proof. Suppose $\bar{D} \in M_X[f(\delta^X)]$ is a dense subset of $\pi_X^{-1}(\mathbb{P}_\beta)$. Then $\bar{D} = \bar{D}_n$ for some $n < \omega$. Let G be $\mathbb{P}_\beta$-generic with $q \in G$. Note that $p_{n+1} = \dot{p}_{n+1}^G \in G$: For all $n < m$ we have $\dot{p}_m^G \leqslant p_n$ and $\dot{p}_m^G \restriction \alpha_m \in G_{\alpha_m}$ by $(A.v)$(d), hence

$$p_{n+1} \restriction \alpha_m \in G_{\alpha_m}$$

Thus

$$p_{n+1} \restriction \sup(\beta \cap X) \in G_{\sup(\beta \cap X)}$$

Also $\operatorname{supp}(p_{n+1}) \subseteq X$ as $p_{n+1} \in X$ and $\operatorname{supp}(p_{n+1})$ is countable. Moreover, $p_{n+1} \in D_n$ by $(A.v)$(c), so that $G \cap D_n \neq \varnothing$. It follows that $\pi_X^{-1}[G] \cap \bar{D} \neq \varnothing$ and hence $\pi_X^{-1}[G]$ is generic over $M_X[f(\delta^X)]$. □

From the argument above it follows that $q \Vdash \dot{p}_n \in \dot{G}$ for all $n < \omega$, in particular for $n = 0$. Thus q is as required. □

Theorem 3.20 follows immediately from Lemma 3.22.

The next lemma can be proven exactly as for proper forcings, see Theorem 2.12 in [Abr10].

Lemma 3.24. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$$

is a countable support iteration of f-proper forcings of length $\gamma \leqslant \omega_2$. Assume CH *holds and $\Vdash_{\mathbb{P}_\alpha} |\dot{\mathbb{Q}}_\alpha| = \omega_1$ for any $\alpha < \gamma$. Then*

(*i*) *$\mathbb{P}$ is ω_2-c.c. and*

(*ii*) CH *holds in $V^{\mathbb{P}_\alpha}$ for $\alpha < \gamma$.*

Lemma 3.25. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Then there is a f-proper (even f-complete) forcing $\mathbb{P}$ so that f witnesses $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}}$.*

Proof. For a dense $D \subseteq \mathbb{B}$ consider the variant $\mathbb{P}_D(f)$ of $\mathbb{P}(f)$ that only shoots a club through the stationary set

$$S_D := \{\alpha < \omega_1 \mid f(\alpha) \cap D \neq \varnothing\}.$$

That is, conditions in $\mathbb{P}_D(f)$ are countable closed sets $c \subseteq S_D$ (with a maximum) and are ordered by end-extension. If CH holds then $\mathbb{P}_D(f)$ is a f-complete forcing of size ω_1. We may assume that CH and $2^{\omega_1} = \omega_2$ holds in V, otherwise we can first force it with σ-closed forcing. We define

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \omega_2, \beta < \omega_2 \rangle$$

as a countable support iteration so that

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \mathbb{P}_{\dot{D}_\alpha}(\check{f})$$

for all $\alpha < \omega_2$, where $\dot{D}_\alpha$ is some $\mathbb{P}_\alpha$-name for a dense subset of $\mathbb{B}$ given by some bookkeeping so that any such dense set in any $V^{\mathbb{P}_\beta}$, $\beta < \omega_2$, is considered at some point. Note that by Lemma 3.24 we have

(*i*) $\mathbb{P}_{\omega_2}$ is ω_2-c.c.,

(*ii*) $V^{\mathbb{P}_\alpha} \models 2^{\omega_1} = \omega_2$ and

(*iii*) $\mathcal{P}(\omega_1)^{V^{\mathbb{P}}} = \bigcup_{\alpha<\omega_2} \mathcal{P}(\omega_1)^{V^{\mathbb{P}_\alpha}}$.

By (*ii*), the proposed bookkeeping exists and by (*iii*), it follows that f witnesses $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}}$. □

While all c.c.c. forcings are proper, it turns out that many prominent c.c.c. forcings are not f-proper in general, for example random forcing, Hechler forcing, instances of almost disjoint coding forcing, sometimes there are Suslin trees which are not f-proper considered as forcings and also Baumgartner's specializing forcing at some Aronszajn tree might not be f-proper. However, Cohen forcing is always f-proper, simply because Cohen forcing is absolute to any transitive model of sufficiently much of ZFC. We will show in Subsection 11.1 that if $\diamondsuit(\mathbb{B})$ holds then there is always an extension in which $\diamondsuit^+(\mathbb{B})$ holds and the continuum is arbitrarily large (together with a reasonable fragment of Martin's axiom).

Recall that proper forcings are exactly those forcings which preserve stationary subsets of $[A]^\omega$ for all uncountable sets A. Next, we will provide an analogous characterization of f-properness.

Definition 3.26. Suppose A is uncountable and f witnesses $\diamondsuit(\mathbb{B})$. A set $\mathcal{S} \subseteq [A]^\omega$ is f-stationary if for any club $\mathcal{C} \subseteq [A]^\omega$ there is some regular $\theta \geqslant \omega_2$ with $A \in H_\theta$ and some f-slim $X \prec H_\theta$ with $X \cap A \in \mathcal{S} \cap \mathcal{C}$.

Lemma 3.27. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. The following are equivalent for any forcing $\mathbb{P}$:*

($\mathbb{P}.i$) *$\mathbb{P}$ is f-proper.*

($\mathbb{P}.ii$) *$\mathbb{P}$ preserves f-stationary subsets of $[A]^\omega$ for all uncountable sets A.*

Proof. ($\mathbb{P}.i$)$\Rightarrow$($\mathbb{P}.ii$): Let A be uncountable and $\mathcal{S} \subseteq [A]^\omega$ be f-stationary. We have to show that $\mathcal{S}$ is f-stationary in $V^{\mathbb{P}}$. Let $\dot{\mathcal{C}}$ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ with

$$p \Vdash \text{“}\dot{\mathcal{C}} \text{ is club in } [\check{A}]^\omega\text{”}.$$

Let θ be sufficiently large and regular so that $A \in H_\theta$. As $\mathcal{S}$ is f-stationary, we can find some f-slim $X \prec H_\theta$ so that

($X.i$) $\dot{\mathcal{C}} \in X$ and

($X.ii$) $X \cap A \in \mathcal{S}$.

As $\mathbb{P}$ is f-proper, there is $q \leqslant p$ that is $(X, \mathbb{P}, f)$-generic. Let $G \subseteq \mathbb{P}$ be generic with $q \in G$. Then $X[G] \prec H_\theta^{V[G]}$ is f-slim and

$$X[G] \cap A = X \cap A \in \mathcal{S}.$$

Moreover, $\mathcal{C} := \dot{\mathcal{C}}^G \in X[G]$ and it follows that $X[G] \cap A \in \mathcal{C}$.
($\mathbb{P}.ii$)$\Rightarrow$($\mathbb{P}.i$): Let θ be sufficiently large and regular. Suppose towards a contradiction that $\mathbb{P}$ is not f-proper. Then the set

$$\mathcal{S} := \{X \prec H_\theta \mid X \text{ is } f\text{-slim} \wedge \exists p \in X \neg\exists q \leqslant p\ q \text{ is } (X, \mathbb{P}, f)\text{-generic}\}$$

is f-stationary in $[H_\theta]^\omega$. In fact, there must be some $p \in \mathbb{P}$ so that

$$\mathcal{S}_p := \{X \prec H_\theta \mid X \text{ is } f\text{-slim} \wedge p \in X \wedge \neg\exists q \leqslant p\ q \text{ is } (X, \mathbb{P}, f)\text{-generic}\}$$

is not f-stationary. We leave this detail to the reader. Let $G \subseteq \mathbb{P}$ be generic and note that in $V[G]$, $\mathcal{S}_p$ is still a f-stationary subset of $[H_\theta^V]^\omega$ by ($\mathbb{P}.ii$). It follows that we can find some f-slim $X \prec H_\theta^{V[G]}$ with $Y := X \cap H_\theta^V \in \mathcal{S}_p$. Hence $Y[G] \cap V = Y$ and $Y[G]$ is f-slim. But then there must be some $q \leqslant p$, $q \in G$ that is $(Y, \mathbb{P}, f)$-generic, contradiction. $\square$

We may now define the appropriate forcing axiom corresponding to our iteration theorem.

Definition 3.28. Suppose f witnesses $\diamondsuit(\mathbb{B})$. $\mathrm{PFA}(f)$ holds iff for any $\mathbb{P}$ that is f-proper and any sequence $\langle D_\alpha \mid \alpha < \omega_1\rangle$ of dense subsets of $\mathbb{P}$, there is a $\mathbb{P}$-filter G with $D_\alpha \cap G \neq \varnothing$ for all $\alpha < \omega_1$.

Theorem 3.29. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and κ is supercompact. Then there is a f-proper generic extension in which* $\mathrm{PFA}(f)$ *holds.*

Proof. Essentially the same argument as to force PFA from a supercompact works using Theorem 3.20 instead of the iteration theorem for proper forcings. $\square$

Theorem 3.30. *Under* $\mathrm{PFA}(f)$ *the following holds:*

(*i*) $\neg$CH.

(*ii*) *f witnesses $\diamondsuit^+(\mathbb{B})$.*

This is, admittedly, not a very impressive list of consequences of $\mathrm{PFA}(f)$. Indeed, there are some issues generalizing the known proofs of the interesting structural consequences of PFA. Dealing with this now would lead us too far astray, but we will come back to this issue later in Subsection 11.2.

3.3 f-semiproper forcing

Next up, we introduce f-semiproper forcings. Roughly speaking we have

$$\frac{f\text{-semiproper}}{\text{semiproper}} = \frac{f\text{-proper}}{\text{proper}}.$$

A decent amount of arguments carry over from the section before, and we will not present them again. The main result of this section will be that f-semiproperness can be preserved along certain countable support-style iterations.

Definition 3.31. Suppose $\mathbb{P}$ is a forcing and f witnesses $\diamondsuit(\mathbb{B})$.

(*i*) Let θ be a sufficiently large regular cardinal and $X \prec H_\theta$ f-slim with $\mathbb{P} \in X$. A condition $q \in \mathbb{P}$ is $(X, \mathbb{P}, f)$-*semigeneric* if q is $(X, \mathbb{P})$-semigeneric[24] and
$$q \Vdash \text{“}\check{X}[\dot{G}] \text{ is } f\text{-slim”}.$$

(*ii*) $\mathbb{P}$ is *f-semiproper* if for any sufficiently large regular θ and any f-slim $X \prec H_\theta$ with $\mathbb{P} \in X$ as well as all $p \in \mathbb{P} \cap X$ there is $q \leqslant p$ that is $(X, \mathbb{P}, f)$-semigeneric.

[24] Recall that q is $(X, \mathbb{P})$-semigeneric if $q \Vdash \check{X} \sqsubseteq \check{X}[\dot{G}]$.

Trivially, we have:

Proposition 3.32. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Any f-proper forcing is f-semiproper.*

Once again, similar to $(X, \mathbb{P}, f)$-genericity, there is a reformulation of $(X, \mathbb{P}, f)$-semigenericity, this time slightly more complicated.

Proposition 3.33. *Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is a forcing and $X \prec H_\theta$ is f-slim with $\mathbb{P} \in X$. The following are equivalent for $q \in \mathbb{P}$:*

(i) *q is $(X, \mathbb{P}, f)$-semigeneric.*

(ii) *q is $(X, \mathbb{P})$-semigeneric,*

$$q \Vdash \text{“}\check{X}[\dot{G}] \cap V \textit{ is } f\textit{-slim”}$$

and

$$q \Vdash \text{“}\pi_{\dot{Y}}^{-1}[\dot{G}] \textit{ is generic over } M_{\dot{Y}}[\check{f}(\delta^{\check{X}})]\text{”}$$

where $\dot{Y}$ is a name for $\check{X}[\dot{G}] \cap V$.

In the upcoming iteration theorem, it is *(ii)* that we will try to achieve to establish f-semiproperness of the iteration.

Lemma 3.34. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is f-semiproper. Then $\mathbb{P}$ preserves f-stationary sets. In particular, $\mathbb{P}$ preserves f.*

Proof. An argument essentially identical to the proof of Lemma 3.17 works. □

Proposition 3.35. *Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is f-semiproper and*

$$\Vdash_{\mathbb{P}} \text{“}\dot{\mathbb{Q}} \textit{ is } f\textit{-semiproper”}$$

Let θ be sufficiently large and regular, $X \prec H_\theta$ f-slim with $\mathbb{P} * \dot{\mathbb{Q}} \in X$. If $(p, \dot{r}) \in \mathbb{P} * \dot{\mathbb{Q}}$ so that

(i) *p is $(X, \mathbb{P}, f)$-semigeneric and*

(ii) *$p \Vdash \dot{r} \in \check{X}[\dot{G}]$*

then there is $\dot{q}$ such that $(p, \dot{q})$ is $(X, \mathbb{P} * \dot{\mathbb{Q}}, f)$-semigeneric and $(p, \dot{q}) \leqslant (p, \dot{r})$. In particular, $\mathbb{P} * \dot{\mathbb{Q}}$ is f-semiproper.

Proof. Same as Proposition 3.21. □

For technical reasons, we will not prove that f-semiproperness is preserved along RCS-iterations. Instead we will use a different type of support. For convenience, we include a proof that f-semiproperness is preserved along full support iterations of length ω. The main ideas on how to arrange preservation of f-semiproperness are present in that argument and the reader may simply choose to believe the more general iteration theorem.

Lemma 3.36. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ is a full support iteration of f-semiproper forcings. Then $\mathbb{P}$ is f-semiproper.*

Proof. Let θ be sufficiently large, regular and $X \prec H_\theta$ be f-slim with $\mathbb{P} \in X$. Let us write $g := f(\delta^X)$ and let $p \in \mathbb{P} \cap X$. For $i < \omega$, let $\dot{X}_i$ be a $\mathbb{P}_i$-name for[25]

$$\check{X}[\dot{G}_i] \cap V$$

and let

$$\langle \dot{D}_{i,j} \mid j < \omega \rangle$$

be a list of $\mathbb{P}_i$-names for all dense subsets of

$$\pi^{-1}_{\dot{X}_i}(\check{\mathbb{P}}) \text{ in } M_{\dot{X}_i}[\check{g}].$$

Fix a surjection

$$h \colon \omega \to \omega \times \omega$$

with $i \leqslant n$ whenever $h(n) = (i, j)$. We write

$$\dot{D}_n := \dot{D}_{i,j}$$

whenever $(i, j) = h(n)$. Note that we can consider $\dot{D}_n$ as a $\mathbb{P}_n$-name.
We will define sequences $\langle q_n \mid n < \omega \rangle$ and $\langle \dot{p}_n \mid n < \omega \rangle$ with the following properties for all $n < \omega$:

$(A.i)$ q_n is $(X, \mathbb{P}_n, f)$-semigeneric.

$(A.ii)$ $q_{n+1} \upharpoonright n = q_n$.

$(A.iii)$ $\dot{p}_0$ is the $\mathbb{P}_0$-check-name for p.

$(A.iv)$ $\dot{p}_{n+1} \in V^{\mathbb{P}_n}$.

$(A.v)$ q_n forces

- (a) $\dot{p}_{n+1} \in (\check{\mathbb{P}} \cap \check{X}[\dot{G}_n])$,
- (b) $\dot{p}_{n+1} \leqslant \dot{p}_n$,
- (c) $\dot{p}_{n+1} \in \pi_{\dot{X}_n}[\mu^+_{i,n}(\dot{D}_n)]$ where $h(n) = (i, j)$ for some j and
- (d) $\dot{p}_{n+1} \upharpoonright n \in \dot{G}_n$.

Here, $\mu^+_{i,j}$ denotes $\mu^+_{\dot{X}_i, \dot{X}_j}$ for $i \leqslant j < \omega$. Recall this notation from Proposition 2.18. Also note that we can consider $\dot{p}_n$ as a $\mathbb{P}_n$-name.
q_0 is the unique condition in $\mathbb{P}_0$. Suppose q_n and $\dot{p}_n$ have been constructed. Let G be generic for $\mathbb{P}_n$ with $q_n \in G$ and let $p_n = \dot{p}_n^G$. Then by $(A.i)$ and $(A.v)$:

[25] $\dot{G}_i$ is the canonical $\mathbb{P}_i$-name for the generic filter.

- $\delta^{X[G]} = \delta^X$,
- $p_n \in \mathbb{P} \cap X[G]$ and
- $p_n \upharpoonright n \in G$.

As $q_n \in G$, g is generic over $M_{X[G]}$. For $i \leqslant n$ let

- $X_i := X[G_i] \cap V = \dot{X}_i^{G_i}$ as well as
- $M_i := M_{X_i}$.

Let $h(n) = (i, j)$ and $D_{i,j} = (\dot{D}_{i,j})^G$. By elementarity,

$$M_n[g] \models \mu^+_{i,n}(D_{i,j}) \text{ is dense in } \pi^{-1}_{X_n}(\mathbb{P})$$

and thus indeed

$$\pi_{X_n}[\mu^+_{i,n}(D_{i,j})] \text{ is dense in } X[G_n] \cap \mathbb{P},$$

so we may find $p_{n+1} \leqslant p_n$ with

$$p_{n+1} \in \pi_{X_n}[\mu^+_{i,n}(D_{i,j})]$$

and

$$p_{n+1} \upharpoonright n \in G.$$

Thus by fullness, we can construe a $\mathbb{P}_n$-name $\dot{p}_{n+1}$ which is forced by q_n to have properties $(A.v)(a)$-(d). Using Proposition 3.35, we may now find $q_{n+1} \in \mathbb{P}_{n+1}$ that is $(X, \mathbb{P}_{n+1}, f)$-generic with $q_{n+1} \upharpoonright n = q_n$ and

$$q_{n+1} \Vdash \dot{p}_{n+1} \upharpoonright n + 1 \in \dot{G}_{n+1}$$

This completes the construction.
Let q be the limit of $(q_n)_{n<\omega}$.

Claim 3.37. *$q \Vdash \dot{p}_n \in \dot{G}$ for all $n < \omega$.*

Proof. Trivial by $(A.v)$ and as the iteration is of length ω. □

We will now show that q is $(X, \mathbb{P}, f)$-semigeneric. Let G be $\mathbb{P}$-generic with $q \in G$. By Proposition 3.33, we need to show

$(X.i)$ $X \sqsubseteq X[G]$,

$(X.ii)$ $X[G] \cap V$ is f-slim and

$(X.iii)$ $\pi^{-1}_{X[G] \cap V}[G]$ is generic over $M_{X[G]\cap V}[f(\delta^X)]$.

Let X_ω denote $X[G] \cap V$ and $X_n = \dot{X}_n^{G_n} = X[G_n] \cap V$ for $n < \omega$.

Claim 3.38. $X_\omega = \bigcup_{n<\omega} X_n$.

Proof. Let $\dot{x} \in M_X$ be a $\pi_X^{-1}(\mathbb{P})$-name with

$$\Vdash_{\mathbb{P}} \pi_X(\dot{x}) \in V.$$

Then the set of conditions deciding the value of $\dot{x}$ is $D_{0,j} := (\dot{D}_{0,j})^G$ for some $j < \omega$. Find n with $h(n) = (0, j)$. Then by $(A.v)(c)$

$$\dot{p}_{n+1}^G \in \pi_{X_n}[\mu_{0,j}^+(D_{0,j})],$$

but the latter set is the set of conditions in $X_n \cap \mathbb{P}$ deciding $\pi_X(\dot{x})$. Consequently

$$\dot{p}_{n+1}^G \Vdash \pi_X(\dot{x}) \in \dot{X}_n.$$

As $p_{n+1} \in G$ by Claim 3.37, $\pi_X(\dot{x})^G \in X_n$. □

$(X.i)$ and $(X.ii)$ follow immediately from the above together with $(A.i)$. It is left to prove the following claim.

Claim 3.39. $\pi_{X_\omega}^{-1}[G]$ *is generic over* $M_{X_\omega}[f(\delta^X)]$.

Proof. Set $M_\omega = M_{X_\omega}$ and for $i < \omega$, let $\mu_{i,\omega}^+$ denote μ_{X_i,X_ω}^+. Similarly, define $\mu_{i,j}$ for $i \leqslant j \leqslant \omega$ as μ_{X_i,X_j}. It follows from Claim 3.38 that

$$\langle M_\omega[g], \mu_{i,\omega}^+ \mid i < \omega \rangle$$

is the direct limit along

$$\langle M_i[g], \mu_{i,j}^+ \mid i \leqslant j < \omega \rangle$$

for some $(\mu_{i,\omega}^+)_{i<\omega}$. So let $D \in M_\omega[g]$ be dense in $\pi_{X_\omega}^{-1}(\mathbb{P})$. Then, for some $i, j < \omega$,

$$D = \mu_{i,\omega}^+(D_{i,j})$$

where $D_{i,j} = (\dot{D}_{i,j})^G$. Find n with $h(n) = (i, j)$. Then

$$\dot{p}_{n+1}^G \in G \cap \pi_{X_n}[\mu_{i,n}^+(D_{i,j})]$$

by $(A.v)(c)$ and Claim 3.37. But

$$\pi_{X_n} \circ \mu_{i,n} = \pi_{X_i} = \pi_{X_\omega} \circ \mu_{i,\omega}$$

and hence

$$\dot{p}_{n+1}^G \in G \cap \pi_{X_\omega}[\mu_{i,\omega}^+(D_{i,j})] = G \cap \pi_{X_\omega}[D].$$

□

This shows that q is indeed $(X, \mathbb{P}, f)$-semigeneric. □

An easy argument now gives that f-semiproperness is preserved along countable support iterations of length ω_1. Eventually though, we run into the usual problem that along an iteration of length γ, the set $X[G_\alpha] \cap \gamma$ can grow as α increases and $X[G] \cap \gamma$ can even be cofinal in γ even if γ has uncountable cofinality in V. The theory of RCS iterations deals with this problem in the case of semiproper forcings. f-semiproper forcings are indeed fully iterable with RCS-support, however we choose to use Miyamoto's nice support. The main reason is that the iteration theorem we prove in Section 7 is significantly smoother when using nice support and we will use nice iterations anyway when quoting iteration theorems of Miyamoto in Sections 9 and 10. We prefer to make use of only one type of support when iterating semiproper-style forcings.

3.4 Miyamoto's theory of nice iterations

For all our intents and purposes, it does not matter in applications how the limit our iterations look like as long as we can prove a preservation theorem about it.

We give a brief introduction to Miyamoto's theory of nice iterations. These iterations are an alternative to RCS-itertaions when dealing with the problem described above. In the proof of the iteration theorem for (f-)proper forcings, one constructs a generic condition q by induction as the limit of a sequence $\langle q_n \mid n < \omega\rangle$. In case of ($f$-)semiproper forcings, the length of the iteration may have uncountable cofinality in V but become ω-cofinal along the way. In this case, a sequence $\langle q_n \mid n < \omega\rangle$ with the desired properties cannot be in V. The key insight to avoid this issue is that one should give up linearity of this sequence and instead build a tree of conditions in the argument. Nice supports follow the philosophy of form follows function, i.e. its definitions takes the shape of the kind of arguments it is intended to be involved in. The conditions allowed in a nice limit are represented by essentially the kind of trees that this inductive nonlinear constructions we hinted at above produces.

Miyamoto works with a general notion of iteration. For our purposes, we will simply define nice iterations by induction on the length. Successor steps are defined as usual, that is if $\mathbb{P}_\gamma = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ is a nice iteration of length γ and $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for a forcing then $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma+1, \beta \leqslant \gamma\rangle$ is a nice iteration of length $\gamma + 1$ where $\mathbb{P}_{\gamma+1} \cong \mathbb{P}_\gamma * \dot{\mathbb{Q}}_\gamma$.

Definition 3.40 (Miyamoto, [Miy02]). Let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma\rangle$ be a potential nice iteration, that is

($\vec{\mathbb{P}}.i$) $\mathbb{P}_\alpha$ is a nice iteration of length α for all $\alpha < \gamma$,

($\vec{\mathbb{P}}.ii$) $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for all $\alpha + 1 < \gamma$ and

($\vec{\mathbb{P}}.iii$) $\mathbb{P}_\beta \upharpoonright \alpha = \mathbb{P}_\alpha$ for all $\alpha \leqslant \beta < \gamma$.

A *nested antichain* in $\vec{\mathbb{P}}$ is of the form

$$(T, \langle T_n \mid n < \omega\rangle, \langle \mathrm{suc}^n_T \mid n < \omega\rangle)$$

so that for all $n < \omega$ the following hold[26]:

(i) $T = \bigcup_{n<\omega} T_n$.

(ii) $T_0 = \{a_0\}$ for some $a_0 \in \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$.

(iii) $T_n \subseteq \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$ and $\mathrm{suc}^n_T \colon T_n \to \mathcal{P}(T_{n+1})$.

(iv) For $a \in T_n$ and $b \in \mathrm{suc}^n_T(a)$, $\mathrm{lh}(a) \leqslant \mathrm{lh}(b)$ and $b \upharpoonright \mathrm{lh}(a) \leqslant a$.

(v) For $a \in T_n$ and distinct $b, b' \in \mathrm{suc}^n_T(a)$, $b \upharpoonright \mathrm{lh}(a) \perp b' \upharpoonright \mathrm{lh}(a)$.

(vi) For $a \in T_n$, $\{b \upharpoonright \mathrm{lh}(a) \mid b \in \mathrm{suc}^n_T(a)\}$ is a maximal antichain below a in $\mathbb{P}_{\mathrm{lh}(a)}$.

(vii) $T_{n+1} = \bigcup\{\mathrm{suc}^n_T(a) \mid a \in T_n\}$.

Abusing notation, we will usually identify T with

$$(T, \langle T_n \mid n < \omega\rangle, \langle \mathrm{suc}^n_T \mid n < \omega\rangle).$$

If $b \in \mathrm{suc}^n_T(a)$ then we also write $a = \mathrm{pred}^n_T(b)$. If $\beta < \gamma$ then $p \in \mathbb{P}_\beta$ is a *mixture of T up to β* iff for all $\alpha < \beta$, $p \upharpoonright \alpha$ forces

($p.i$) $p(\alpha) = a_0(\alpha)$ if $\alpha < \mathrm{lh}(a_0)$ and $a_0 \upharpoonright \alpha \in G_\alpha$,

($p.ii$) $p(\alpha) = b(\alpha)$ if there are $a, b \in T$, $n < \omega$ with $b \in \mathrm{suc}^n_T(a)$, $\mathrm{lh}(a) \leqslant \alpha < \mathrm{lh}(b)$ and $b \upharpoonright \alpha \in G_\alpha$,

($p.iii$) $p(\alpha) = \mathbb{1}_{\dot{\mathbb{Q}}_\alpha}$ if there is a sequence $\langle a_n \mid n < \omega\rangle$ with $a_{n+1} \in \mathrm{suc}^n_T(a_n)$, $\mathrm{lh}(a_n) \leqslant \alpha$ and $a_n \in G_{\mathrm{lh}(a_n)}$ for all $n < \omega$.

If $\xi \leqslant \gamma$ is a limit, and q is a sequence of length ξ (may or may not be in $\mathbb{P}_\xi$), q is *(T, ξ)-nice* if for all $\beta < \xi$, $q \upharpoonright \beta \in \mathbb{P}_\beta$ is a mixture of T up to β.

We refer to [Miy02] for basic results on nested antichains and mixtures. We go on and define nice limits.

Definition 3.41 (Miyamoto, [Miy02])**.** Suppose $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma\rangle$ is a potential nice iteration of limit length γ. Let $\bar{\mathbb{P}}$ denote the inverse limit along $\vec{\mathbb{P}}$. The *nice limit* of $\vec{\mathbb{P}}$ is defined as

$$\mathrm{nicelim}(\vec{\mathbb{P}}) = \{p \in \bar{\mathbb{P}} \mid \exists T \text{ a nested antichain of } \vec{\mathbb{P}} \text{ and } p \text{ is } (T, \gamma)\text{-nice}\}.$$

$\mathrm{nicelim}(\vec{\mathbb{P}})$ inherits the order from $\bar{\mathbb{P}}$.

[26] Usually, we identify the nested antichain with T, its first component and write $\mathrm{suc}(a)$ instead of $\mathrm{suc}^n_T(a)$ if n, T are clear from context.

Finally, if $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a potential nice iteration then

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$$

is a *nice iteration* of length γ where $\mathbb{P}_\gamma = \mathrm{nicelim}(\vec{\mathbb{P}})$.

The fundamental property of nice iterations is:

Fact 3.42 (Miyamoto,[Miy02])**.** *Suppose* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *is a nice iteration and* T *is a nested antichain in* $\mathbb{P}$*. Then there is a mixture of* T*.*

Definition 3.43 (Miyamoto,[Miy02])**.** Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration. If S, T are nested antichains in $\mathbb{P}$ then $S \angle\, T$ iff for any $n < \omega$ and $a \in S_n$ there is $b \in T_{n+1}$ with

$$\mathrm{lh}(b) \leqslant \mathrm{lh}(a) \text{ and } a \upharpoonright \mathrm{lh}(b) \leqslant b.$$

Fact 3.44 (Miyamoto, [Miy02])**.** *Let* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *be a nice iteration of limit length* γ*. Suppose that*

(*i*) T *is a nested antichain in* $\mathbb{P}$*,*

(*ii*) p *is a mixture of* T *and* $s \in \mathbb{P}$*,*

(*iii*) $r \in T_1$*,*

(*iv*) $s \leqslant r^\frown p \upharpoonright [\mathrm{lh}(r), \gamma)$ *and*

(*v*) $A \subseteq \gamma$ *is cofinal.*

Then there is a nested antichain S *in* $\mathbb{P}$ *with*

(*a*) s *is a mixture of* S*,*

(*b*) *If* $S_0 = \{c\}$ *then* $\mathrm{lh}(r) \leqslant \mathrm{lh}(c) \in A$ *and* $c \upharpoonright \mathrm{lh}(r) \leqslant r$ *and*

(*c*) $S \angle\, T$*.*

The following describes the tool we use to construct conditions.

Definition 3.45 (Miyamoto, [Miy02])**.** Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration of limit length γ. A *fusion structure* in $\mathbb{P}$ is

$$T, \langle p^{(a,n)},\ T^{(a,n)} \mid n < \omega,\ a \in T_n \rangle$$

where

(*i*) T is a nested antichain in $\mathbb{P}$

and for all $n < \omega$ and $a \in T_n$

(*ii*) $T^{(a,n)}$ is a nested antichain in $\mathbb{P}$,

(*iii*) $p^{(a,n)} \in \mathbb{P}$ is a mixture of $T^{(a,n)}$,

(*iv*) $a \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(a)$ and if $T_0^{(a,n)} = \{c\}$ then $\mathrm{lh}(a) = \mathrm{lh}(c)$ and

(*v*) for any $b \in \mathrm{suc}_T^n(a)$, $T^{(b,n+1)} \angle\, T^{(a,n)}$, thus $p^{(b,n+1)} \leqslant p^{(a,n)}$.

If $q \in \mathbb{P}$ is a mixture of T then q is called a *fusion* of the fusion structure.

Fact 3.46 (Miyamoto, [Miy02]). *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration of limit length γ. If $q \in \mathbb{P}$ is a fusion of a fusion structure*

$$T, \langle p^{(a,n)},\ T^{(a,n)} \mid n < \omega,\ a \in T_n \rangle$$

and G is $\mathbb{P}$-generic with $q \in G$ then the following holds in $V[G]$: There is a sequence $\langle a_n \mid n < \omega \rangle$ so that for all $n < \omega$

(*i*) $a_0 \in T_0$,

(*ii*) $a_n \in G_{\mathrm{lh}(a_n)}$,

(*iii*) $a_{n+1} \in \mathrm{suc}_T^n(a_n)$ *and*

(*iv*) $p^{(a_n,n)} \in G$.

We mention one more convenient fact:

Fact 3.47 (Miyamoto, [Miy03]). *Suppose κ is an inaccessible cardinal, $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$ is a nice iteration so that*

(*i*) $|\mathbb{P}_\alpha| < \kappa$ *for all* $\alpha < \kappa$ *and*

(*ii*) $\mathbb{P}$ *preserves* ω_1.

Then $\mathbb{P}$ is κ-c.c..

Miyamoto proves this for so called simple iterations of semiproper forcings. The proof works just as well for nice iterations of semiproper forcings and finally the proof can be made to work with assuming only $\mathbb{P}$ preserves ω_1 instead of $\mathbb{P}$ being a semiproper iteration.

3.5 The iteration theorem for f-semiproper forcing

Theorem 3.48. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. If $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a nice iteration of f-semiproper forcings then $\mathbb{P}$ is f-semiproper.*

As usual, the iteration theorem will be a consequence from a more technical lemma.

Lemma 3.49. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ be a nice iteration of f-semiproper forcings. Assume that*

(A.i) θ *is sufficiently large and regular,*

(A.ii) $p \in \mathbb{P}$ *is a mixture of a nested anitchain* T^p,

(A.iii) $\alpha < \gamma$ *and if* $T^p_0 = \{r\}$ *then* $\mathrm{lh}(r) = \alpha$,

(A.iv) $q_0 \in \mathbb{P}_\alpha$, $\dot{Y} \in V^{\mathbb{P}_\alpha}$ *with*

$$q_0 \Vdash_\alpha \text{“}\dot{Y} \prec H_{\check{\theta}}^{V[\dot{G}_\alpha]} \text{ is } \check{f}\text{-slim”},$$

(A.v) $q_0 \Vdash_\alpha \check{T}^p, \check{p}, \check{\mathbb{P}}, \dot{G}_\alpha \in \dot{Y}$ *and*

(A.vi) $q_0 \leqslant p \upharpoonright \alpha$.

Then there is $q \in \mathbb{P}$ with

(q.i) $q \Vdash_\gamma \text{“}\dot{Y} \sqsubseteq \dot{Y}[\dot{G}_{\check{\alpha},\check{\gamma}}] \prec H_{\check{\theta}}^{V[\dot{G}_\gamma]} \text{ is } \check{f}\text{-slim”}$,

(q.ii) $q \upharpoonright \alpha = q_0$ *and*

(q.iii) $q \leqslant p$.

Proof. The proof will be by induction on γ, the length of the iteration. The successor step is handled with Proposition 3.35, so we will focus on the case $\gamma \in \mathrm{Lim}$. Let

$$h\colon \omega \to \omega \times \omega$$

be a surjection with $i \leqslant n$ whenever $h(n) = (i, j)$.
Let $\dot{\delta}$ be a $\mathbb{P}_\alpha$-name for $\delta^{\dot{Y}}$. We now construct a fusion structure

$$T, \langle p^{(a,n)}, T^{(a,n)} \mid a \in T_n,\ n < \omega\rangle$$

in $\mathbb{P}$ as well as names

$$\left\langle \dot{Y}^{(a,n)}, \dot{Z}^{(a,n)}, \left(\dot{D}^{(a,n)}_j\right)_{j<\omega} \mid a \in T_n, n < \omega \right\rangle$$

so that for any $n < \omega$ and $a \in T_n$

(F.i) $T_0 = \{q_0\}$, $p^{(q_0,0)} = p$,

(F.ii) $T^{(q_0,0)} = T^p$,

(F.iii) $\dot{Y} = \dot{Y}^{(q_0,0)} = \dot{Z}^{(q_0,0)}$,

(F.iv) $a \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(a)$,

$(F.v)$ $a \Vdash_{\mathrm{lh}(a)} \check{T}^{(a,n)}, \dot{G}_{\mathrm{lh}(a)} \in \dot{Y}^{(a,n)}$,

$(F.vi)$ $a \Vdash_{\mathrm{lh}(a)}$ "$\dot{Y} \sqsubseteq \dot{Y}^{(a,n)} \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(a)}]}$ is f-slim",

$(F.vii)$ $a \Vdash_{\mathrm{lh}(a)} \check{p}^{(a,n)}, \check{T}^{(a,n)}, \dot{G}_{\mathrm{lh}(\check{a})} \in \dot{Y}^{(a,n)}$,

$(F.viii)$ $\dot{Z}^{(a,n)}$ is a $\mathbb{P}_{\mathrm{lh}(a)}$-name for $\dot{Y}^{(a,n)} \cap V[\dot{G}_\alpha]$ and

$(F.ix)$ $\left(\dot{D}_j^{(a,n)}\right)_{j<\omega}$ is forced by a to be an enumeration of all dense subsets of $\pi_{\dot{Z}^{(a,n)}}^{-1}(\check{\mathbb{P}})$ in

$$M_{\dot{Z}^{(a,n)}}\left[\check{f}(\dot{\delta})\right].$$

Moreover, for any $b \in \mathrm{suc}_T^n(a)$

$(F.x)$ $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)} \check{p}^{(b,n+1)}, \check{T}^{(b,n+1)}, \mathrm{lh}(b) \in \dot{Y}^{(a,n)}$,

$(F.xi)$ $b \Vdash_{\mathrm{lh}(b)} \dot{Y}^{(b,n+1)} = \dot{Y}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}]$ and

$(F.xii)$ if $h(n) = (i,j)$ and $c = \mathrm{pred}_T^i(b)$ then

$$b \Vdash_{\mathrm{lh}(b)} \check{p}^{(b,n+1)} \in \pi_{\dot{Z}^{(a,n)}}\left[\mu_{c,a}^+\left(\dot{D}_j^{(c,i)}\right)\right].$$

Here, $\mu_{c,a}^+$ denotes a name for

$$\mu^+_{\dot{Z}^{(c,i)},\dot{Z}^{(a,n)}} : M_{\dot{Z}^{(c,i)}}\left[\check{f}(\dot{\delta})\right] \to M_{\dot{Z}^{(a,n)}}\left[\check{f}(\dot{\delta})\right].$$

Recall this notation from Proposition 2.18. We define all objects by induction on $n < \omega$.

$$T_0 = \{q_0\}, p^{(q_0,0)}, T^{(q_0,0)}, \dot{Y}^{(q_0,0)}, \dot{Z}^{(q_0,0)}, \left(\dot{D}_j^{(q_0,0)}\right)_{j<\omega}$$

are given by $(F.i)$-$(F.iii)$, $(F.viii)$ and $(F.ix)$. Suppose we have already defined

$$T_n, \left\langle p^{(a,n)}, T^{(a,n)}, \dot{Y}^{(a,n)}, \dot{Z}^{(a,n)}, \left(\dot{D}_j^{(a,n)}\right)_{j<\omega} \mid a \in T_n \right\rangle$$

and we will further construct

$$T_{n+1}, \left\langle p^{(b,n+1)}, T^{(b,n+1)}, \dot{Y}^{(b,n+1)}, \dot{Z}^{(b,n+1)}, \left(\dot{D}_j^{(b,n+1)}\right)_{j<\omega} \mid b \in T_{n+1} \right\rangle.$$

Fix $a \in T_n$. Let E be the set of all b with the following properties:

$(E.i)$ $b \in \mathbb{P}_{\mathrm{lh}(b)}$ and $\mathrm{lh}(b) < \gamma$.

$(E.ii)$ $\mathrm{lh}(a) \leqslant \mathrm{lh}(b)$ and $b \upharpoonright \mathrm{lh}(a) \leqslant a$.

Furthermore, there are a nested antichain S in $\mathbb{P}$, $s \in \mathbb{P}$ and a name $\dot{Y}^b$ with

$(E.iii)$ $S \angle\, T^{(a,n)}$,

$(E.iv)$ $s \leqslant p^{(a,n)}$ is a mixture of S,

$(E.v)$ if $h(n) = (i,j)$ and $c = \mathrm{pred}^i_T(a)$ then

$$b \Vdash_{\mathrm{lh}(b)} \check{s} \in \pi_{\dot{Z}^{(a,n)}}\left[\dot{\mu}^+_{c,a}\left(\dot{D}^{(c,i)}_j\right)\right],$$

$(E.vi)$ $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)} \check{s}, \check{S} \in \dot{Y}^{(a,n)}$,

$(E.vii)$ $b \Vdash_{\mathrm{lh}(b)} \check{s} \upharpoonright \mathrm{lh}(b) \in \dot{G}_{\mathrm{lh}(b)}$,

$(E.viii)$ $b \Vdash_{\mathrm{lh}(b)}$ "$\dot{Y}^{(a,n)} \sqsubseteq \dot{Y}^b = \dot{Y}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}] \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(\check{b})}]}$ is $\check{f}$-slim" and

$(E.ix)$ if $S_0 = \{c_0\}$ then $\mathrm{lh}(b) = \mathrm{lh}(c_0)$ and $b \leqslant c_0$.

Claim 3.50. *$E \upharpoonright \mathrm{lh}(a) := \{b \upharpoonright \mathrm{lh}(a) \mid b \in E\}$ is dense in $\mathbb{P}_{\mathrm{lh}(a)}$.*

Proof. Let $a' \leqslant a$ and let G be $\mathbb{P}_{\mathrm{lh}(a)}$-generic with $a' \in G$. Let $\delta = \dot{\delta}^G$. By $(F.iv)$, $p^{(a,n)} \upharpoonright \mathrm{lh}(a) \in G$. Let $h(n) = (i,j)$ and $c = \mathrm{pred}^i_T(a)$. Let

$$Y^{(c,i)} := \left(\dot{Y}^{(c,i)}\right)^{G_{\mathrm{lh}(c)}} \text{ and } Y^{(a,n)} := \left(\dot{Y}^{(a,n)}\right)^{G}.$$

Moreover, set $Z^{(c,i)} := Y^{(c,i)} \cap V[G_\alpha]$, $Z^{(a,n)} := Y^{(a,n)} \cap V[G_\alpha]$. $\mu^+_{c,a}$ is then the canonical embedding

$$\mu^+_{c,a} \colon M_{Z^{(c,i)}}[f(\delta)] \to M_{Z^{(a,n)}}[f(\delta)].$$

Note that this exists by $(F.vi)$ and Proposition 2.18. Find $r \in T^{(a,n)}_1$ with $r \upharpoonright \mathrm{lh}(a) \in G$. As $p^{(a,n)}$ is a mixture of $T^{(a,n)}$, we have

$$r \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(r).$$

Let $\hat{r} = r^\frown(p^{(a,n)} \upharpoonright [\mathrm{lh}(r), \gamma))$. Note that $\hat{r} \in Y^{(a,n)}$ as

$$p^{(a,n)}, T^{(a,n)}, G \in Y^{(a,n)}$$

by $(F.vii)$. Moreover, $\hat{r} \upharpoonright \mathrm{lh}(a) \in G$. Note that by $(F.vi)$ and the product lemma, $\pi^{-1}_{Z^{(a,n)}}[G]$ is generic over $M_{Z^{(a,n)}}[f(\delta)]$. Thus we may now find $s \leqslant \hat{r}$ with

$$s \in \pi_{Z^{(a,n)}}[\mu^+_{c,a}(D)] \text{ and } s \upharpoonright \mathrm{lh}(a) \in G$$

where

$$D = \left(\dot{D}^{(c,i)}_j\right)^{G_{\mathrm{lh}(c)}} \in M_{Z^{(c,i)}}[f(\delta)].$$

Note that $s \in Y^{(a,n)}$. We can now apply Fact 3.44 in $Y^{(a,n)}$ and get a nested antichain $S \in Y^{(a,n)}$ with

(a) s is a mixture of S,

(b) if $S_0 = \{d\}$ then $\mathrm{lh}(r) \leqslant \mathrm{lh}(d)$ and $d \upharpoonright \mathrm{lh}(r) \leqslant r$ and

(c) $S \angle\, T^{(a,n)}$.

Let $\dot{Y}^b$ be a name for $Y^{(a,n)}[\dot{G}_{\mathrm{lh}(a)\mathrm{lh}(d)}]$. As $\mathrm{lh}(d) < \gamma$, by induction there is now $b \in \mathbb{P}_{\mathrm{lh}(d)}$ with

$(b.i)$ $b \leqslant d$,

$(b.ii)$ $b \upharpoonright \mathrm{lh}(a) \leqslant a'$ and

$(b.iii)$ $b \Vdash$ "$\dot{Y}^{(a,n)} \sqsubseteq \dot{Y}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(d)}]$ is $\check{f}$-slim".

We can arrange $(b.ii)$ as G is a filter and $d \upharpoonright \mathrm{lh}(a) = s \upharpoonright \mathrm{lh}(a), a' \in G$. It is now easily checked that $b \in E$ and $b \upharpoonright \mathrm{lh}(a) \leqslant a'$. $\square$

To define T_{n+1}, fix a maximal antichain $A \subseteq E \upharpoonright \mathrm{lh}(a)$ and for any $e \in A$ choose $b_e \in E$ with $b_e \upharpoonright \mathrm{lh}(a) = e$. We set $\mathrm{suc}^n_T(a) = \{b_e \mid e \in A\}$. For any $b \in \mathrm{suc}^n_T(a)$, let $S, s, \dot{Y}^b$ witness $b \in E$. We then set

$$p^{(b,n+1)} = s,\ T^{(b,n+1)} = S,\ \dot{Y}^{(b,n+1)} = \dot{Y}^b$$

and $\dot{Z}^{(b,n+1)}$ a name for $\dot{Y}^{(b,n+1)} \cap V[\dot{G}_\alpha]$. Further, let $\left(\dot{D}^{(b,n+1)}_j\right)_{j<\omega}$ be a sequence of names that are forced by b to enumerate all dense subsets of $\pi^{-1}_{\dot{Z}^{(b,n+1)}}(\mathbb{P})$ in $M_{\dot{Z}^{(b,n+1)}}\left[\check{f}(\dot{\delta})\right]$. This finishes the construction.

By Fact 3.44, there is a mixture of q of T and we may assume, as $T_0 = \{q_0\}$, that $q \upharpoonright \mathrm{lh}(q_0) = q_0$. Let G be $\mathbb{P}$-generic with $q \in T$. By Fact 3.46, in $V[G]$ there is a sequence $\langle a_n \mid n < \omega\rangle$ so that for all $n < \omega$

(i) $a_0 = q_0$,

(ii) $a_{n+1} \in \mathrm{suc}^n_T(a_n)$ and

(iii) $p^{(a_n,n)} \in G$.

Let $\alpha_n = \mathrm{lh}(a_n) < \gamma$ and $\alpha_\omega = \gamma$. For $i \leqslant n < \omega$ we let

- $Y := \dot{Y}^G$,
- $Y_n := \left(\dot{Y}^{(a_n,n)}\right)^G$,
- $Z_n := Y_n \cap V[G_\alpha]$,
- $M_n := M_{Z_n}$ and $\pi_n := \pi_{Z_n}$,
- $Y_\omega := \dot{Y}^{G_{\alpha_0}}[G_{\alpha_0,\gamma}] = Y_0[G_{\alpha_0,\gamma}]$,

- $Z_\omega := Y_\omega \cap V[G_\alpha]$,
- $M_\omega := M_{Z_\omega}$ and $\pi_\omega := \pi_{Z_\omega}$.

We also set $\delta := \dot{\delta}^G$. With Proposition 3.33 in mind, we have to show the following that

$(Y.i)$ $Y \sqsubseteq Y_\omega$,

$(Y.ii)$ Z_ω is f-slim and

$(Y.iii)$ $\pi_\omega^{-1}[G_{\alpha,\gamma}]$ is generic for $\pi_\omega^{-1}(\dot{\mathbb{P}}^{G_\alpha}_{\alpha,\gamma})$ over $M_\omega[f(\delta)]$.

Claim 3.51. $Z_\omega = \bigcup_{n<\omega} Z_n$.

Proof. Let $\dot{x} \in Y_0$ be a $\dot{\mathbb{P}}^{G_\alpha}_{\alpha,\gamma}$-name for a set in $V[G_\alpha]$. Then the set D of conditions deciding $\dot{x}$ is $\pi_Y\left(\left(\dot{D}_j^{(a_0,0)}\right)^{G_\alpha}\right)$ for some $j < \omega$. Find n with $h(n) = (0, j)$. Then

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{0,n}(D)]$$

by $(F.xii)$. Since $p^{(a_{n+1},n+1)} \in Z_n$ by $(F.x)$, we can conclude $\dot{x}^G \in Z_n$. This shows " $\subseteq$ ".
To see " $\supseteq$ ", note that

$$Y_n = Y_0[G_{\alpha_1}] \dots [G_{\alpha_n}] \subseteq Y_\omega$$

for any $0 < n < \omega$ by $(F.x)$ and $(F.xi)$. Hence

$$Z_n = Y_n \cap V[G_\alpha] \subseteq Y_\omega \cap V[G_\alpha] = Z_\omega.$$

□

$(Y.i)$ and $(Y.ii)$ follow immediately from this as $X \sqsubseteq Y_n$ is f-slim for all $n < \omega$ by $(F.vi)$.

It is left to prove the following.

Claim 3.52. *$\pi_\omega^{-1}[G_{\alpha,\gamma}]$ is generic over $M_\omega[f(\delta)]$.*

Proof. Let $E \subseteq \pi_\omega^{-1}(\dot{\mathbb{P}}^{G_\alpha}_{\alpha,\gamma})$ be dense, $E \in M_\omega[f(\delta)]$. By Claim 3.51 we have that

$$\langle M_\omega[f(\delta)], \mu^+_{n,\omega} \mid n < \omega\rangle = \varinjlim\langle M_i[f(\delta)], \mu^+_{i,n} \mid i \leqslant n < \omega\rangle$$

for some maps $(\mu^+_{n,\omega})_{n<\omega}$. Thus, for some $i, j < \omega$

$$E = \mu^+_{i,\omega}(D)$$

where $D = \left(\dot{D}_j^{(a_i,i)}\right)^{G_{\alpha_i}}$. If we let $n < \omega$ so that $h(n) = (i, j)$, we get by $(F.xii)$,

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{i,n}(D)] \subseteq \pi_\omega[\mu^+_{i,\omega}(D)] = \pi_\omega[E]$$

where the inclusion follows from:

$$\pi_\omega \circ \mu_{i,\omega} = \pi_\omega \circ \pi_\omega^{-1} \circ \pi_i = \pi_i = \pi_n \circ \pi_n^{-1} \circ \pi_i = \pi_n \circ \mu_{i,n}.$$

As $p^{(a_{n+1},n+1)} \in G$, we have $E \cap \pi_\omega^{-1}[G_{\alpha,\gamma}] \neq \varnothing$. $\square$

This shows that q indeed satisfies $(q.i)$-$(q.iii)$. $\square$

In fact we have shown the following useful strengthening of Theorem 3.48.

Corollary 3.53. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ is a nice iteration of f-semiproper forcings. Then for any $\alpha \leqslant \gamma$*

$$V^{\mathbb{P}_\alpha} \models \text{“}\dot{\mathbb{P}}_{\alpha,\gamma}^{\dot{G}_\alpha} \textit{ is } f\textit{-semiproper”}.$$

3.6 NS_{ω_1} saturated together with $\diamondsuit^+(\mathbb{B})$ from a Woodin cardinal

Suppose δ is a Woodin cardinal. Shelah has shown that there is a forcing extension in which NS_{ω_1} is saturated. We will show that we can additionally turn a witness f of $\diamondsuit(\mathbb{B})$ into a witness of $\diamondsuit^+(\mathbb{B})$ as well as make ψ_{AC} hold in the extension.

Definition 3.54. ψ_{AC} states that for any stationary, costationary $S, T \subseteq \omega_1$ there is a canonical function[27] η_ξ for some $\xi < \omega_2$ so that

$$S = \eta_\xi^{-1}[T] \mod \mathrm{NS}_{\omega_1}.$$

Proposition 3.55 (Folklore)**.** *Suppose that κ is a measurable cardinal and $\theta > \kappa$ is regular. For any countable $X \prec H_\theta$ with $\kappa \in X$ there is $Y \prec H_\theta$ with*

$(Y.i)$ $X \subseteq Y$,

$(Y.ii)$ $\sup(X \cap \kappa) < \sup(Y \cap \kappa)$ *and*

$(Y.iii)$ $Y \cap \sup(X \cap \kappa) = X \cap \kappa$.

Proof. As X is countable there is a wellorder $\trianglelefteq$ of H_θ with

$$X \prec (H_\theta; \in, \trianglelefteq) =: \mathcal{H}.$$

[27] A canonical function for $\xi < \omega_2$ is a function $\eta\colon \omega_1 \to \omega_1$ with

$$\mathrm{otp}(\prod_{\alpha<\omega_1} \eta(\alpha) + 1/\sim, \lhd) = \xi + 1$$

where $g \sim h$ iff g equals h on a club and $[g]_\sim \lhd [h]_\sim$ if g is strictly less than h on a club.

Let $U \in X$ be a normal measure that witnesses κ to be measurable and let $A = \bigcap(U \cap X)$. As X is of size $<\kappa$, $A \in U$ so we can pick some $\xi \in A$. We will show that

$$Y = \mathrm{Hull}^{\mathcal{H}}(X \cup \{\xi\})$$

has the required properties. $(Y.i)$ is clear and $(Y.ii)$ holds as $\xi > \sup(X \cap \kappa)$. To see $(Y.iii)$, let $\gamma \in Y \cap \sup(X \cap \kappa)$ and find a term τ, and $x \in X$ so that

$$\gamma = \tau^{\mathcal{H}}(x, \xi).$$

We have that the function

$$f\colon \kappa \to H_\theta,\ \alpha \mapsto \tau(x, \alpha)$$

is in H_θ as θ is regular and is definable over $\mathcal{H}$ from x, thus $f \in X$. Let $B = \{\alpha < \kappa \mid f(\alpha) \in \alpha\} \in X$. As $\xi \in B$ and U is an ultrafilter, we must have $B \in U$. By normality of U there is $C \subseteq B$, $C \in U$ and $\alpha < \kappa$ so that $f[C] = \{\alpha\}$. We may assume $C, \alpha \in X$ by elementarity, so that $\xi \in C$ and hence $f(\xi) = \alpha \in X$. □

Proposition 3.56. *Suppose f witnesses $\diamondsuit(\mathbb{B})$, $\mathbb{P}$ is a forcing, θ is sufficiently large, $X, Y \prec H_\theta$ both f-slim, $X \sqsubseteq Y$ with $\mathbb{P} \in X$. If q is $(Y, \mathbb{P}, f)$-semigeneric then q is $(X, \mathbb{P}, f)$-semigeneric.*

Proof. Let $\delta := \delta^X = \delta^Y$. It is clear that q is $(X, \mathbb{P})$-semigeneric, so we must show that

$$q \Vdash \text{“}\check{f}(\check{\delta}) \text{ is generic over } M_{\check{X}}\text{”}.$$

Let G be $\mathbb{P}$-generic with $q \in G$. Look at the canonical elementary embedding

$$\mu := \mu_{X[G], Y[G]}\colon M_{X[G]} \to M_{Y[G]}.$$

By assumption, $f(\delta)$ is generic over $M_{Y[G]}$. Moreover, μ is the inclusion on $\mathcal{P}(\delta) \cap M_{X[G]}$, as $\mathrm{crit}(\mu) > \delta$ if it exists (this follows from the semigenericity of q). Thus $f(\delta)$ meets all dense subsets of $\bar{\mathbb{B}}$ in $M_{X[G]}$. □

The following is a slight modification of Lemma 10.95 in [Woo10].

Lemma 3.57. *Suppose there is a measurable cardinal κ, f witnesses $\diamondsuit(\mathbb{B})$ and $S, T \subseteq \omega_1$ are stationary costationary. Then there is a f-semiproper forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$*

$$S = \eta_\xi^{-1}[T] \mod \mathrm{NS}_{\omega_1}$$

where η_ξ is a canonical function for some $\xi < \omega_2^{V^{\mathbb{P}}}$.

Proof. Let conditions in $\mathbb{P}$ be of the form $p = (g, c)$ with

$(p.i)$ $c \subseteq \omega_1$ is closed bounded with maximum $\alpha = \alpha^p < \omega_1$,

($p.ii$) $g : \alpha \to \kappa$ and

($p.iii$) $S \cap c = \{\beta \leqslant \alpha \mid \mathrm{otp}(g[\beta]) \in T\}$.

The order on $\mathbb{P}$ is given by $q = (h, d) \leqslant (g, c) = p$ iff

($\leqslant .i$) $\alpha^q \geqslant \alpha^p$,

($\leqslant .ii$) $h \upharpoonright \alpha^p = g$ as well as

($\leqslant .iii$) $d \cap (\alpha^p + 1) = c$.

We will only show that $\mathbb{P}$ is f-semiproper and leave the rest to the reader. Let $\theta > \kappa$ be regular and let $X \prec H_\theta$ be f-slim with $S, T, \mathbb{P} \in X$ and let $p \in \mathbb{P} \cap X$. By Proposition 3.55, we can build an increasing continuous sequence $\vec{X} = \langle X_\alpha \mid \alpha < \omega_1 \rangle$ of f-slim elementary substructures of H_θ so that

($\vec{X}.i$) $X_0 = X$,

($\vec{X}.ii$) $X \sqsubseteq X_\alpha$ for all $\alpha < \omega_1$ and

($\vec{X}.iii$) $\{\mathrm{otp}(X_\alpha \cap \kappa) \mid \alpha < \omega_1\}$ is a club.

As T is stationary costationary, there is thus some $\alpha < \omega_1$ so that:

$$\delta^{X_\alpha} = \delta^X \in S \Leftrightarrow \mathrm{otp}(X_\alpha \cap \kappa) \in T$$

We may now find a decreasing sequence $\vec{p} = \langle p_n \mid n < \omega \rangle$ of conditions in p with

($\vec{p}.i$) $p_0 = p$,

($\vec{p}.ii$) $p_n \in X$ for all $n < \omega$ and

($\vec{p}.iii$) for all dense $D \subseteq \pi_{X_\alpha}^{-1}(\mathbb{P})$, $D \in M_{X_\alpha}[f(\delta^X)]$ there is $n < \omega$ with $p_n \in \pi_{X_\alpha}[D]$.

Let us write $p_n = (g_n, c_n)$ for $n < \omega$. Let $g = \bigcup_{n<\omega} g_n$,

$$c = \bigcup_{n<\omega} c_n \cup \{\delta^X\}$$

and observe that c is closed bounded with maximum δ^X, $\mathrm{dom}(g) = \delta^X$ as well as $\mathrm{ran}(g) = X_\alpha \cap \kappa$. By choice of α, $q = (g, c)$ is a condition in $\mathbb{P}$ and the properties of $\vec{p}$ give that $q \leqslant p$ and q is $(X_\alpha, \mathbb{P}, f)$-semigeneric. By Proposition 3.56, q is $(X, \mathbb{P}, f)$-semigeneric as well. □

Definition 3.58. If $S, T \subseteq \omega_1$ are stationary, costationary and there is a measurable cardinal, then we denote the forcing $\mathbb{P}$ in the argument above relative to the least measurable cardinal by $\mathbb{P}(S, T)$.

Definition 3.59. Suppose $\mathcal{A}$ is a maximal antichain in $\mathrm{NS}_{\omega_1}^+$. The *antichain sealing forcing at $\mathcal{A}$* is denoted by $\mathbb{P}_{\mathcal{A}}$ and consists of conditions $p = (c, h)$ where

($p.i$) $h\colon \alpha \to \mathcal{A}$ and

($p.ii$) $c \subseteq \omega_1$ is a closed set with $\max(c) = \alpha$ and $\beta \in \bigcup h[\beta]$ for all $\beta \in c$

for some $\alpha = \alpha^p < \omega_1$. The order on $\mathbb{P}_{\mathcal{A}}$ is given by $q = (d, k) \leqslant (c, h) = p$ if $\alpha^q \geqslant \alpha^p$, $d \cap (\alpha^p + 1) = c$ and $k \upharpoonright \alpha^p = h$.

Theorem 3.60. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and δ is Woodin. There is then a f-semiproper forcing so that in the extension*

(*i*) $\delta = \omega_2$,

(*ii*) *f witnesses $\diamondsuit^+(\mathbb{B})$,*

(*iii*) *ψ_{AC} holds and*

(*iv*) *NS_{ω_1} is saturated.*

The argument is generally very similar to Shelah's proof of forcing "NS_{ω_1} is saturated" from a Woodin cardinal, cf. Schindler's write-up [Sch11]. As we will make use of the above theorem quite a lot, we give the details.

Definition 3.61. A cardinal δ is *Woodin with $\diamondsuit$* if there is a sequence $\langle a_\beta \mid \beta < \delta\rangle$ with $a_\beta \subseteq V_\beta$ so that for any $A \subseteq V_\delta$ there are stationarily many $\kappa < \delta$ with

($\kappa.i$) $a_\kappa = A \cap V_\kappa$ and

($\kappa.ii$) κ is $<\delta$-A-strong[28].

Proof of Theorem 3.60. We may assume that δ is Woodin with $\diamondsuit$ as we can otherwise force with $\mathrm{Add}(\delta, 1)$ first, see [Sch11]. Say this is witnessed by $\vec{a} = \langle a_\beta \mid \beta < \delta\rangle$. Define a nice iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \delta, \beta < \delta\rangle$$

of f-semiproper forcings as follows. At inaccessible $\kappa < \delta$, we choose $\dot{\mathbb{Q}}_\kappa$ so that

$$\Vdash_{\mathbb{P}_\kappa} \text{“If } a_\kappa \text{ is a maximal antichain in } \mathrm{NS}_{\omega_1}^+ \text{ and } \mathbb{P}_{a_\kappa} \text{ is } f\text{-semiproper then } \dot{\mathbb{Q}}_\kappa = \mathbb{P}_{a_\kappa}, \text{ otherwise } \dot{\mathbb{Q}}_\kappa = \mathrm{Col}(\omega_1, 2^{\omega_1})\text{”}$$

[28] κ is $<\delta$-A-strong if for any $\lambda < \delta$ there is an elementary embedding $j\colon V \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $j(A) \cap V_\lambda = A \cap V_\lambda$.

where where we consider a_κ as a $\mathbb{P}_\kappa$-name.
At accessible ordinals $\alpha < \delta$ we alternate between

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha = \mathrm{Col}(\omega_1, \omega_2)\text{”}, \ \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha = \mathbb{P}(\check{f})\text{” and } \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha = \mathbb{P}(\dot{S}, \dot{T})\text{”}$$

where $\dot{S}$, $\dot{T}$ are some $\mathbb{P}_\alpha$-names for stationary costationary subsets of ω_1 in $V^{\mathbb{P}_\alpha}$ given by some suitable bookkeeping. Recall the forcings $\mathbb{P}(f), \mathbb{P}(S,T)$ from Definitions 3.6, 3.58 respectively.
$\mathbb{P}$ is then f-semiproper by Lemma 3.57 and Theorem 3.48, also note that f witnesses $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}_\kappa}$ for inaccessible $\kappa \leqslant \delta$. Moreover $\mathbb{P}$ is δ-c.c. by Fact 3.47. Now suppose that G is $\mathbb{P}$-generic over V.

Claim 3.62. $V[G] \models \psi_{\mathrm{AC}}$.

Proof. If $(S,T) \in V[G]$ is a pair or stationary costationary subsets of ω_1 then S,T exist already in an intermediate extension. Our bookkeeping makes sure that there is $\alpha < \delta$ so that

$$V[G_\alpha] \models \dot{\mathbb{Q}}_\alpha^{G_\alpha} = \mathbb{P}(S,T)$$

and hence the instance of ψ_{AC} corresponding to (S,T) is true in $V[G_{\alpha+1}]$. This is preserved in the extension from $V[G_{\alpha+1}]$ to $V[G]$, simply because ω_1 is not collapsed. □

Note that δ is preserved by $\mathbb{P}$ and as we have thrown enough collapses into $\mathbb{P}$, $\delta = \omega_2^{V[G]}$. It is left to show that NS_{ω_1} is saturated in $V[G]$. Suppose that NS_{ω_1} is not saturated so that there is a maximal antichain

$$\vec{A} := \langle A_i \mid i < \delta \rangle$$

of stationary sets in $V[G]$ of length δ. We can find a nice $\mathbb{P}$-name $\dot{\tau} \in V_{\delta+1}$ for $\vec{A}$. It follows that there is a club $C \subseteq \delta$ so that

$(C.i)$ $\vec{A} \upharpoonright \alpha = (\tau \cap V^{\mathbb{P}_\alpha})^{G_\alpha}$ and

$(C.ii)$ $\vec{A} \upharpoonright \alpha$ is a maximal antichain in $V[G_\alpha]$

for all $\alpha \in C$. As $\mathbb{P}$ is δ-c.c., there is a club $C' \subseteq C$ in V. We can now find $\kappa \in C'$ so that

$(\kappa.i)$ κ is $<\delta$-$\dot{\tau}$-strong and

$(\kappa.ii)$ $a_\kappa = \dot{\tau} \cap V^{\mathbb{P}_\kappa}$.

Note that κ is inaccessible in V and thus

$(\kappa.iii)$ $\kappa = \omega_2^{V[G_\kappa]}$ and

$(\kappa.iv)$ $V[G_\kappa] \models$ “f witnesses $\diamondsuit^+(\mathbb{B})$”.

Let $\mathcal{A} = \{A_i \mid i < \kappa\}$ and observe that

$$V[G_\kappa] \models \text{“}\dot{\mathbb{Q}}_\kappa^{G_\kappa} = \mathbb{P}_\mathcal{A} \text{ if this is } f\text{-semiproper, else } \dot{\mathbb{Q}}_\kappa^{G_\kappa} = \mathrm{Col}(\omega, 2^{\omega_2})\text{”}$$

as $\kappa \in C$ and by $(\kappa.ii)$.

Claim 3.63. *$\mathbb{P}_\mathcal{A}$ is f-semiproper in $V[G_\kappa]$.*

Proof. Suppose not and work in $V[G_\kappa]$. We can find $p \in \mathbb{P}_\mathcal{A}$ so that

$$\mathcal{S} = \{X \prec H_{\kappa^+} \mid \mathbb{P}_\mathcal{A}, p \in X \in [H_{\kappa^+}]^\omega \wedge \neg\exists q \leqslant p \; q \text{ is } (X, \mathbb{P}_\mathcal{A}, f)\text{-semigeneric}\}$$

is f-stationary in $[H_{\kappa^+}]^\omega$. Moreover, $\dot{\mathbb{Q}}_\kappa^{G_\kappa} = \mathrm{Col}(\omega, 2^\kappa)$. In $V[G_{\kappa+1}]$, there is a surjection $g \colon \omega_1 \to (H_{\kappa^+})^{V[G_\kappa]}$. As $\mathcal{S}$ is still f-stationary in $[(H_{\kappa^+})^{V[G_\kappa]}]^\omega$ in $V[G_{\kappa+1}]$ by Lemma 3.27, we have that

$$S = \{\alpha < \omega_1 \mid g[\alpha] \in \mathcal{S}\}$$

is f-stationary in $V[G_{\kappa+1}]$. By Corollary 3.53, the extension $V[G_{\kappa+1}] \subseteq V[G]$ preserves f-stationary sets, hence S is f-stationary in $V[G]$. It follows that $S \cap A_i$ is stationary for some $i < \delta$. In fact, $S \cap A_i$ is f-stationary as f witnesses $\diamondsuit^+(\mathbb{B})$ in $V[G]$. Find some V-inaccessible λ with $\kappa, i < \lambda \in C$. Back in V, we can find an elementary embedding

$$j \colon V \to M$$

with M transitive and

$(j.i)$ $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$,

$(j.ii)$ $V_\lambda \subseteq M$,

$(j.iii)$ $M^\kappa \subseteq M$,

$(j.iv)$ $j(\dot\tau) \cap V_\lambda = \dot\tau \cap V^{\mathbb{P}_\lambda}$ and $j(\mathbb{P}_\kappa)_\lambda = \mathbb{P}_\lambda$.

In $V[G_{\kappa+1}]$, build a continuous increasing chain

$$\langle Y_\alpha \mid \alpha < \omega_1 \rangle$$

of countable elementary substructures of $H_{\lambda^+}^{V[G_{\kappa+1}]}$ so that

$(\vec{Y}.i)$ $g, \dot\tau \cap V^{\mathbb{P}_\lambda}, G_{\kappa+1}, \mathcal{S}, \mathbb{P}_\lambda, V^{\mathbb{P}_\lambda} \cap V_\lambda \in Y_0$ and

$(\vec{Y}.ii)$ $j[Y_\alpha \cap V] \subseteq Y_\alpha$ for all $\alpha < \omega_1$.

There is $\alpha < \omega_1$ such that

$$\delta^{Y_\alpha} \in S \cap A_i$$

and $Y_\alpha \sqsubseteq Y_\alpha[G_{\kappa+1,\lambda}]$ is f-slim. Let $Y := Y_\alpha$. As $\dot{\mathbb{Q}}_\kappa^{G_\kappa}$ is σ-closed in $V[G_\kappa]$, $X := Y \cap (H_{\kappa^+})^{V[G_\kappa]} \in V[G_\kappa]$. Note that $X = g[\delta^Y]$ and consequently $X \in \mathcal{S}$. Clearly, G_λ is $\mathbb{P}_\lambda$-generic over M.

Subclaim 3.64. $Y[G_{\kappa+1,\lambda}] \in M[G_\lambda]$.

Proof. As $\mathbb{P}_\kappa$ is of size κ in V,

$$M[G_\kappa]^\kappa \subseteq M[G_\kappa]$$

holds in $V[G_\kappa]$ by $(j.iii)$. Similarly,

$$V_\lambda^{V[G_\kappa]} \subseteq M[G_\kappa]$$

is a consequence of $(j.ii)$ and the regularity of λ. Now $Z := Y \cap V[G_\kappa] \in V[G_\kappa]$ as the extension $V[G_\kappa] \subseteq V[G_{\kappa+1}]$ is σ-distributive. As Z is a countable subset of $V_\lambda^{V[G_\kappa]}$, $Z \in M[G_\kappa]$ and thus

$$Y[G_{\kappa+1,\lambda}] = Z[G_{\kappa,\lambda}] \in M[G_\kappa].$$

□

By Corollary 3.53, the tail $j(\mathbb{P}_\kappa)^{G_\lambda}_{\lambda,j(\kappa)}$ is f-semiproper in $M[G_\lambda]$. As $Y[G_{\kappa+1,\lambda}]$ is f-slim, (in some outer model) we can find a filter H that is $j(\mathbb{P}_\kappa)^{G_\lambda}_{\lambda,j(\kappa)}$-generic over $M[G_\lambda]$ such that

$$Y \sqsubseteq Y[G_{\kappa+1,\lambda}][H] \text{ is } f\text{-slim.}$$

Let $Y_* := Y[G_{\kappa+1,\lambda}][H]$. As κ is V-inaccessible, $\mathbb{P}_\kappa$ is κ-c.c. in V and $\mathbb{P}_\kappa$ is (isomorphic to) the direct limit along $\langle \mathbb{P}_\alpha \mid \alpha < \kappa\rangle$. It follows that $j[G_\kappa] = G_\kappa$. Thus j lifts to

$$j^+ \colon V[G_\kappa] \to M[G_\lambda][H].$$

We find $j^+(X) \in j^+(\mathcal{S})$ and furthermore

$$j^+(X) = j^+[X] \sqsubseteq Y_*$$

follows from $(\vec{Y}.ii)$. In $M[G_\lambda][H]$, we can easily find $q \leqslant j^+(p)$ that is $(Y_*, j(\mathbb{P}_\mathcal{A}), f)$-generic: Build a descending sequence

$$\vec{p} = \langle p_n \mid n < \omega\rangle$$

of conditions in $j(\mathbb{P}_\mathcal{A}) \cap Y_*$ with

$(\vec{p}.i)$ $p_0 = j^+(p)$ and

$(\vec{p}.ii)$ For any dense $D \subseteq \pi_{Y_*}^{-1}(j^+(\mathbb{P}_\mathcal{A}))$, $D \in M_{Y_*}[f(\delta^Y)]$, we have $p_n \in \pi_{Y_*}[D]$ for some $n < \omega$.

As $\delta^{Y_*} = \delta^Y \in A_i \in Y_*$ and

$$A_i \in (\dot\tau \cap V^{\mathbb{P}_\lambda})^{G_\lambda} \subseteq j(\dot\tau \cap V^{\mathbb{P}_\kappa})^{G_\lambda * H} = j^+(\mathcal{A})$$

by $(j.iv)$, there is $q \in j^+(\mathbb{P}_{\mathcal{A}})$ a lower bound of $\vec{p}$. It follows that q is $(Y_*, j^+(\mathbb{P}_{\mathcal{A}}), f)$-generic and hence $(j^+(X), j^+(\mathbb{P}_{\mathcal{A}}), f)$-semigeneric by Proposition 3.56. But then $j^+(X) \notin j^+(\mathcal{S})$, contradiction. $\square$

It follows that in $V[G_{\kappa+1}]$, $\triangledown_{i<\kappa} A_i$ contains a club. But then already $\vec{A} \upharpoonright \kappa$ is maximal in $V[G]$, contradiction. $\square$

We note that "NS_{ω_1} is saturated" has an enhancing effect on the principles $\diamondsuit(\mathbb{B})$.

Definition 3.65. For a forcing $\mathbb{B} \subseteq \omega_1$ and $S \subseteq \omega_1$ stationary, $\diamondsuit^+_S(\mathbb{B})$ holds if there is a function f witnessing $\diamondsuit(\mathbb{B})$ so that

$$\{\alpha \in S \mid f(\alpha) \cap D = \varnothing\} \in \mathrm{NS}_{\omega_1}$$

for all dense $D \subseteq \mathbb{B}$.

If NS_{ω_1} is saturated and I is a normal uniform ideal on ω_1, then for any $S \in I^+$ we have $I \upharpoonright T = \mathrm{NS}_{\omega_1} \upharpoonright T$ for some I-positive $T \subseteq S$ [29]. If f witnesses $\diamondsuit(\mathbb{B})$ then NS_f is a normal uniform ideal on ω_1 by Lemma 2.20 and hence for any f-stationary S there is some f-stationary $T \subseteq S$ with $\mathrm{NS}_f \upharpoonright T = \mathrm{NS}_{\omega_1} \upharpoonright T$. Hence f witnesses $\diamondsuit^+_T(\mathbb{B})$ for such T.
This is not true without any further assumptions: $\diamondsuit(\mathbb{B})$ alone does not prove "$\diamondsuit^+_S(\mathbb{B})$ for some $S \in \mathrm{NS}^+_{\omega_1}$" in case $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$, this is implicit in Corollary 5.18. Moreover, the conclusion cannot be strengthened to $\diamondsuit^+(\mathbb{B})$ in general: "NS_{ω_1} is saturated"+$\diamondsuit(\mathbb{B})$ does not imply $\diamondsuit^+(\mathbb{B})$ in case $\mathbb{B}$ is Cohen forcing, see Theorem 5.32.

3.7 $\mathrm{MM}^{++}(f)$ and related forcing axioms

We will formulate a new MM-style forcing axiom called $\mathrm{MM}(f)$, together with their bounded and ++-counterparts. One can think of $\mathrm{MM}(f)$ as essentially MM conditioned on "f witnesses $\diamondsuit(\mathbb{B})$".

Definition 3.66. Suppose Γ is a class of forcings and f witnesses $\diamondsuit(\mathbb{B})$.

(i) f-$\mathrm{FA}^{++}(\Gamma)$ holds if for any $\mathbb{P} \in \Gamma$ and sets

- $\mathcal{D}$ of at most ω_1-many dense subsets of $\mathbb{P}$,
- $\mathcal{S}$ of at most ω_1-many $\mathbb{P}$-names for f-stationary subsets of ω_1,

there is a filter $g \subseteq \mathbb{P}$ so that

[29] $I \upharpoonright T = \{U \subseteq \omega_1 \mid U \cap T \in I\}$.

$(g.i)$ $g \cap D \neq \varnothing$ for all $D \in \mathcal{D}$ and

$(g.ii)$ $\dot{S}^g := \{\beta < \omega_1 \mid \exists p \in g\ p \Vdash \check{\beta} \in \dot{S}\}$ is f-stationary for all $\dot{S} \in \mathcal{S}$.

(ii) SPFA(f) is FA(f-semiproper).

(iii) SPFA$^{++}(f)$ is f-FA^{++}(f-semiproper).

(iv) MM(f) is FA(f-stationary set preserving).

(v) MM$^{++}(f)$ is f-FA^{++}(f-stationary set preserving).

Our goal is to prove MM$^{++}(f)$ consistent from a supercompact cardinal. Similar as to the construction of a model of MM^{++}, the argument goes through the axiom SPFA$^{++}(f)$. Thus we aim to prove the following Theorem first.

Theorem 3.67. *Suppose f witnesses $\diamondsuit(\mathbb{B})$.*

(i) SPFA$(f) \Leftrightarrow$ MM(f).

(ii) SPFA$^{++}(f) \Leftrightarrow$ MM$^{++}(f)$.

We will prove this via the principle SRP.

Definition 3.68 (Todorčević, [Tod87])**.**

(i) For θ an uncountable cardinal and $\mathcal{S} \subseteq [H_\theta]^\omega$ we define

$$\mathcal{S}^\perp = \{X \in [H_\theta]^\omega \mid \forall Y \in [H_\theta]^\omega (X \sqsubseteq Y \rightarrow Y \notin S)\}.$$

(ii) The Strong Reflection Principle (SRP) holds if: Whenever $\theta \geqslant \omega_2$ is regular, $a \in H_\theta$ and $S \subseteq [H_\theta]^\omega$ then $\mathcal{S} \cup \mathcal{S}^\perp$ contains a continuous increasing ω_1-chain of countable elementary substructures of H_θ containing a, i.e. there is $\langle X_\alpha \mid \alpha < \omega_1\rangle$ so that for all $\alpha < \omega_1$

$(\vec{X}.i)$ $X_\alpha \prec H_\theta$ is countable,

$(\vec{X}.ii)$ $X_\alpha \in \mathcal{S} \cup \mathcal{S}^\perp$,

$(\vec{X}.iii)$ $a \in X_0$,

$(\vec{X}.iv)$ $X_\alpha \in X_{\alpha+1}$ and

$(\vec{X}.v)$ if $\alpha \in \mathrm{Lim}$ then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

Lemma 3.69. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Then* SPFA$(f) \Rightarrow$ SRP*.*

Proof. We argue that the proof of SRP from SPFA still goes through. Let $\theta \geqslant \omega_2$ be regular and suppose $a \in H_\theta$ and $\mathcal{S}' \subseteq [H_\theta]^\omega$. Let $2^\theta < \lambda$ be regular. Let

$$\mathcal{S} := \{X \prec H_\lambda \mid X \text{ countable and } X \cap H_\theta \in \mathcal{S}'\}.$$

Let $\mathbb{P}$ be the forcing that shoots a continuous increasing ω_1-chain of countable elementary substructures of H_λ containing a through $\mathcal{S} \cup \mathcal{S}^\perp$. Conditions in $\mathbb{P}$ are countable initial segments of such a sequence of non-limit length, that is a sequence $\langle X_\alpha \mid \alpha < \gamma + 1\rangle$ for some countable γ, that satisfies $(\vec{X}.i)$-$(\vec{X}.v)$ for all $\alpha \leqslant \gamma$, ordered by end-extension. We will show that $\mathbb{P}$ is f-semiproper.
Let $\mu > \lambda$ be sufficiently large and regular, $Y \prec H_\mu$ countable with $f, p, \mathcal{S}, \theta, \lambda \in Y$. Let $p_0 \in \mathbb{P} \cap Y$. We may assume that for some wellorder $\lhd$ of H_μ,

$$(Y; \in, \lhd \cap Y) \prec (H_\mu; \in, \lhd)$$

Claim 3.70. *There is a countable $Y \sqsubseteq Z \prec H_\mu$ with $Z \cap H_\lambda \in \mathcal{S} \cup \mathcal{S}^\perp$.*

The argument is due to Shelah, we give a proof for the convenience of the reader.

Proof. If $Y \cap H_\lambda \in \mathcal{S}^\perp$, we can take $Z = Y$, so let us assume otherwise. Then there is some $Y \cap H_\lambda \sqsubseteq Z' \prec H_\lambda$ with $Z' \in \mathcal{S}$. We set

$$Z = \mathrm{Hull}^{(H_\mu; \in, \lhd)}(Y \cup (Z' \cap H_\theta)).$$

Subclaim 3.71. $Z \cap H_\theta = Z' \cap H_\theta$.

Proof. Suppose $\tau(v_0, v_1)$ is a term, $p \in Z' \cap H_\theta, q \in Y$ are parameters and

$$r := \tau(p, q)^{(H_\theta; \in, \lhd)} \in H_\theta$$

Next, we define a function

$$h\colon H_\theta \to H_\theta,\ h(x) = \begin{cases} \tau(x, q)^{(H_\mu; \in, \lhd)} & \text{if this is in } H_\theta \\ \varnothing & \text{otherwise.} \end{cases}$$

We have $h \in Y \cap H_\lambda \subseteq Z'$ and thus $r = h(p) \in Z'$. □

It follows from the definition of $\mathcal{S}$ and from $Z' \in \mathcal{S}$ that $Z \cap H_\lambda \in \mathcal{S}$. □

Let $\delta = \delta^Z$. Note that f witnesses $\diamondsuit^+(\mathbb{B})$ by Theorem 3.30 and thus $f(\delta)$ is generic over M_Z. We now build a descending sequence $\langle p_n \mid n < \omega\rangle$ in $\mathbb{P} \cap Z$ so that for any dense set D of $\pi_Z^{-1}(\mathbb{P})$ in $M_Z[f(\delta)]$,

$$p_n \in \pi_Z[D]$$

for some n. Let $\langle X_\alpha \mid \alpha < \delta\rangle$ be the limit of $(p_n)_{n<\omega}$. The point is that

$$X_\delta := \bigcup_{\alpha<\delta} X_\alpha = Z \cap H_\lambda \in \mathcal{S} \cup \mathcal{S}^\perp$$

and thus $q = \langle X_\alpha \mid \alpha < \delta + 1\rangle$ is a condition in $\mathbb{P}$. Clearly q is $(Z, \mathbb{P}, f)$-generic by construction and thus by Proposition 3.56 also $(Y, \mathbb{P}, f)$-semigeneric. Applying $\mathrm{SPFA}(f)$ to $\mathbb{P}$ yields a continuous chain

$$\vec{X} := \langle X_\alpha \mid \alpha < \omega_1\rangle \in V$$

of elementary substructures of H_λ with all $X_\alpha \in \mathcal{S} \cup \mathcal{S}^\perp$ and $a \in X_0$. □

Lemma 3.72. *Suppose f witnesses $\diamondsuit^+(\mathbb{B})$ and* SRP *holds. The following are equivalent for any forcing $\mathbb{P}$:*

(*i*) *$\mathbb{P}$ is f-semiproper.*

(*ii*) *$\mathbb{P}$ preserves f-stationary sets.*

We will prove something stronger in Lemma 7.15, so we will skip the proof for now.

Theorem 3.67 follows immediately from Lemma 3.69 and 3.72.

Theorem 3.73. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and κ is supercompact. Then there is a generic extension by f-semiproper forcing in which* $\mathrm{MM}^{++}(f)$ *holds.*

Proof. Do the usual construction that forces SPFA^{++} but replace semiproper by f-semiproper, stationary by f-stationary and build a nice iteration instead of a RCS-iteration. Our iteration theorem for f-semiproper forcing yields a f-semiproper $\mathbb{P}$ with $V^{\mathbb{P}} \models \mathrm{SPFA}^{++}(f)$. Here we use that tails of the iteration $\mathbb{P}$ are f-stationary set preserving, i.e. Corollary 3.53. By Theorem 3.67, $\mathrm{MM}^{++}(f)$ is valid in $V^{\mathbb{P}}$. □

Fact 3.74 (Woodin, [Woo10, Lemma 5.14])**.** $\psi_{\mathrm{AC}} \Rightarrow 2^\omega = 2^{\omega_1} = \omega_2$.

Theorem 3.75. *Assume* $\mathrm{MM}(f)$. *Then we have*

(*i*) $2^\omega = 2^{\omega_1} = \omega_2$,

(*ii*) *f witnesses $\diamondsuit^+(\mathbb{B})$,*

(*iii*) NS_{ω_1} *is saturated,*

(*iv*) ψ_{AC} *and*

(*v*) $\neg\square_\kappa$ *for all $\kappa \geqslant \omega_2$.*

Proof. (*i*) follows from (*iv*) by Fact 3.74, (*ii*) follows already from $\mathrm{PFA}(f)$ by Theorem 3.30 and (*iii*)-(*v*) are consequences of SRP which holds by Lemma 3.69.

□

We note here the following consequence for later use:

Corollary 3.76. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and there is a supercompact cardinal. Then there is a f-stationary set preserving forcing $\mathbb{P}$ with $V^{\mathbb{P}} \models$* SRP.

We finally show that MM(f) is a maximal forcing axiom in the sense that the class of forcings it applies to cannot be increased given that f is a witness of $\diamondsuit(\mathbb{B})$. To make the ride smoothly we introduce the relevant bounded forcing axioms.

Definition 3.77. Let Γ be a class of forcings.

(*i*) f-BFA$^{++}(\Gamma)$ holds iff for any $\mathbb{P} \in \Gamma$ we have

$$(H_{\omega_2}; \in, \mathrm{NS}_f)^V \prec_{\Sigma_1} (H_{\omega_2}; \in, \mathrm{NS}_f)^{V^{\mathbb{P}}}.$$

(*ii*) For f a witness of $\diamondsuit(\mathbb{B})$, BMM(f) is BFA(f-stationary set preserving) and BMM$^{++}(f)$ is f-BFA$^{++}(f -$ stationary set preserving).

Joan Bagaria has proven (a stronger theorem than) the following in [Bag00]:

Fact 3.78 (Bagaria, [Bag00])**.** *Let $\mathbb{P}$ be any forcing. Then* FA$^{++}(\mathbb{P}) \Rightarrow$ BFA$^{++}(\{\mathbb{P}\})$.

Moreover, his methods show the following:

Lemma 3.79. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. For any forcing $\mathbb{P}$, we have* FA$^{++}(\{\mathbb{P}\}) \Rightarrow f$-BFA$^{++}(\{\mathbb{P}\})$.

Lemma 3.80. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and $\mathbb{P}$ is a forcing which is not f-stationary set preserving. Then* BFA$(\{\mathbb{P}\})$ *is false.*

Proof. There is then some $p \in \mathbb{P}$ and S f-stationary so that

$$p \Vdash \text{“}\check{S} \text{ is not } f\text{-stationary”}.$$

Let G be $\mathbb{P}$-generic with $p \in G$. We have:

$$(H_{\omega_2}^{V[G]}, \in) \models \text{“}\exists \langle D_\alpha \mid \alpha < \omega_1\rangle \text{ a sequence of dense subsets of } \mathbb{B} \text{ with } \{\alpha \in S \mid \forall \beta < \alpha\ f(\alpha) \cap D_\beta \neq \varnothing\} \text{ nonstationary”}.$$

Note that this is a Σ_1-statement in parameters $\omega_1, \mathbb{B}, f$ which would be true in $(H_{\omega_2}^V; \in)$ if BFA$(\{\mathbb{P}\})$ would hold by Fact 3.78. This would contradict the f-stationarity of S in V. $\square$

4 Blueprints for Instances of "$\mathrm{MM}^{++} \Rightarrow (*)$"

We modify the $(*)$-forcing method of Asperó-Schindler in a way that allows us to prove a variety of instances of $\mathrm{MM}^{++} \Rightarrow (*)$. The results of this section also play a crucial role in answering Woodin's question as we use they provide tools to prove Lemma 7.41.

Definition 4.1. Let $\mathbb{P} \in L(\mathbb{R})$ be a forcing. $\mathbb{P}$-$(*)$ asserts that AD holds in $L(\mathbb{R})$ and there is a filter $g \subseteq \mathbb{P}$ with

(i) g is $\mathbb{P}$-generic over $L(\mathbb{R})$ and

(ii) $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g]$.

$(*)$ is $\mathbb{P}_{\max}$-$(*)$. $\mathbb{P}_{\max}$ is the most prominent of a number of similar forcing notions defined and analyzed by Woodin in [Woo10]. A central notion to all of them is that of a generically iterable structure.

Definition 4.2. Suppose the following holds:

$(M.i)$ $(M; \in, I)$ is a countable transitive model of (sufficiently much of) ZFC where I is allowed as a class parameter in the schemes.

$(M.ii)$ $(M; \in, I) \models$ "I is a normal uniform ideal on ω_1".

$(M.iii)$ $a_0, \ldots, a_n \in M$.

In this case, we call $(M, I, a_0, \ldots, a_n)$ a *potentially iterable structure*. A *generic iteration* of $(M, I, a_0, \ldots, a_n)$ is a sequence

$$\langle (M_\alpha, I_\alpha, a_{0,\alpha}, \ldots, a_{n,\alpha}), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$$

with

- $(M_0, I_0) = (M, I)$,
- $a_{i,\alpha} = \mu_{0,\alpha}(a_i)$ for $i \leqslant n$,
- $\mu_{\alpha,\alpha+1} \colon (M_\alpha; \in, I_\alpha) \to (M_{\alpha+1}; \in, I_{\alpha+1})$ is a generic ultrapower of M_α w.r.t I_α and
- if $\alpha \in \mathrm{Lim}$ then

$$\langle (M_\alpha; \in, I_\alpha), \mu_{\beta,\alpha} \mid \beta < \alpha \rangle = \varinjlim \langle (M_\beta; \in, I_\beta), M_{\beta,\xi} \mid \beta \leqslant \xi < \alpha \rangle$$

for all $\alpha \leqslant \gamma$. $(M, I, a_0, \ldots, a_n)$ is a generically iterable structure if all (countable) generic iterations of $(M, I, a_0, \ldots, a_n)$ produce wellfounded models. Note that this only depends on (M, I) and that we do not require $I \in M$.

Remark 4.3. A generic iteration $\langle (M_\alpha, I_\alpha, a_{0,\alpha}, \ldots, a_{n,\alpha}), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$ can be read off from the final map $\mu_{0,\gamma} \colon M_0 \to M_\gamma$, so we will frequently identify one with the other. We also reserve the right to call generic iterations simply iterations.

Definition 4.4. $\mathbb{P}_{\max}$-conditions are generically iterable structures (M, I, a) with $a \in \mathcal{P}(\omega_1)^M$ and $M \models \omega_1^{L[a]} = \omega_1$. $\mathbb{P}_{\max}$ is ordered by $q = (N, J, b) <_{\mathbb{P}_{\max}} p$ iff there is a generic iteration

$$\mu \colon p \to p^* = (M^*, I^*, a^*)$$

of length $\omega_1^q + 1$ in q so that

($<_{\mathbb{P}_{\max}} .i$) $I^* = J \cap M^*$ and

($<_{\mathbb{P}_{\max}} .ii$) $a^* = b$.

There are a number of ways this definition can be varied, leading to different partial orders. We will work with such variants in a general context.

4.1 $\mathbb{P}_{\max}$-variations and the $\mathbb{V}_{\max}$-multiverse view

Definition 4.5. A $\mathbb{P}_{\max}$-variation is a nonempty projective preorder $(\mathbb{V}_{\max}, \leqslant_{\mathbb{V}_{\max}})$ with the following properties:

($\mathbb{V}_{\max}.i$) Conditions in $\mathbb{V}_{\max}$ are generically iterable structures $(M, I, a_0, \ldots, a_n)$ for some fixed $n = n^{\mathbb{V}_{\max}}$[30].

($\mathbb{V}_{\max}.ii$) There is a first order formula $\varphi^{\mathbb{V}_{\max}}$ in the language[31] $\{\in, \dot{I}, \dot{a}_0, \ldots, \dot{a}_n\}$ so that $q = (N, J, b_0, \ldots b_n) <_{\mathbb{V}_{\max}} (M, I, a_0, \ldots a_n)$ iff there is a generic iteration
$$j \colon p \to p^* = (M^*, I^*, a_0^*, \ldots, a_n^*)$$
in N of length $\omega_1^N + 1$ with
$$(N; \in, J, b_0, \ldots, b_n) \models \varphi^{\mathbb{V}_{\max}}(p^*).$$

($\mathbb{V}_{\max}.iii$) If $\mu \colon p \to p^*$ witnesses $q <_{\mathbb{V}_{\max}} p$ and $\sigma \colon q \to q^*$ witnesses $r <_{\mathbb{V}_{\max}} q$ then $\sigma(\mu) \colon p \to \sigma(p^*)$ witnesses $r <_{\mathbb{V}_{\max}} p$.

($\mathbb{V}_{\max}.iv$) Suppose (M, I) is generically iterable, $j \colon (M, I) \to (M^*, I^*)$ is a generic iteration of (M, I) of countable length and $a_0, \ldots a_n \in M$. Then
$$(M, I, a_0, \ldots, a_n) \in \mathbb{V}_{\max} \Leftrightarrow (M^*, I^*, j(a_0), \ldots, j(a_n)) \in \mathbb{V}_{\max}.$$

[30] Of course, not all structures of this form are necessarily conditions.

[31] When dealing with $\mathbb{P}_{\max}$-variations, we stick to the convention that capitalized symbols are unary predicates symbols which are lower case are constants.

($\mathbb{V}_{\max}.v$) $\mathbb{V}_{\max}$ has no minimal conditions.

We always consider $\mathbb{P}_{\max}$-variations as a class defined by a projective formula, rather then the set itself. So if we mention $\mathbb{V}_{\max}$ in, e.g. a forcing extension of V, then we mean the evaluation of the projective formula in that model[32].

Remark 4.6. Typically, $\varphi^{\mathbb{V}_{\max}}$ dictates e.g. one or more of the following:

- $a_0^* = b_0, \ldots, a_n^* = b_n$.
- $I^* = J \cap M^*$.
- Some first order property is absolute between M^* and N.

We want to relate forcing axioms to star axioms of the form $\mathbb{V}_{\max}$-$(*)$ for $\mathbb{P}_{\max}$-variations $\mathbb{V}_{\max}$. To explain this relationship heuristically we present the $\mathbb{V}_{\max}$-Multiverse View:
Suppose $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variation (with $n^{\mathbb{V}_{\max}} = 0$ for convenience) and

- $V = (V_\kappa)^{\mathcal{V}}$ for some large cardinal κ in some larger model $\mathcal{V}$ and
- there are a proper class of Woodin cardinals both in V and $\mathcal{V}$.

We will take the point of view of $\mathcal{V}^{\mathrm{Col}(\omega,\kappa)}$. Note that our assumptions imply generic projective absoluteness (and more) in $\mathcal{V}$, in particular $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variation also in $\mathcal{V}^{\mathrm{Col}(\omega,\kappa)}$ and $\mathbb{V}_{\max}^W = \mathbb{V}_{\max} \cap W$ for any generic extension of V. Pick some $\vec{A} = (A_0, \ldots, A_{n^{\mathbb{V}_{\max}}}) \in H_{\omega_2}^V$. Let $\mathcal{M}(V)$ denote the closure of V under generic extensions and grounds containing $\vec{A}$. Points $W \in \mathcal{M}(V)$ may be considered as $\mathbb{V}_{\max}$-conditions if

$$(W, \mathrm{NS}_{\omega_1}^W, A_0, \ldots, A_{n^{\mathbb{V}_{\max}}}) \in \mathbb{V}_{\max}.$$

In this case we identify W with this condition. In practice, this can only reasonably hold if $\omega_1^W = \omega_1^V$ so we make this an explicit condition. The $\mathbb{V}_{\max}$-multiverse of V (w.r.t. $\vec{A}$) is

$$\mathcal{M}_{\mathbb{V}_{\max}}(V) = \{W \in \mathcal{M}(V) \mid W \in \mathbb{V}_{\max} \wedge \omega_1^W = \omega_1^V\}.$$

If we $\vec{A}$ picked with sufficient care then $\mathcal{M}_{\mathbb{V}_{\max}}(V)$ should be nonempty. If $W[G]$ is a generic extension of W, both in $\mathcal{M}_{\mathbb{V}_{\max}}(V)$, then it is a good extension if

$$W[G] \leqslant_{\mathbb{V}_{\max}} W.$$

Here, $p \leqslant_{\mathbb{V}_{\max}} q$ means $p \Vdash_{\mathbb{V}_{\max}} \check{q} \in \dot{G}$. The existence of a proper class of Woodin cardinals in V should guarantee that $\mathcal{M}_{\mathbb{V}_{\max}}(V)$ reversely ordered

[32] In practice this extension will be projectively absolute so it does not matter which projective formula we choose. Also all the variations we consider will have a Π_2^1-definition.

by good extensions is "as rich as" $\mathbb{V}_{\max}$.
In this sense, iterated forcing along good extensions corresponds to building descending sequences in $\mathbb{V}_{\max}$. In practice, $\mathbb{P}_{\max}$-variations are σ-closed. From this point of view, σ-closure of $\mathbb{V}_{\max}$ becomes roughly equivalent to a forcing iteration theorem: If

$$\langle W[G_\alpha] \mid \alpha < \gamma \rangle$$

is a chain of good extensions $W[G_\alpha] \subseteq W[G_\beta]$ of points

$$W[G_\alpha], W[G_\beta] \in \mathcal{M}_{\mathbb{V}_{\max}}(V),\ \alpha \leqslant \beta < \gamma \in V$$

then this constitutes a countable decreasing chain[33] in $\mathbb{V}_{\max}$ in $\mathcal{V}^{\mathrm{Col}(\omega,\kappa)}$. σ-closure of $\mathbb{V}_{\max}$ suggests that there should be a further point

$$W[G_\gamma] \in \mathcal{M}_{\mathbb{V}_{\max}}(V)$$

below all $W[G_\alpha]$, $\alpha < \gamma$. Thus the "forcing iteration along $\langle W[G_\alpha] \mid \alpha < \gamma \rangle$" preserves ω_1 and enough structure to be able to be extended to a $\mathbb{V}_{\max}$-condition below all $W[G_\alpha]$ without collapsing ω_1.
We should be able to find points satisfying $\mathbb{V}_{\max}$-$(*)$ by constructing "closure points" $W \in \mathcal{M}_{\mathbb{V}_{\max}}(V)$ of sufficiently generic $\leqslant_{\mathbb{V}_{\max}}$-decreasing sequences

$$\langle W_\alpha \mid \alpha < \gamma \rangle$$

in $\mathcal{M}_{\mathbb{V}_{\max}}(V)$. To make that precise, we want:

$$\text{If } D \in L(\mathbb{R})^W \text{ is dense open in } \mathbb{V}_{\max}^W \text{ then } W_\alpha \in D^* \text{ for some } \alpha < \gamma. \quad (\bigstar)$$

Here, D^* is the reinterpretation of the universally Baire D in $\mathcal{V}^{\mathrm{Col}(\omega,\kappa)}$. The degree of closure of $W \in \mathcal{M}_{\mathbb{V}_{\max}}(V)$ under this procedure is measured by

$$g^W = \{p \in \mathbb{V}_{\max} \mid W <_{\mathbb{V}_{\max}} p\}$$

which should be a filter if W is "sufficiently closed". g^W can be defined in W via

$$g^W = \{p \in \mathbb{V}_{\max} \mid \exists \mu \colon p \to p^* \text{ of length } \omega_1 + 1 \text{ with } \varphi^{\mathbb{V}_{\max}}(p^*)\}^W$$

if $\mathbb{V}_{\max}$ has unique iterations.

Definition 4.7. $\mathbb{V}_{\max}$ has unique iterations if whenever $q <_{\mathbb{V}_{\max}} p$ then there is a unique generic iteration of p witnessing this.

[33] Note that the size of γ in V does not matter here.

Under reasonable assumptions, ($\bigstar$) implies that g^W is generic over $L(\mathbb{R})^W$. Finally, an additional property[34] like $W \models$ "NS_{ω_1} is saturated" should imply $\mathcal{P}(\omega_1)^W \subseteq L(\mathbb{R})^W[g^W]$.

Taking a step back, forcing a forcing axiom related to good extensions via iterated forcing looks like it should produce such sequences $\langle W_\alpha \mid \alpha < \gamma\rangle$ with ($\bigstar$) and NS_{ω_1} saturated in W, so $\mathbb{V}_{\max}$-$(*)$ should follow from such a forcing axiom.

On the other hand, W looks like an endpoint of an iteration liberally incorporating forcings leading to good extensions: For $\alpha < \gamma$, if $D \in L(\mathbb{R})^{W_\alpha}$ is dense open in $\mathbb{V}_{\max}^{W_\alpha}$ then D^* is dense open in the full $\mathbb{V}_{\max}$. D^* can also be considered as a dense subset of $\mathcal{M}_{\mathbb{V}_{\max}}(V)$. As $D^* \cap \mathbb{V}_{\max}^W \in L(\mathbb{R})^W$, by ($\bigstar$), there will be some later $\alpha \leqslant \beta < \gamma$ with $W_\beta \in D^*$. Thus one might expect a forcing axiom to hold at W. This suggest that $\mathbb{V}_{\max}$ should in fact be *equivalent* to a forcing axiom related to good extensions. The consistency of this forcing axiom should follow from the iteration theorem suggested by the σ-closure of $\mathbb{V}_{\max}$.

If we look at the case $\mathbb{V}_{\max} = \mathbb{P}_{\max}$ and let A be some subset of ω_1 so that $\omega_1^{L[A]} = \omega_1^V$ then stationary set preserving extensions are exactly the generic extensions intermediate to a good extension. The $\mathbb{P}_{\max}$-Multiverse View is roughly correct in the sense that:

- (Woodin) $\mathbb{P}_{\max}$ is σ-closed assuming $\mathrm{AD}^{L(\mathbb{R})}$.
- (Shelah) Semiproper forcings can be iterated and the class of stationary set preserving forcings and semiproper forcings coincide under MM.
- (Asperó-Schindler) If there is a proper class of Woodin cardinals then
$$(*) \Leftrightarrow (\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))\text{-}\mathrm{BMM}^{++}.$$

The rest of this section distills this heuristic into rigorous mathematics that relates more $\mathbb{P}_{\max}$-variations to forcing axioms. We will assume (two-step) generic absoluteness in this section, though this is not fully necessary. Note that in this case, if $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variation then we have

$$V^{\mathbb{P}} \models \text{“}\mathbb{V}_{\max} \text{ is a } \mathbb{P}_{\max}\text{-variation”}$$

in any generic extension $V^{\mathbb{P}}$, where $\mathbb{V}_{\max}$ is to be understood as defined by a projective formula. Usually, $\mathbb{P}_{\max}$-variations are Π^1_2.

We will from now on work with some fixed $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ and assume $n^{\mathbb{V}}_{\max} = 0$ to ease notation.

[34] Often, simply $(\neg\mathrm{CH})^W$ is enough. Woodin [Woo] (see also [Sch]) has shown that if $\mathrm{AD}^{L(\mathbb{R})}$ holds, there is a filter $g \subseteq \mathbb{P}_{\max}$ generic over $L(\mathbb{R})$ and CH fails then g witnesses $(*)$.

Definition 4.8. We say that a structure $\mathcal{H}$ is *almost a* $\mathbb{V}_{\max}$*-condition* if

$$V^{\mathrm{Col}(\omega,\mathcal{H})} \models \check{\mathcal{H}} \in \mathbb{V}_{\max}.$$

For $A \in H_{\omega_2}$, $\mathcal{H}_A$ denotes the structure:

$$\mathcal{H}_A := (H_{\omega_2}, \mathrm{NS}_{\omega_1}, A)$$

Suppose that for some fixed $A \in H_{\omega_2}$ we have that $\mathcal{H} := \mathcal{H}_A$ is almost a $\mathbb{V}_{\max}$-condition. We may define

$$g_A = \{p \in \mathbb{V}_{\max} \mid V^{\mathrm{Col}(\omega,2^{\omega_1})} \models \mathcal{H} <_{\mathbb{V}_{\max}} p\}.$$

Our goal is to show that g_A witnesses $\mathbb{V}_{\max}$-$(*)$ under favorable circumstances. At the very least, it should be a filter.

Proposition 4.9. *Suppose g_A meets all projective dense $D \subseteq \mathbb{V}_{\max}$. Then g_A is a filter.*

Proof. It is easy to see that if $q <_{\mathbb{V}_{\max}} p$ and $q \in g_A$ then $p \in g_A$. So assume $p, q \in g_A$ and we have to find some $r \in g_A$ with $r \leqslant_{\mathbb{V}_{\max}} p, q$. Consider

$$D = \{r \in \mathbb{V}_{\max} \mid r \leqslant_{\mathbb{V}_{\max}} p, q \vee r \perp p \vee r \perp q\}$$

and note that D is a projective dense subset of $\mathbb{V}_{\max}$, so by assumption we can find some $r \in D \cap g_A$. Now in $V^{\mathrm{Col}(\omega,2^{\omega_1})}$ we have $r, p, q \leqslant_{\mathbb{V}_{\max}} \mathcal{H}$ and thus r is compatible with both p and q. By generic absoluteness, this is true in V as well so that $r \leqslant_{\mathbb{V}_{\max}} p, q$ as $r \in D$. □

Even assuming that g_A is a fully generic over $L(\mathbb{R})$, we still have to arrange $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g_A]$.

Definition 4.10. Suppose that

(*i*) $g \subseteq \mathbb{V}_{\max}$ is a filter,

(*ii*) $p \in g$ and

(*iii*) $\langle p_\alpha, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma\rangle$ is a generic iteration of $p_0 = p$.

Then we say that $\langle p_\alpha, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma\rangle$ is guided by g if $p_\alpha \in g$ for all countable $\alpha \leqslant \gamma$.

Lemma 4.11. *Suppose $\mathbb{V}_{\max}$ has unique iterations and $g \subseteq \mathbb{V}_{\max}$ is a filter meeting all projective dense $D \subseteq \mathbb{V}_{\max}$. For any $p \in g$ and any $\gamma \leqslant \omega_1$, there is a unique iteration*

$$\langle p_\alpha, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma\rangle$$

of $p_0 = p$ of length $\gamma + 1$ guided by g.

Proof. First, we prove existence for all $\gamma < \omega_1$.

Claim 4.12. *There is $q \in g$ with $\omega_1^q > \gamma$.*

Proof. Let $D = \{q \in \mathbb{V}_{\max} \mid \omega_1^q > \gamma\}$. Clearly, D is projective and we will show that D is dense. Let $q \in \mathbb{V}_{\max}$ and using ($\mathbb{V}_{\max}.v$), find $r <_{\mathbb{V}_{\max}} q$ as witnessed by

$$\sigma\colon q \to q^*.$$

Now let

$$\nu\colon r \to r^*$$

be any generic iteration of r of length $\gamma+2$, consequently $\omega_1^{r^*} > \gamma$. We have $r^* \in \mathbb{V}_{\max}$ by ($\mathbb{V}_{\max}.iv$). Note that the iteration $\nu \circ \sigma$ witnesses $r^* <_{\mathbb{V}_{\max}} q$. Again applying ($\mathbb{V}_{\max}.v$), there is $s <_{\mathbb{V}_{\max}} r^*$ and thus $s <_{\mathbb{V}_{\max}} q$ and $s \in D$. Thus $g \cap D \neq \emptyset$. □

As g is a filter, we can find $q <_{\mathbb{V}_{\max}} p$ with $\omega_1^q > \gamma$. Thus if $\mu\colon p \to p^*$ witnesses this then μ is an iteration

$$\langle p_{\alpha,\beta}, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1^q\rangle$$

of length $\omega_1^q + 1 > \gamma + 1$ by ($\mathbb{V}_{\max}.ii$).

Claim 4.13. *$\langle p_{\alpha,\beta}, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma\rangle$ is guided by g.*

Proof. Let $\alpha \leqslant \gamma$. Then μ_{α,ω_1^q} is an iteration of length $\omega_1^q + 1$ in q and $q \models \varphi^{\mathbb{V}_{\max}}(p_{\omega_1^q})$, thus $q <_{\mathbb{V}_{\max}} p_\alpha$ and $p_\alpha \in g$. □

Next we prove uniqueness. By proceeding by induction on $\gamma \leqslant \omega_1$, it is in fact enough to verify the case $\gamma = 1$. Suppose that $\mu_i\colon p \to p_i^*$ is a generic ultrapower of p with $p_i^* \in g$ for $i < 2$. As g is a filter and by ($\mathbb{V}_{\max}.v$), there is $q \in g$ with $q <_{\mathbb{V}_{\max}} p_i^*$ as witnessed by some

$$\mu_i^*\colon p_i^* \to p_i^{**}$$

for $i < 2$ as well as $q <_{\mathbb{V}_{\max}} p$ as witnessed by

$$\mu\colon p \to p^{**}.$$

Let $i < 2$. We have that p, p_i^* are countable in q. As

"p_i^* is a generic ultrapower of p"

is a true $\Sigma_1^1(p, p_i^*)$-statement, it is true in q as well. Thus there is a generic ultrapower

$$\mu_i'\colon p \to p_i^*$$

in q. Both $\mu, \mu_i^* \circ \mu_i'$ witness $q <_{\mathbb{V}_{\max}} p$ and as $\mathbb{V}_{\max}$ has unique iterations, $\mu = \mu_i^* \circ \mu_i'$. It follows that $p_0^* = p_1^*$.

Claim 4.14. $\mu_0^* = \mu_1^*$.

Proof. Assume this fails, then

$$\text{“There are distinct generic ultrapower maps } p \to p_0^*\text{”}$$

is another true $\Sigma_1^1(p, p_0^*)$-statement which accordingly must hold in q. Thus there is a generic ultrapower map $\mu_0'': p \to p_0^*$ in q different from μ_0'. But then both $\mu_0^* \circ \mu_0'$ and $\mu_0^* \circ \mu_0''$ witness $q <_{\mathbb{V}_{\max}} p$, which contradicts that $\mathbb{V}_{\max}$ has unique iterations. □

Finally, existence of a generic iteration of p of length $\omega_1 + 1$ guided by g follows from existence and uniqueness of generic iterations of p guided by g of any countable length. □

This suggests the following definition:

Definition 4.15. Suppose $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variation with unique iterations and $g \subseteq \mathbb{V}_{\max}$ is a filter. For $p \in g$, the *g-iteration* of p is the unique generic iteration of p of length $\omega_1 + 1$ that is guided by g (if it exists).

Corollary 4.16. *Suppose that*

(*i*) AD *holds in* $L(\mathbb{R})$,

(*ii*) $\mathbb{V}_{\max}$ *has unique iterations,*

(*iii*) $\mathcal{H}_A$ *is almost a* $\mathbb{V}_{\max}$*-condition,*

(*iv*) $g_A \cap D \neq \emptyset$ *for all dense* $D \subseteq \mathbb{V}_{\max}$, $D \in L(\mathbb{R})$ *and*

(*v*) $\mathcal{P}(\omega_1) = \bigcup\{\mathcal{P}(\omega_1) \cap p^* \mid p \in g_A \wedge \mu\colon p \to p^* \text{ is guided by } g_A\}$.

Then $\mathbb{V}_{\max}$-$(*)$ *holds and* g_A *witnesses this.*

Proof. g_A is a filter by Proposition 4.9 and thus $L(\mathbb{R})$-generic by assumption. To see that $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g_A]$, notice that for any $p \in g_A$, $L(\mathbb{R})$ knows of all countable generic iterations of p. Hence, $L(\mathbb{R})[g_A]$ can piece together the g_A-iteration of p from the countable iterations of p that are guided by g_A. $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g_A]$ now follows immediately from (v). □

The biggest obstacle by far is to get into a situation where $g_A \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{V}_{\max}$, $D \in L(\mathbb{R})$. The main idea is:

Lemma 4.17. *Suppose that all of the following hold:*

(*i*) $D \subseteq \mathbb{V}_{\max}$ *is dense.*

(*ii*) $\mathcal{H}_A$ *is almost a* $\mathbb{V}_{\max}$*-condition.*

(iii) $\mathbb{P}$ *is a forcing and* D *is* $|\mathbb{P}|$*-universally Baire.*

(iv) *In* $V^{\mathbb{P}}$ *there is* $q \in D^*$ *and an iteration* $\sigma\colon q \to q^*$ *with*

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A)^{V^{\mathbb{P}}} \models \varphi^{\mathbb{V}_{\max}}(q^*).$$

(v) Γ *is a set of formulas in the language* $\{\in, \dot{I}, \dot{a}, \dot{D}\}$ *so that*

($\Gamma.i$) $\varphi^{\mathbb{V}_{\max}} \in \Gamma$,

($\Gamma.ii$) $\Sigma_0 \subseteq \Gamma$, *where* Σ_0 *is computed in the language* $\{\in, \dot{D}\}$ *and*

($\Gamma.iii$) Γ *is closed under* $\exists$ *and* $\wedge$.

(vi) $(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A, D)^V \prec_\Gamma (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A, D^*)^{V^{\mathbb{P}}}$.

Then $g_A \cap D \neq \varnothing$.
If additionally

(vii) $H_{\omega_2}^V \subseteq q^*$

then $\mathcal{P}(\omega_1) = \bigcup\{\mathcal{P}(\omega_1) \cap p^* \mid p \in g_A \wedge \mu\colon p \to p^*$ *is guided by* $g_A\}$.

Proof. Observe that $(H_{\omega_2}; \in) \prec_{\Sigma_1} (H_{\omega_2}; \in)^{V^{\mathbb{P}}}$ implies that $\mathbb{P}$ preserves ω_1. The statement

$$\exists q \in \dot{D}\ \exists \sigma\colon q \to q^* \text{ an iteration of length } \omega_1 + 1 \text{ and } \varphi^{\mathbb{V}_{\max}}(q^*)$$

is in Γ and thus is true in

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A, D)^V$$

as witnessed by some $p \in D$ and iteration $\mu\colon p \to p^*$. It follows that μ witnesses $\mathcal{H}_A <_{\mathbb{V}_{\max}} q$ in $V^{\mathrm{Col}(\omega, 2^{\omega_1})}$ so that $p \in D \cap g_A$.
Now assume *(vii)*, it is our duty to show

$$\mathcal{P}(\omega_1) = \bigcup\{\mathcal{P}(\omega_1) \cap p^* \mid p \in g_A \wedge \mu\colon p \to p^* \text{ is guided by } g_A\}.$$

Let $X \subseteq \omega_1$. As above,

$$\exists q \in \mathbb{V}_{\max}\ \exists \sigma\colon q \to q^* \text{ an iteration of length } \omega_1 + 1 \text{ and } \varphi^{\mathbb{V}_{\max}}(q^*) \wedge X \in q^*$$

reflects down to V. The iteration witnessing this in V is guided by g_A by the same argument that showed $p \in g_A$ above. □

Condition *(vi)* is a typical consequence of a (bounded) forcing axiom. It is left to construct forcings $\mathbb{P}$ with property *(iv)* to which hopefully a broad range of forcing axioms may apply.

4.2 Asperó-Schindler $(*)$-forcing

We describe the results of Asperó-Schindler[AS21]. Their results carry over to any $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ though they were originally proven in the case of $\mathbb{V}_{\max} = \mathbb{P}_{\max}$. Suppose that

(*i*) NS_{ω_1} is saturated,

(*ii*) $A \in H_{\omega_2}$ is so that $\mathcal{H} = (H_{\omega_2}, \mathrm{NS}_{\omega_1}, A)$ is almost a $\mathbb{V}_{\max}$-condition and

(*iii*) $D \subseteq \mathbb{V}_{\max}$ is a 2^{ω_1}-universally Baire dense subset of $\mathbb{V}_{\max}$ whose reinterpretation is still dense in extensions by forcings of size $\leqslant 2^{\omega_1}$, as witnessed by trees T, S with $D = p[T]$.

Asperó-Schindler construct a partial order $\mathbb{P} = \mathbb{P}(\mathbb{V}_{\max}, A, D)$ so that in $V^{\mathbb{P}}$ the following picture

$$\begin{tikzcd}
 & p[T] \arrow[d, phantom, "\Cup"] & \\
 & q_0 = (N, I, b) \arrow[r, "\sigma_{0,\omega_1}"] \arrow[d, phantom, "\Cup"] & q_{\omega_1} = (N^*, I^*, b) \arrow[d, phantom, "\Cup"] \\
p_0 \arrow[r, "\mu_{0,\omega_1^N}"] \arrow[d, phantom, "\Cap"] & p_{\omega_1^N} \arrow[r, "\mu_{\omega_1^N,\omega_1}"] & p_{\omega_1} \arrow[d, phantom, "\parallel"] \\
\mathbb{V}_{\max} & & ((H_{\omega_2})^V, \mathrm{NS}^V_{\omega_1}, A) = \mathcal{H}
\end{tikzcd}$$

exists so that

($\mathbb{P}$.*i*) $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ are generic iterations of p_0, q_0 respectively,

($\mathbb{P}$.*ii*) μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$,

($\mathbb{P}$.*iii*) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$ and

($\mathbb{P}$.*iv*) the generic iteration $\sigma_{0,\omega_1} : q_0 \to q_{\omega_1}$ is *correct*, i.e. $I^* = \mathrm{NS}^{V^{\mathbb{P}}}_{\omega_1} \cap N^*$.

If $\varphi^{\mathbb{V}_{\max}}((M, J, a))$ implies $J = \dot{I} \cap M$ then $\mathrm{NS}^{p_{\omega_1^N}}_{\omega_1} = I \cap p_{\omega_1^N}$. This gets transported upwards along σ_{0,ω_1} and shows $\mathrm{NS}^V_{\omega_1} = I^* \cap H^V_{\omega_2}$. Together with ($\mathbb{P}$.*iv*), this yields $\mathrm{NS}^V_{\omega_1} = \mathrm{NS}^{V^{\mathbb{P}}}_{\omega_1} \cap V$, i.e. $\mathbb{P}$ preserves stationary sets. If MM^{++} holds in V then

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A, D)^V \prec_{\Sigma_1} (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A, D^*)^{V^{\mathbb{P}}}$$

and it follows from Lemma 4.17 that $g_A \cap D \neq \varnothing$ (note that $\varphi^{\mathbb{P}_{\max}}((M, I, a))$"$= I = \dot{I} \cap M \wedge a = \dot{a}$"). This is how Asperó-Schindler prove MM$^{++} \Rightarrow (*)$.
An important observation is the following: To invoke a forcing axiom in the case of $\mathbb{P}$ or variants thereof, typically $\mathbb{P}$ needs to preserve certain structure, like stationary sets in the example above. This preservation is proven in two steps:

(*i*) Preservation between q_{ω_1} and $V^{\mathbb{P}}$. This is governed by the iteration σ_{0,ω_1} having certain properties in $V^{\mathbb{P}}$, e.g. correctness.

(*ii*) Preservation between p_{ω_1} and q_{ω_1}. This is governed by the nature of $\mathbb{V}_{\max}$, specifically the formula $\varphi^{\mathbb{V}_{\max}}$.

We will modify the construction of $\mathbb{P}$ and get a forcing $\mathbb{P}^{\diamondsuit}$ which strengthens ($\mathbb{P}.iv$) so that $\mathbb{P}^{\diamondsuit}$ can have a variety of preservation properties depending on the $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ in question, for example

- preserving stationary sets as well as all Suslin trees ($\rightsquigarrow$ $\mathrm{SM}^{++} \Rightarrow \mathbb{S}_{\max}$-$(*)$, Section 6) or
- preserving a witness f of $\diamondsuit(\omega_1^{<\omega})$ ($\rightsquigarrow$ $\mathrm{QM} \Rightarrow \mathbb{Q}_{\max}$-$(*)$, Section 7).

4.3 $\diamondsuit$-iterations

We introduce the concept that is roughly the equivalent of $\diamondsuit$-forcing in the world of generic iterations.

Definition 4.18. Suppose (N, I) is generically iterable. A generic iteration

$$\langle (N_i, I_i), \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1 \rangle$$

of $(N, I) = (N_0, I_0)$ is a $\diamondsuit$-*iteration* if for any

(*i*) sequence $\langle D_i \mid i < \omega_1 \rangle$ of dense subsets of $((\mathcal{P}(\omega_1)/I_{\omega_1})^+)^{N_{\omega_1}}$ and

(*ii*) $S \in \mathcal{P}(\omega_1)^{N_{\omega_1}} - I_{\omega_1}$

the set

$$\{\xi \in S \mid \forall i < \xi \ g_\xi \cap \sigma^{-1}_{\xi,\omega_1}[D_i] \neq \emptyset\}$$

is stationary. Here, g_ξ is the generic ultrafilter applied to N_ξ for $\xi < \omega_1$.

If (N, I) is generically iterable and $\diamondsuit$ holds then there is a $\diamondsuit$-iteration of (N, I), see Lemma 8.4. But this is not generally the case. Paul Larson noted that if (M, I) is generically iterable and

$$\langle M_\alpha, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$$

is a *generic* generic iteration of $(M, I) = (M_0, I_0)$ of length ω_1 then this is a $\diamondsuit$-iteration. By this we mean that this iteration has been constructed generically by forcing with countable approximations ordered by endextension.

Lemma 4.19. *Suppose*

$$\langle (N_i, I_i), \sigma_{i,j}, g_i \mid i \leqslant j \leqslant \omega_1 \rangle$$

is a $\diamondsuit$*-iteration. If*

$$N_{\omega_1} \models \text{“}f \textit{ witnesses } \diamondsuit^+_{I_{\omega_1}}(\mathbb{B})\text{”}$$

then $I_{\omega_1} = \mathrm{NS}_f \cap N_{\omega_1}$. *In particular,* f *witnesses* $\diamondsuit(\mathbb{B})$.

Proof. Let $S \in \mathcal{P}(\omega_1)^{N_{\omega_1}} - I_{\omega_1}$, we have to show that S is f-stationary. Let $\langle D'_i \mid i < \omega_1 \rangle$ be a sequence of dense subsets of $\mathbb{B}$. As f witnesses $\diamondsuit^+_{I_{\omega_1}}(\mathbb{B})$ in N_{ω_1}, we have

$$N_{\omega_1} \models \text{“}\eta_f \colon \mathbb{B} \to (\mathcal{P}(\omega_1)/I_{\omega_1})^+ \text{ is a complete embedding”}$$

and notice that η_f is a complete embedding in V as well. Thus $D_i = \eta_f[D'_i]$ is dense for $i < \omega_1$. As $\sigma_{0,\omega_1} : N_0 \to N_{\omega_1}$ is a $\diamondsuit$-iteration,

$$T := \{\xi \in S \mid \forall i < \xi \ g_\xi \cap \sigma^{-1}_{\xi,\omega_1}[D_i] \neq \varnothing\}$$

is stationary. Thus if $C \subseteq \omega_1$ is club, we can find $\xi \in C \cap T$ with $\omega_1^{N_\xi} = \xi$ and $f \in \operatorname{ran}(\sigma_{\xi,\omega_1})$. It follows that

$$f(\xi) = \eta^{-1}_{\sigma^{-1}_{\xi,\omega_1}(f)}[g_\xi]$$

so that $f(\xi) \cap D'_i \neq \varnothing$ for all $i < \xi$. □

4.4 $\diamondsuit$-$(*)$-forcing

Theorem 4.20. *Suppose that*

- *(i)* *generic projective absoluteness holds for generic extensions by forcings of size* 2^{ω_1},
- *(ii)* $\mathbb{V}_{\max}$ *is a* $\mathbb{P}_{\max}$*-variation,*
- *(iii)* NS_{ω_1} *is saturated and* $\mathcal{P}(\omega_1)^\sharp$ *exists,*
- *(iv)* $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, A_0, \ldots, A_{n^{\mathbb{V}_{\max}}})$ *is almost a* $\mathbb{V}_{\max}$*-condition and*
- *(v)* $D \subseteq \mathbb{V}_{\max}$ *is* 2^{ω_1}*-universally Baire and dense in* $\mathbb{V}_{\max}$ *in any generic extension by a forcing of size* 2^{ω_1}*, as witnessed by trees* T, S *with* $p[T] = D$.

Then there is a forcing $\mathbb{P}^\diamondsuit$ *so that in* $V^{\mathbb{P}^\diamondsuit}$ *the following picture*

$$\begin{tikzcd}
 & p[T] & \\
 & q_0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\sigma_{0,\omega_1}"] & q_{\omega_1} \\
p_0 \arrow[r, "\mu_{0,\omega_1^N}"] & p_{\omega_1^N} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\mu_{\omega_1^N,\omega_1}"] & p_{\omega_1} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, equal] \\
\mathbb{V}_{\max} \arrow[u, phantom, "\rotatebox{-90}{$\in$}"] & & ((H_{\omega_2})^V, \mathrm{NS}^V_{\omega_1}, A_0, \ldots, A_{n^{\mathbb{V}_{\max}}}) = \mathcal{H}
\end{tikzcd}$$

exists so that

- $(\mathbb{P}^\diamondsuit.i)$ $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ *are generic iterations of* p_0, q_0 *respectively,*

($\mathbb{P}^\diamondsuit$.*ii*) μ_{0,ω_1^N} *witnesses* $q_0 <_{\mathbb{V}_{\max}} p_0$,

($\mathbb{P}^\diamondsuit$.*iii*) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$ *and*

($\mathbb{P}^\diamondsuit$.*iv*) *the generic iteration* $\sigma_{0,\omega_1} \colon q_0 \to q_{\omega_1}$ *is a* $\diamondsuit$*-iteration.*

For the remainder of this section, ω_1 will always denote ω_1^V.
So suppose (i)-(v) holds. We will assume $n^{\mathbb{V}_{\max}} = 0$ for notational purposes. For the most part, we will follow the construction of $\mathbb{P}$ in [AS21] but will put additional constraints on the certificates. The idea that guides us here is:

> In order for $\sigma_{0,\omega_1} \colon q \to q^*$ to be a $\diamondsuit$-iteration, the forcing $\mathbb{P}^\diamondsuit$ will have to anticipate dense subsets of the forcing $(I^+)^{N_{\omega_1}}$ so that they have been "hit before". This should be captured by the map $K \to C$. Formulating this correctly produces a strengthened version of the "genericity condition" put onto semantic certificates.

A reader who can compile the above paragraph without syntax error can probably safely skip most the definition of $\mathbb{P}$ and go straight to (Σ.8).

We try to keep our notation here consistent with the notation in the paper [AS21]. For this reason, we will identify a condition $p = (M, I, a) \in \mathbb{V}_{\max}$ with its first coordinate M. Additionally, by even more abuse of notation:

Convention 4.21. If (N, J, b) is (almost) a condition in $\mathbb{V}_{\max}$, then

- I^N denotes J,
- $(I^+)^N$ denotes $\mathcal{P}(\omega_1)^M - J$ and
- a^N denotes b.

We will additionally assume both $2^{\omega_1} = \omega_2$ and $\diamondsuit_{\omega_3}$ to hold. Otherwise, first force with $\mathrm{Add}(\omega_2, 1) * \mathrm{Add}(((2^{\omega_1})^+)^V, 1)$ and note that (i) and (v) still hold for forcing with $\mathrm{Col}(\omega, \omega_2)$, which is all we need. Moreover, observe that this preserves "NS_{ω_1} is saturated".
We will denote ω_3 by κ and pick a $\diamondsuit_\kappa$-sequence $\langle \bar{A}_\lambda \mid \lambda < \kappa \rangle$.

We may find $T_0 \subseteq T$ of size ω_2 so that

$$V^{\mathrm{Col}(\omega,\omega_2)} \models \exists q \in p[T_0]\ q <_{\mathbb{V}_{\max}} \mathcal{H}.$$

Here we use that $\mathcal{H}$ is almost a $\mathbb{V}_{\max}$-condition as well as ($\mathbb{V}_{\max}.v$). Note that $p[T_0] \subseteq p[T]$ in any outer model. Without loss of generality, we may assume that T_0 is a tree on $\omega \times \omega_2$.
Fix a bijection

$$c \colon \kappa \to H_\kappa.$$

For $\lambda < \kappa$ let
$$Q_\lambda := c[\lambda] \text{ and } A_\lambda := c\left[\bar{A}_\lambda\right].$$
There is then a club $C \subseteq \kappa$ with

(i) $T_0, p \in Q_\lambda$ and $\omega_2 + 1 \subseteq Q_\lambda$,

(ii) $Q_\lambda \cap \mathrm{Ord} = \lambda$ and

(iii) $(Q_\lambda; \in) \prec (H_\kappa; \in)$

for all $\lambda \in C$. We now have

$$(\diamondsuit) \qquad \begin{array}{l}\text{For all } P, B \subseteq H_\kappa \text{ the set}\\ \{\lambda \in C \mid (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B)\}\\ \text{is stationary.}\end{array}$$

We will also define Q_κ as H_κ. The forcing $\mathbb{P}$ will add some
$$(N_0, I_0, a_0) \in D^*$$
together with a generic iteration
$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1 \rangle$$
by Henkin-style finite approximations. By abuse of notation, we let $N_i = (N_i; I_i, a_i)$. For readability we will also write
$$N_{\omega_1} = (N_{\omega_1}, I^*, a^*).$$
$\mathbb{P}^\diamondsuit$ will be the last element of an increasing sequence $\langle \mathbb{P}^\diamondsuit_\lambda \mid \lambda \in C \cup \{\kappa\} \rangle$ of forcings which we define inductively. We will have:

(i) $\mathbb{P}^\diamondsuit_\lambda \subseteq Q_\lambda$,

(ii) conditions in $\mathbb{P}^\diamondsuit_\lambda$ will be finite sets of formulae in a first order language $\mathcal{L}_\lambda$ and

(iii) the order on $\mathbb{P}^\diamondsuit_\lambda$ is reverse inclusion.

Suppose now that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}^\diamondsuit_\nu$ is defined for all $\nu \in C \cap \lambda$.
We will make use of the same convention as Asperó-Schindler.

Convention 4.22. $x \subseteq \omega$ is a real code for $N_0 = (N, I_0, a_0)$ if there is a surjection $f \colon \omega \to N$ so that x is the monotone enumeration of Gödel numbers of all expressions of the form
$$\ulcorner \dot{N} \models \varphi(\dot{n}_1, \ldots, \dot{n}_l, \dot{I}, \dot{a}) \urcorner$$
where φ is a first order formula of the language associated to (N_0, I_0, a_0)(see below) and
$$N \models \varphi(f(n_1), \ldots, f(n_l), I_0, a_0)$$
holds.

We will have conditions in $\mathbb{P}^{\diamondsuit}_{\lambda}$ be certified in a concrete sense by objects $\mathfrak{C}$ which exist in generic extensions of V that satisfies projective absoluteness w.r.t. V. They are of the form

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k_n, \alpha_n) \mid n < \omega\rangle, \langle \lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

where

($\mathfrak{C}$.1) M_0, $N_0 \in \mathbb{V}_{\max}$,

($\mathfrak{C}$.2) $x = \langle k_n \mid n < \omega\rangle$ is a real code for $N_0 = (N_0; \in, I, a_0)$ and $\langle (k_n, \alpha_n) \mid n < \omega\rangle$ is a branch through T_0,

($\mathfrak{C}$.3) $\langle M_i, \mu_{i,j} \mid i \leqslant j \leqslant \omega_1^{N_0}\rangle \in N_0$ is a generic iteration of M_0 witnessing $N_0 <_{\mathbb{V}_{\max}} M_0$,

($\mathfrak{C}$.4) $\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle$ is a generic iteration of N_0,

($\mathfrak{C}$.5) $\langle M_i, \mu_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle = \sigma_{0,\omega_1}(\langle M_i, \mu_{i,j} \mid i \leqslant j \leqslant \omega_1^{N_0}\rangle)$ and

$$M_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathrm{NS}_{\omega_1})^V, A),$$

($\mathfrak{C}$.6) $K \subseteq \omega_1$ and for all $\xi \in K$

($\mathfrak{C}$.6.a) $\lambda_\xi \in \lambda \cap C$, and if $\gamma < \xi$ is in K then $\lambda_\gamma < \lambda_\xi$ and $X_\gamma \cup \{\lambda_\gamma\} \subseteq X_\xi$,

($\mathfrak{C}$.6.b) $X_\xi \prec (Q_{\lambda_\xi}; \in, \mathbb{P}^{\diamondsuit}_{\lambda_\xi}, A_{\lambda_\xi})$ and $\delta^{X_\xi} = \xi$.

If $\mathfrak{C}$ has these properties, we call $\mathfrak{C}$ a *potential certificate*.

Next up, we will define a certain first order language $\mathcal{L}$. $\mathcal{L}$ will have the following distinguished constants

- $\underline{x}$ for any $x \in H_\kappa$,
- $\dot{n}$ for any $n < \omega$,
- $\dot{M}_i$ for $i < \omega_1$,
- $\dot{\mu}_{i,j}$ for $i \leqslant j \leqslant \omega_1$,
- $\dot{\vec{M}}$,
- $\dot{N}_i$ for $i < \omega_1$,
- $\dot{\sigma}_{i,j}$ for $i \leqslant j < \omega_1$,
- $\dot{I}$, $\dot{a}$ and
- $\dot{X}_\xi$ for $\xi < \omega_1$.

The constants $\dot{n}$ will eventually produce "Henkin-style" term models for the N_i. Formulas in the language $\mathcal{L}$ are of the form

$$\ulcorner \dot{N}_i \models \varphi(\underline{\gamma_1}, \ldots, \underline{\gamma_k}, \dot{n}_1, \ldots, \dot{n}_l, \dot{I}, \dot{a}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\mu}_{q_1,r_1}, \ldots, \dot{\mu}_{q_s,r_s}, \dot{\vec{M}}) \urcorner$$

where

- $i < \omega_1$,
- $\gamma_1, \ldots \gamma_k < \omega_1$,
- $n_1, \ldots, n_l < \omega$,
- $j_1, \ldots, j_m < \omega_1$,
- $q_t \leqslant r_t < \omega_1$ for $t \in \{1, \ldots, s\}$

and φ is a first order $\in$-formula. Moreover we allow as formulas

- $\ulcorner \dot{\mu}_{i,\omega_1}(\dot{n}) = \underline{x} \urcorner$ for $i < \omega_1, n < \omega$ and $x \in H_{\omega_2}$,
- $\ulcorner \dot{\mu}_{\omega_1,\omega_1}(\underline{x}) = \underline{x} \urcorner$ for $x \in H_{\omega_2}$,
- $\ulcorner \dot{\sigma}_{i,j}(\dot{n}) = \dot{m} \urcorner$ for $i \leqslant j < \omega_1$ and $n, m < \omega$,
- $\ulcorner (\underline{\vec{k}}, \underline{\vec{\alpha}}) \in \underline{T} \urcorner$ for $\vec{k} \in \omega^{<\omega}$ and $\vec{\alpha} \in \omega_2^{<\omega}$,
- $\ulcorner \underline{\xi} \mapsto \underline{\nu} \urcorner$ for $\xi < \omega_1$ and $\nu < \kappa$ and
- $\ulcorner \underline{x} \in \dot{X}_{\xi} \urcorner$ for $\xi < \omega_1$ and $x \in H_\kappa$.

$\mathcal{L}^\lambda$ is the set of $\mathcal{L}$-formulae φ so that if $\underline{x}$ appears in φ for some $x \in H_\kappa$ then $x \in Q_\lambda$. We assume formulae in $\mathcal{L}^\lambda$ to be coded in a reasonably way (ultimately uniform in λ) so that $\mathcal{L}^\lambda = \mathcal{L} \cap Q_\lambda$. We will not make this precise.

A potential certificate

$$\mathfrak{C} = \langle \langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1 \rangle, \langle (k_n, \alpha_n) \mid n < \omega \rangle, \langle \lambda_\xi, X_\xi \mid \xi \in K \rangle \rangle$$

is (λ-)*precertified* by $\Sigma \subseteq \mathcal{L}^\lambda$ if there are surjections $e_i \colon \omega \to N_i$ for $i < \omega_1$ so that

(Σ.1) $\ulcorner \dot{N}_i \models \varphi(\underline{\gamma_1}, \ldots, \underline{\gamma_k}, \dot{n}_1, \ldots, \dot{n}_l, \dot{I}, \dot{a}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\mu}_{q_1,r_1}, \ldots, \dot{\mu}_{q_s,r_s}, \dot{\vec{M}}) \urcorner \in \Sigma$ iff

(a) $i < \omega_1$,
(b) $\gamma_1, \ldots, \gamma_k \leqslant \omega_1^{N_i}$,
(c) $n_1, \ldots, n_l < \omega$,
(d) $j_1, \ldots, j_m \leqslant \omega_1^{N_i}$,

(e) $q_t \leqslant r_t \leqslant \omega_1^{N_i}$ for $t \in \{1, \ldots, s\}$

and

$$N_i \models \varphi(\gamma_1, \ldots, \gamma_k, e_i(n_1), \ldots, e_i(n_l), I^{N_i}, a^{N_i},\\ M_{j_1}, \ldots, M_{j_m}, \mu_{q_1,r_1}, \ldots, \mu_{q_s,r_s}, \vec{M})$$

where $\vec{M} = \langle M_j, \mu_{j,j'} \mid j \leqslant j' \leqslant \omega_1^{N_i} \rangle$,

(Σ.2) $\ulcorner \dot{\mu}_{i,\omega_1}(\dot{n}) = \underline{x} \urcorner \in \Sigma$ iff $i < \omega_1$, $n < \omega$ and $\mu_{i,\omega_1}(e_i(n)) = x$,

(Σ.3) $\ulcorner \dot{\mu}_{\omega_1,\omega_1}(\underline{x}) = \underline{x} \urcorner \in \Sigma$ for all $x \in H_{\omega_2}$,

(Σ.4) $\ulcorner \dot{\sigma}_{i,j}(\dot{n}) = \dot{m} \urcorner \in \Sigma$ iff $i \leqslant j < \omega_1$ and $\sigma_{i,j}(e_i(n)) = e_j(m)$,

(Σ.5) $\ulcorner (\underline{\vec{l}}, \underline{\vec{\beta}}) \in \underline{T} \urcorner \in \Sigma$ iff for some $n < \omega$, $\mathrm{lh}(\vec{l}) = n = \mathrm{lh}(\vec{\beta})$ and for all $m < n$ $l_m = k_m$, $\beta_m = \alpha_m$,

(Σ.6) $\ulcorner \underline{\xi} \mapsto \underline{\nu} \urcorner \in \Sigma$ iff $\xi \in K$ and $\nu = \lambda_\xi$ and

(Σ.7) $\ulcorner \underline{x} \in \dot{X}_\xi \urcorner \in \Sigma$ iff $\xi \in K$ and $x \in X_\xi$.

Note that $\mathfrak{C}$ can be "read off" from Σ in a unique way via a Henkin-style construction. For $i < \omega_1$ and $n, m < \omega$, let

$$n \sim_i m \Leftrightarrow \ulcorner N_i \models \dot{n} = \dot{m} \urcorner \in \Sigma$$

and denote the equivalence class of n modulo $\sim_i$ by $[n]_i^\Sigma$. We will usually drop the superscript Σ if it is clear from context. Also let

$$n \tilde{\in}_i m \Leftrightarrow \ulcorner N_i \models \dot{n} \in \dot{m} \urcorner \in \Sigma.$$

Then $(N_i, \in) \cong (\omega, \tilde{\in}_i)/\sim_i$. We call the latter model the term model producing N_i. See Lemma 3.7 in [AS21] for more details. For $x \in N_i$ we say x is represented by n if x gets mapped to $[n]_i$ by the unique isomorphism of N_i to the term model. The term model for N_{ω_1} is then the direct limit along the term models producing the N_i, $i < \omega_1$ and elements can then be represented by pairs (i, n), $i < \omega_1, n < \omega$ in the natural way.

To define certificates, we make use of the following concept:

Definition 4.23. For $\bar{\lambda} \in C \cap \lambda$,

$$Z \subseteq \mathbb{P}^{\diamondsuit}_{\bar{\lambda}} \times \omega_1 \times \omega$$

is a $\bar{\lambda}$-code for a dense subset of $(I^+)^{\dot{N}_{\omega_1}}$ given that

(i) if $(p, i, n) \in Z$ then

$$\ulcorner \dot{N}_i \models \text{“}\dot{n} \in \dot{I}_i^{+}\text{”}\urcorner \in p,$$

(ii) for any $(q, j, m) \in \mathbb{P}_{\bar{\lambda}} \times \omega_1 \times \omega$ with

$$\ulcorner \dot{N}_j \models \text{“}\dot{m} \in \dot{I}_j^{+}\text{”}\urcorner \in q$$

there is $(p, i, n) \in Z$ with

(a) $p \leqslant q$, $j \leqslant i$ and

(b) $\ulcorner \dot{N}_i \models \text{“}\dot{n} \subseteq \dot{k} \mod \dot{I}_i\text{”}\urcorner, \ulcorner \dot{\sigma}_{j,i}(\dot{m}) = \dot{k}\urcorner \in p$ for some $k < \omega$,

(iii) and if $(p, i, n) \in Z$ as well as $q \leqslant p$ then $(q, i, n) \in Z$.

Suppose that

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

is $(\lambda-)$precertified by $\Sigma \subseteq \mathcal{L}^\lambda$ as witnessed by $(e_i)_{i<\omega_1}$. For $Z_0 \subseteq Z$ we define the evaluation of Z_0 by Σ as

$$Z_0^\Sigma := \{S \in N_{\omega_1} \mid \exists p \in [\Sigma]^{<\omega} \exists i < \omega_1 \exists n < \omega\, ((p, i, n) \in Z_0 \wedge S = \sigma_{i,\omega_1}(e_i(n)))\}.$$

A potential certificate $\mathfrak{C}$ is (λ-)*certified* by a collection $\Sigma \subseteq \mathcal{L}^\lambda$ if $\mathfrak{C}$ is (λ-)precertified by Σ and additionally

(Σ.8) whenever $\xi \in K$ and Z is a λ_ξ-code for a dense subset of $(I^+)^{\dot{N}_{\omega_1}}$ definable over

$$(Q_{\lambda_\xi}; \in, \mathbb{P}^\diamondsuit_{\lambda_\xi}, A_{\lambda_\xi})$$

from parameters in X_ξ, then there is $S \in (Z \cap X_\xi)^\Sigma$ with $\xi \in S$.

Definition 4.24. In the case that (Σ.8) is satisfied, we call $\mathfrak{C}$ a semantic certificate, and Σ a syntactic certificate, relative to

$$\mathbb{V}_{\max}, A, H_{\omega_2}, T_0, \langle A_\nu \mid \nu \in C \cap \lambda\rangle \text{ and } \langle \mathbb{P}^\diamondsuit_\nu \mid \nu \in C \cap \lambda\rangle.$$

Remark 4.25. The genericity condition in [AS21] that is replaced here with (Σ.8) (adapted to our context) is:

(Σ.8)$^{\mathrm{AS}}$ If $\xi \in K$ and $E \subseteq \mathbb{P}^\diamondsuit_{\lambda_\xi}$ is dense and definable over

$$(Q_{\lambda_\xi}; \in, \mathbb{P}^\diamondsuit_{\lambda_\xi}, A_{\lambda_\xi})$$

from parameters in X_ξ then

$$[\Sigma]^{<\omega} \cap E \cap X_\xi \neq \emptyset.$$

Condition (Σ.8) is stronger than $(\Sigma.8)^{\mathrm{AS}}$: From any such E,

$$Z = \{(p, i, n) \in \mathbb{P}^{\diamondsuit}_{\lambda} \times \omega_1 \times \omega \mid \exists q \in E\ p \leqslant q \wedge \ulcorner \dot{N}_i \models \text{“}\dot{n} \in \dot{I}^{+}_{i}\text{”}\urcorner \in p\}$$

is a λ_ξ-code for a dense subset of $(I^+)^{\dot{N}_{\omega_1}}$ definable over the same structure from the same parameters. If $(Z \cap X_\xi)^\Sigma \neq \varnothing$, it follows that

$$[\Sigma]^{<\omega} \cap E \cap X_\xi \neq \varnothing.$$

Suppose Σ is a certificate that certifies

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle,$$

$\xi \in K$ and Z is a λ_ξ-code for a dense subset of $(I^+)^{\dot{N}_{\omega_1}}$ definable over

$$(Q_{\lambda_\xi}; \in, \mathbb{P}^{\diamondsuit}_{\lambda_\xi}, A_{\lambda_\xi}).$$

Z is supposed to represent a dense subset of $(I^+)^{N_{\omega_1}}$ (w.r.t. inclusion mod $I^{N_{\omega_1}}$) in $V^{\mathbb{P}^{\diamondsuit}_{\lambda}}$. Σ may not be “generic over V”, so it may not be the case that Z^Σ is dense in $(I^+)^{N_{\omega_1}}$. Nonetheless, already $(\Sigma.8)^{\mathrm{AS}}$ implies that

$$D = \sigma^{-1}_{\xi,\omega_1}[(Z \cap X_\xi)^\Sigma] \subseteq (I^+)^{N_\xi}$$

is dense. D may not be in N_ξ, so it is not guaranteed that D is hit by the ultrapower $\sigma_{\xi,\xi+1}\colon N_\xi \to N_{\xi+1}$ just from genericity over N_ξ alone, however (Σ.8) makes sure that this happens (observe that $\omega_1^{N_\xi} = \xi$). So in essence, the idea of (Σ.8) is that any dense subset of $(I^+)^{N_{\omega_1}}$ that exists in the final $V^{\mathbb{P}^{\diamondsuit}_{\kappa}}$ has been “hit” before at some point along the iteration of N_0 to N_{ω_1}.

Remark 4.26. Note that for any syntactic certificate, there is a unique semantic certificate it corresponds to. Given a semantic certificate, its corresponding syntactic certificate is unique modulo the choice of the maps $(e_i)_{i<\omega}$.

A finite set p of $\mathcal{L}^\lambda$-formulas is *certified* by Σ iff Σ is a syntactic certificate and $p \subseteq \Sigma$. If $\mathfrak{C}$ is a semantic certificate then we also say p is certified by $\mathfrak{C}$ in case there is a syntactic certificate λ certifying both $\mathfrak{C}$ and p.

Definition 4.27. Conditions $p \in \mathbb{P}^{\diamondsuit}_{\lambda}$ are finite sets of $\mathcal{L}^\lambda$ formulae so that

$$V^{\mathrm{Col}(\omega,\omega_2)} \models \text{“}\exists \Sigma \subseteq \mathcal{L}^\lambda\ \Sigma \text{ certifies } p\text{”}.$$

This completes the construction of $\mathbb{P}^{\diamondsuit}_{\lambda}$.

Proposition 4.28. *Let $p \in [\mathcal{L}^\lambda]^{<\omega}$. If p is certified in some outer model, then p is certified in $V^{\mathrm{Col}(\omega,\omega_2)}$.*

Proof. Let g be $\mathrm{Col}(\omega, \omega_2)$-generic. If there is some outer model in which p is certified, then by Shoenfield absoluteness we can find in $V[g]$ a set of $\mathcal{L}^\lambda$-formulas Σ with $p \in [\Sigma]^{<\omega}$ such that if

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

is the corresponding semantic interpretation then

(*i*) Σ satisfies $(\Sigma.1)$-$(\Sigma.8)$,

(*ii*) $\mathfrak{C}$ satisfies $(\mathfrak{C}.2)$ as well as $(\mathfrak{C}.4)$-$(\mathfrak{C}.6)$ and

(*iii*) $\mathfrak{C}$ satisfies $(\mathfrak{C}.3)$ in the sense that $\mu_{0,\omega_1^{N_0}} \in N_0$ and $N_0 \models \varphi^{\mathbb{V}_{\max}}(M_{\omega_1^{N_0}})$,

as this can be expressed by a $\mathbf{\Sigma}^1_2$-formula. It remains to show that $(\mathfrak{C}.1)$ holds true as well, i.e. $M_0, N_0 \in \mathbb{V}_{\max}$. For N_0 this follows as $N_0 \in p[T_0]$ and by assumption (v), $p[T_0] \subseteq \mathbb{V}_{\max}$ in $V[g]$. To see that $M_0 \in \mathbb{V}_{\max}$, note that $\mathcal{H} \in \mathbb{V}_{\max}$ as $\mathcal{H}$ is almost a $\mathbb{V}_{\max}$-condition in V. By $(\mathbb{V}_{\max}.iv)$, it is enough to see that M_0 is generically iterable. This follows from (the proof of) Theorem 3.16 in [Woo10], here we use $\mathcal{P}(\omega_1)^\sharp$ exists in V. □

We let $\mathbb{P}^\diamondsuit = \mathbb{P}^\diamondsuit_\kappa$. As in Asperó-Schindler, we conclude that there is a club $D \subseteq C$ so that for all $\lambda \in D$

$$\mathbb{P}^\diamondsuit_\lambda = \mathbb{P}^\diamondsuit \cap Q_\lambda$$

and hence we get

$$(\diamondsuit(\mathbb{P}^\diamondsuit)) \qquad \begin{array}{l} \text{for all } B \subseteq H_\kappa \text{ the set} \\ \quad \{\lambda \in C \mid (Q_\lambda; \in, \mathbb{P}^\diamondsuit_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\} \\ \text{is stationary.} \end{array}$$

Lemma 4.29. $\varnothing \in \mathbb{P}^\diamondsuit_{\min(C)}$.

The argument is essentially the same as the proof of Lemma 3.6 in [AS21] modulo some details that arise from replacing $\mathbb{P}_{\max}$ by a general $\mathbb{P}_{\max}$-variation.

Proof. Let g be generic for $\mathrm{Col}(\omega, \omega_2)$. Note that $\mathcal{H} \in \mathbb{V}_{\max}$ as $\mathcal{H}$ is almost a $\mathbb{V}_{\max}$-condition in V. By choice of T_0, we can find $N_0 = (N_0, I_0, a_0) \in D^*$ with $N_0 <_{\mathbb{V}_{\max}} \mathcal{H}$. Let $\langle(k_n, \alpha_n) \mid n < \omega\rangle$ witness $N_0 \in p[T]$. Let us denote $M_0 = \mathcal{H}$ and let

$$\mu_{0,\omega_1^{N_0}} \colon M_0 \to M_{\omega_1^{N_0}}$$

witness $N_0 <_{\mathbb{V}_{\max}} M_0$. Now let

$$\sigma_{0,\kappa} \colon N_0 \to N_\kappa$$

be a generic iteration of N_0 of length $\kappa + 1 = \omega_1^{V[g]} + 1$ as well as

$$\mu_{0,\kappa} := \sigma_{0,\kappa}(\mu_{0,\omega_1^{N_0}})\colon M_0 \to M_\kappa$$

the stretch of $\mu_{0,\omega_1^{N_0}}$ by $\sigma_{0,\kappa}$. Note that this is a generic iteration of M_0 of length $\kappa + 1$.

Claim 4.30. *The generic iteration*

$$\langle M_\alpha, \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \kappa\rangle$$

can be extended to a generic iteration of $M_0^+ := (V, \mathrm{NS}_{\omega_1}^V)$ of length $\kappa + 1$. That is, there is a generic iteration

$$\langle M_\alpha^+, \mu_{\alpha,\beta}^+ \mid \alpha \leqslant \beta \leqslant \kappa\rangle$$

of M_0^+ so that for all $\alpha \leqslant \beta \leqslant \kappa$

(+.i) $M_\alpha = \left(H_{\omega_2}\right)^{M_\alpha^+}$ *and*

(+.ii) $\mu_{\alpha,\beta} = \mu_{\alpha,\beta}^+ \restriction M_\alpha$.

Proof. The iteration $\langle M_\alpha^+, \mu_{\alpha,\beta}^+ \mid \alpha \leqslant \beta \leqslant \kappa\rangle$ arises by applying the same generic ultrafilter g_α which generates $\mu_{\alpha,\alpha+1}\colon M_\alpha \to M_{\alpha+1}$ to M_α^+. By induction on α, as $M_\alpha = \left(H_{\omega_2}\right)^{M_\alpha^+}$, g_α measures all subsets of $\omega_1^{M_\alpha^+}$ in M_α^+. It is a generic ultrafilter as

$$M_\alpha^+ \models \text{“}\mathrm{NS}_{\omega_1} \text{ is saturated”}$$

by elementarity of $\mu_{0,\alpha}^+$, and hence all maximal antichains in $(\mathrm{NS}_{\omega_1}^+)^{M_\alpha^+}$ are already in M_α, hence are met by g_α. Now let

$$\mu_{\alpha,\alpha+1}^+\colon M_\alpha^+ \to M_{\alpha+1}^+ := \mathrm{Ult}(M_\alpha^+, g_\alpha)$$

be the ultrapower. Any $x \in (H_{\omega_2})^{M_{\alpha+1}^+}$ is represented by some function $f\colon \omega_1^{M_\alpha^+} \to \left(H_{\omega_2}\right)^{M_\alpha^+}$ which is an element of $\left(H_{\omega_2}\right)^{M_\alpha^+} = M_\alpha$. It follows that $\mu_{\alpha,\alpha+1} = \mu_{\alpha,\alpha+1}^+ \restriction M_\alpha$. It is easy to see that the properties (+.*i*),(+.*ii*) are stable under taking direct limits. □

The point is that

$$\langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \varnothing\rangle$$

is a semantic certificate for $\varnothing$ in $M^+ := M_\kappa^+$ with respect to

$$\mu^+(\mathbb{V}_{\max}), \mu^+(A), \left(H_{\omega_2}\right)^{M^+}, \mu^+(T_0), \mu^+(\langle A_\nu \mid \nu \in C \cap \lambda\rangle), \mu^+(\langle \mathbb{P}_\nu^\diamondsuit \mid \nu \in C \cap \lambda\rangle)$$

for $\lambda = \min(C)$ and $\mu^+ = \mu_{0,\kappa}^+$. By Proposition 4.28,

$$M^+ \models \varnothing \in \mu^+(\mathbb{P}_{\min(C)}^\diamondsuit)$$

so that $\varnothing \in \mathbb{P}_{\min(C)}^\diamondsuit$ in V by elementarity of μ^+. □

Lemma 4.31. *Suppose $\lambda \in C \cup \{\kappa\}$ and $g \subseteq \mathbb{P}^{\diamondsuit}_{\lambda}$ is a filter with*

(*i*) *$g \cap E \neq \varnothing$ whenever $E \subseteq \mathbb{P}^{\diamondsuit}_{\lambda}$ is dense and definable over*

$$(Q_\lambda; \in, \mathbb{P}^{\diamondsuit}_{\lambda}, A_\lambda),$$

(*ii*) *g is an element of a generic extension of V by a forcing of size $\leqslant 2^{\omega_2}$.*

Then $\bigcup g$ is a semantic certificate.

Proof. Read off the canonical candidate

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k_n, \alpha_n) \mid n < \omega\rangle, \langle \lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

from g. The proof of Lemma 3.7 in [AS21] shows that $\bigcup g$ λ-precertifies $\mathfrak{C}$. Note that the argument from Proposition 4.28 gives that $M_0, N_0 \in \mathbb{V}_{\max}$ and ($\mathfrak{C}$.3) follows from (Σ.1) and ($\mathbb{V}_{\max}$.*ii*). It remains to check (Σ.8). So suppose $\xi \in K$ and Z is a λ_ξ-code for a dense subset of $(I^+)^{\dot{N}_{\omega_1}}$ definable over

$$\mathcal{Q}_\lambda := (Q_{\lambda_\xi}; \in, \mathbb{P}^{\diamondsuit}_{\lambda_\xi}, A_{\lambda_\xi})$$

from a parameter $x \in X_\xi$. Then there is $p \in g$ with

$$\ulcorner \underline{\xi} \mapsto \underline{\lambda_\xi} \urcorner, \ulcorner \underline{x} \in \dot{X}_\xi \urcorner \in p.$$

Let Σ' be a syntactic certificate certifying p (in some extension of V by $\mathrm{Col}(\omega, \omega_2)$) and

$$\mathfrak{C}' = \langle\langle M'_i, \mu'_{i,j}, N'_i, \sigma'_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k'_n, \alpha'_n) \mid n < \omega\rangle, \langle \lambda'_\rho, X'_\rho \mid \rho \in K'\rangle\rangle$$

the corresponding semantic certificate. We have $\xi \in K$ and $\lambda'_\xi = \lambda_\xi$ as well as $x \in X'_\xi$. Thus Z is definable over $\mathcal{Q}_\lambda$ from parameters in X'_ξ. As Σ' satisfies (Σ.8), there is $S \in (Z \cap X'_\xi)^{\Sigma'}$ with $\xi \in S$. We may now find $(q, i, n) \in Z \cap X'_\xi$ so that

$$S = \sigma_{i,\omega_1}([n]^{\Sigma'}_i).$$

Note that $i < \xi$ as $\delta^{X'_\xi} = \xi$. Let $\sigma_{i,\xi+1}([n]^{\Sigma'}_i] = [m]^{\Sigma'}_{\xi+1}$. It follows that

$$\ulcorner \dot{N}_{\xi+1} \models \text{“}\underline{\xi} \in \dot{m}\text{”} \urcorner, \ulcorner \dot{\sigma}_{i,\xi+1}(\dot{n}) = \dot{m} \urcorner \in \Sigma'.$$

This is a density argument that shows: There are $s \geqslant r \in g$, $j < \xi$, $l < \omega$ so that

(*i*) $(s, j, l) \in Z$,

(*ii*) $\ulcorner \underline{s} \in \dot{X}_\xi \urcorner \in r$ and

(*iii*) $\ulcorner \dot{N}_{\xi+1} \models \text{“}\underline{\xi} \in \dot{k}\text{”} \urcorner, \ulcorner \dot{\sigma}_{j,\xi+1}(\dot{l}) = \dot{k} \urcorner \in r$ for some $k < \omega$.

It follows that for $S = \sigma_{j,\omega_1}([l]_j^{\bigcup g})$, we have $S \in (Z \cap X_\xi)^{\bigcup g}$ and $\xi \in S$. $\square$

Lemma 4.32. *Suppose g is generic for $\mathbb{P}^\diamondsuit$ and*

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

is the resulting semantic certificate. Then in $V[g]$,

$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle$$

is a $\diamondsuit$-iteration.

Proof. Let $\dot{S}, \dot{C}$ be $\mathbb{P}^\diamondsuit$-names with

$$p \Vdash \text{“}\dot{C} \subseteq \omega_1 \text{ is club and } \dot{S} \in (\dot{I}^+)^{\dot{N}_{\omega_1}}\text{”}$$

for some $p \in \mathbb{P}^\diamondsuit$. Further suppose $\langle \dot{D}_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of $\mathbb{P}^\diamondsuit$-names for dense subsets of $(I^+)^{\dot{N}_{\omega_1}}$. We may suppose that

$$p \Vdash \dot{S} = \dot{\sigma}_{i_0,\omega_1}([\check{n}]_{\check{i}_0}^{\bigcup \dot{G}})$$

for some $i_0 < \omega_1$ and $n < \omega$ where $\dot{\sigma}_{i_0,\omega_1}$ is a name for σ_{i_0,ω_1} which arises in the semantic certificate corresponding to the generic filter. It is our duty to find $\xi < \omega_1$ and $q \leqslant p$ with

$$q \Vdash \check{\xi} \in \dot{S} \cap \dot{C} \wedge \forall \alpha < \check{\xi}\ \dot{g}_\xi \cap \dot{\sigma}_{\xi,\omega_1}^{-1}[\dot{D}_\alpha] \neq \varnothing \tag{$\spadesuit$}$$

where $\dot{g}_\xi$ is a name for the generic ultrafilter applied to $\dot{N}_\xi$ along the iteration to $\dot{N}_{\omega_1}$. We will replace the $\dot{D}_\alpha$ with codes for them: For $\alpha < \omega_1$, let Z_α be defined by $(q, j, m) \in Z_\alpha$ iff

(Z.i) $(q, j, m) \in \mathbb{P}^\diamondsuit \times \omega_1 \times \omega$,

(Z.ii) $\ulcorner \dot{N}_j \models \text{“}\dot{m} \in \dot{I}_j\text{”}\urcorner \in q$ and

(Z.iii) $q \Vdash \dot{\sigma}_{j,\omega_1}\left([m]_j^{\bigcup \dot{G}}\right) \in \dot{D}_\alpha$.

Further, for $\alpha < \omega_1$, we let

$$E_\alpha = \{q \leqslant p \mid \exists \beta\ \alpha \leqslant \beta \wedge q \Vdash \check{\beta} \in \dot{C}\}$$

and

$$E = \{(q, \alpha) \in \mathbb{P}^\diamondsuit \times \omega_1 \mid q \Vdash \check{\alpha} \in \dot{C}\}.$$

Finally we define

$$\tau = \left(\bigoplus_{\alpha<\omega_1} Z_\alpha\right) \oplus \left(\bigoplus_{\alpha<\omega_1} E_\alpha\right) \oplus E.$$

We may now find $\lambda \in C$ so that $p \in \mathbb{P}^{\diamondsuit}_{\lambda}$ and

$$(Q_\lambda; \in, \mathbb{P}^{\diamondsuit}_{\lambda}, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}^{\diamondsuit}, \tau).$$

Here, $\oplus$ denotes some canonical way of coding at most ω_1-many subsets of H_κ into a subset of H_κ. Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V.

Claim 4.33. *In $V[h]$, there are filters g, G that satisfy the following properties (i)-(iii):*

(i) *g meets every dense subset of $\mathbb{P}^{\diamondsuit}_{\lambda}$ that is definable (with parameters) in*

$$(Q_\lambda; \in, \mathbb{P}^{\diamondsuit}_{\lambda}, A_\lambda).$$

Let

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

denote the semantic certificate corresponding to g.

(ii) *G is $(I^+)^{N_{\omega_1}}$-generic over N_{ω_1} with $\dot{S}^g = [n]^{\bigcup g}_{i_0} \in G$.*

(iii) *G meets $Z^{\bigcup g}$ whenever Z is a λ-code for a dense subset of $(\dot{I}^+)^{\dot{N}_{\omega_1}}$ definable (with parameters) over*

$$(Q_\lambda; \in, \mathbb{P}^{\diamondsuit}_{\lambda}, A_\lambda).$$

Proof. Let $g' \subseteq \mathbb{P}^{\diamondsuit}_{\lambda}$ be generic over V and let

$$\mathfrak{C}' = \langle\langle M'_i, \mu'_{i,j}, N'_i, \sigma'_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k'_n, \alpha'_n) \mid n < \omega\rangle, \langle\lambda'_\rho, X'_\rho \mid \rho \in K'\rangle\rangle$$

be the semantic certificate corresponding to $\bigcup g'$. Let further G' be $(I^+)^{N'_{\omega_1}}$-generic over $V[g']$ (so in particular over N'_{ω_1}) with $\dot{S}^{g'} = [n]^{\bigcup g'}_{i_0} \in G'$. It is clear that g', G' satisfy (i)-(iii) above. The existence of such filters is Σ^1_1 in a real code for $(Q_\lambda; \in, \mathbb{P}^{\diamondsuit}_{\lambda}, A_\lambda)$ so that there are $g, G \in V[h]$ with (i)-(iii) by Shoenfield-absoluteness. $\square$

We now work in $V[h]$. Let G, g be the filters given by the claim above and let

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

be the semantic certificate that comes from g. Let

$$\sigma_{\omega_1,\omega_1+1}\colon N_{\omega_1} \to N_{\omega_1+1} = \mathrm{Ult}(N_{\omega_1}, G)$$

be the generic ultrapower. We can further extend the generic iteration

$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1 + 1\rangle$$

to one of length $\kappa+1$, say

$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle.$$

Further, set

$$\vec{M} = \langle M_i, \mu_{i,j} \mid i \leqslant j \leqslant \kappa\rangle := \sigma_{\omega_1,\kappa}(\langle M_i, \mu_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle).$$

As $\mathfrak{C}$ is certified, $M_{\omega_1} = \mathcal{H}$ and as in Claim 4.30, we can extend the tail of $\vec{M}$ that is an iteration of M_{ω_1} to a generic iteration of $M^+_{\omega_1} := (V, \mathrm{NS}^V_{\omega_1}, A)$, say

$$\langle M^+_i, \mu^+_{i,j} \mid \omega_1 \leqslant i \leqslant j \leqslant \kappa\rangle$$

and have all M^+_i, $i \in [\omega_1, \kappa]$, wellfounded. Let us write

$$\mu^+ := \mu^+_{\omega_1,\kappa} \colon V \to M^+_{\omega_1} =: M^+.$$

Work in M^+. We will now use

$$\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle$$

as part of a certificate. Set

$$q := \mu^+(p) \cup \{\ulcorner \underline{\omega_1} \mapsto \mu^+(\lambda)\urcorner, \ulcorner \dot\sigma_{i_0,\omega_1+1}(\dot n) = \dot m\urcorner, \ulcorner \dot N_{\omega_1+1} \models \text{“}\underline{\omega_1} \in \dot m\text{”}\urcorner\}$$

where $\dot m$ represents $\sigma_{\omega_1,\omega_1+1}(S)$ in the term model for N_{ω_1+1}.

Claim 4.34. *$q \in \mu^+(\mathbb{P}^\diamondsuit)$.*

Proof. Set

$$\mathfrak{C}^* = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle, \langle (k_n, \mu^+(\alpha_n)) \mid n < \omega\rangle, \langle \lambda^*_\xi, X^*_\xi \mid \xi \in K^*\rangle\rangle$$

where

- $K^* = K \cup \{\omega_1\}$,
- for $\xi \in K$, $\lambda^*_\xi = \mu^+(\lambda_\xi)$ and $X^*_\xi = \mu^+[X_\xi]$ and
- $\lambda_{\omega_1} = \mu^+(\lambda)$, $X^*_{\omega_1} = \mu^+[Q_\lambda]$.

We show that $\mathfrak{C}^*$ is a semantic certificate for q in M^+. Note that we have to show that $\mathfrak{C}^*$ is a certificate relative to

$$\mu^+(\mathbb{V}_{\max}), \mu^+(A), \mu^+(H_{\omega_2}) = (H_{\omega_2})^{M^+}, \mu^+(T_0), \mu^+(\langle A_\nu \mid \nu \in C\rangle), \mu^+(\langle \mathbb{P}_\nu \mid \nu \in C\rangle).$$

Observe that we can find a corresponding set of formulae Σ^+ that corresponds to $\mathfrak{C}^*$ with $\mu^+[\bigcup g] \subseteq \Sigma^+$ which we aim to prove to be a syntactic

certificate.
We have $M_\kappa = \left(H_{\omega_2}\right)^{M^+}$. Notice also that

$$\langle (k_n, \mu^+(\alpha_n)) \mid n < \omega \rangle \in [\mu^+(T_0)]$$

and that $(k_n)_{n<\omega}$ is still a real code for N_0. Next, we prove (Σ.8). First assume $\xi \in K$. Then

$$X^*_\xi = \mu^+[X_\xi] \prec (\mu^+(Q_{\lambda_\xi}); \in, \mu^+(\mathbb{P}^\diamondsuit_{\lambda_\xi}), \mu^+(A_{\lambda_\xi}))$$

and $\delta^{X^*_\xi} = \delta^{X_\xi} = \xi$ as $\mathrm{crit}(\mu) = \omega_1 > \xi$. As $\mu^+[X_\xi] = X^*_\xi$, (Σ.8) holds for ξ in $\mathfrak{C}^*$, since it holds for ξ in $\mathfrak{C}$.
Finally, let us consider the case $\xi = \omega_1$. We have

$$X^*_{\omega_1} = \mu^+[Q_\lambda] \prec (\mu^+(Q_\lambda); \in, \mu^+(\mathbb{P}^\diamondsuit_\lambda), \mu^+(A_\lambda))$$

and $\delta^{X^*_{\omega_1}} = \omega_1$ as μ^+ has critical point ω_1. Clearly $X^*_{\omega_1}$ collapses to Q_λ. So if $x \in X^*_{\omega_1}$ and

$$M^+ \models \text{"}\hat{Z} \text{ is a } \mu^+(\lambda)\text{-code for a dense subset of } (\dot{I}^+)^{N_\kappa} \text{ definable over } (\mu^+(Q_\lambda); \in, \mu^+(\mathbb{P}^\diamondsuit_\lambda), \mu^+(A_\lambda)) \text{ with parameter } x\text{"}$$

for some $x \in X^*_{\omega_1}$, then by elementarity, the same definition defines a λ-code Z for a dense subset of $(\dot{I}^+)^{\dot{N}_{\omega_1}}$ over

$$(Q_\lambda; \in, \mathbb{P}^\diamondsuit_\lambda, A_\lambda)$$

with parameter $(\mu^+)^{-1}(x)$ and we have $\mu^+(Z) = \hat{Z}$. Our properties of g, G imply that there is $R \in G \cap Z^{\bigcup g}$. It is not difficult to see

$$(\hat{Z} \cap X^*_{\omega_1})^{\Sigma^+} = \sigma_{\omega_1,\kappa}[Z^{\bigcup g}]$$

and hence $\omega_1 \in \sigma_{\omega_1,\kappa}(R) \in (\hat{Z} \cap X^*_{\omega_1})^{\Sigma^+}$. This shows ($\Sigma$.8) at ω_1.
We conclude that indeed, $\mathfrak{C}^*$ is a semantic certificate for q which exists in some outer model of M^+. This gives $q \in \mu^+(\mathbb{P}^\diamondsuit)$ by Proposition 4.28. $\square$

Thus we have

$$M^+ \models \text{"}\exists \xi < \mu^+(\omega_1) \ \left(\mu^+(p) \cup \{\ulcorner \underline{\xi} \mapsto \underline{\mu^+(\lambda)} \urcorner, \ulcorner \dot{\sigma}_{i_0,\xi+1}(\dot{n}) = \dot{m} \urcorner, \ulcorner \dot{N}_{\xi+1} \models \text{"}\underline{\xi} \in \dot{m}\text{"} \urcorner\} \in \mu^+(\mathbb{P}^\diamondsuit)\right)\text{"}.$$

By elementarity of μ^+, we conclude

$$V \models \text{"}\exists \xi < \omega_1 \ \left(p \cup \{\ulcorner \underline{\xi} \mapsto \underline{\lambda} \urcorner, \ulcorner \dot{\sigma}_{i_0,\xi+1}(\dot{n}) = \dot{m} \urcorner, \ulcorner \dot{N}_{\xi+1} \models \text{"}\underline{\xi} \in \dot{m}\text{"} \urcorner\} \in \mathbb{P}^\diamondsuit\right)\text{"}.$$

Let ξ witness this and set

$$q = p \cup \{\ulcorner \underline{\xi} \mapsto \underline{\lambda} \urcorner, \ulcorner \dot{\sigma}_{i_0,\xi+1}(\dot{n}) = \dot{m} \urcorner, \ulcorner \dot{N}_{\xi+1} \models \text{“}\underline{\xi} \in \dot{m}\text{”} \urcorner\}.$$

We will show that q, ξ witness (♠). From this point on, we work in V again and forget about $h, g, \mathfrak{C}$, etc.

Claim 4.35. *$q \Vdash \check{\xi} \in \dot{C} \cap \dot{S}$.*

Proof. As in Claim 3.17 in [AS21], exploit the components of τ made up from E as well as E_α, $\alpha < \omega_1$. □

Claim 4.36. *$q \Vdash \forall \alpha < \check{\xi}\ \dot{g}_\xi \cap \dot{\sigma}_{\xi,\omega_1}^{-1}[\dot{D}_\alpha] \neq \varnothing$.*

Proof. Let g be $\mathbb{P}^\diamondsuit$-generic with $q \in g$ and let

$$\mathfrak{C} = \langle \langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1 \rangle, \langle (k_n, \alpha_n) \mid n < \omega \rangle, \langle \lambda_\xi, X_\xi \mid \xi \in K \rangle \rangle$$

be the resulting semantic certificate. We have $\xi \in K$ and $\lambda_\xi = \lambda$ as $q \in g$. Fix some $\alpha < \xi$. Clearly,

$$\bar{Z}_\alpha = Z_\alpha \cap Q_\lambda$$

is a λ-code for a dense subset of $(\dot{I}^+)^{\dot{N}_{\omega_1}}$ which is definable over

$$(Q_\lambda; \in, \mathbb{P}_\lambda^\diamondsuit, A_\lambda)$$

from a parameter in X_ξ, namely α. Recall that $\delta^{X_\xi} = \xi$. Using (Σ.8), we find that there is

$$R \in (\bar{Z}_\alpha \cap X_\xi)^{\bigcup g}$$

with $\xi \in R$. Note that there are $r \in g$, $j < \xi = \delta^{X_\xi}$ as well as $k < \omega$ with

(*i*) $(r, j, k) \in \bar{Z}_\alpha \subseteq Z_\alpha$ and

(*ii*) $R = \sigma_{j,\omega_1}([k]_j^{\bigcup g})$.

By definition of Z_α, and as $r \in g$, $R \in D_\alpha$ and since $\xi \in R$, $R \in g_\xi$, where g_ξ is the generic ultrafilter generating $\sigma_{\xi,\xi+1} \colon N_\xi \to N_{\xi+1}$. □

(♠) follows from Claim 4.35 together with Claim 4.36. □

This completes the proof of Theorem 4.20. We denote the forcing $\mathbb{P}^\diamondsuit$ constructed above in the instance of a $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$, the set $A \in H_{\omega_2}$ and appropriate dense $D \subseteq \mathbb{V}_{\max}$ by $\mathbb{P}^\diamondsuit(\mathbb{V}_{\max}, A, D)$ (and forget that $\mathbb{P}^\diamondsuit$ also depends on the choice of T, T_0, etc.).

4.5 The first blueprint

We will formulate a general theorem that will allow us to prove a variety of instances of $\mathrm{MM}^{++} \Rightarrow (*)$. In order to formulate the relevant forcing axioms, we use that in practice $\varphi^{\mathbb{V}_{\max}}$ has a specific form.

Definition 4.37. A $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ is *typical* if $\varphi^{\mathbb{V}_{\max}}$ can be chosen to be the form

$$\begin{aligned}\varphi^{\mathbb{V}_{\max}}(x) = \text{“}\exists M, I, a_0, \dots, a_n\ x = (M, I, a_0, \dots, a_n)\\ \wedge\ \forall y \in M \bigwedge_{\psi \in \Psi} \left[\psi(y) \leftrightarrow (M; \in, I, a_0, \dots, a_n) \models \psi(y)\right]\text{”}\end{aligned}$$

for $n = n^{\mathbb{V}_{\max}}$ and a finite set Ψ of formulae $\psi(y)$ in the language $\{\in, \dot{I}, \dot{a}_0, \dots, \dot{a}_n\}$. Moreover, Ψ contains the formulae $\psi(x) = $ "$x \in \dot{I}$" and $\psi_i(x) = $ "$x = \dot{a}_i$" for all $i \leqslant n^{\mathbb{V}_{\max}}$. We say that Ψ witnesses the typicality of $\mathbb{V}_{\max}$.
This means that $q <_{\mathbb{V}_{\max}} p$ iff there is a generic iteration $\mu\colon p \to p^*$ of p in q of length $\omega_1^q + 1$ so that the formulae in Ψ are absolute between q, p^*.

Remark 4.38. For example, $\mathbb{P}_{\max}$ is (or can be construed as) a typical $\mathbb{P}_{\max}$-variation. We have that typicality of $\mathbb{P}_{\max}$ is witnessed by $\{\psi_0^{\mathbb{P}_{\max}}, \psi_1^{\mathbb{P}_{\max}}\}$ where

- $\psi_0^{\mathbb{P}_{\max}}(y) = $ "$y \in \dot{I}$" and
- $\psi_1^{\mathbb{P}_{\max}}(y) = $ "$y = \dot{a}_0$".

All $\mathbb{P}_{\max}$-variations we will encounter, except for $\mathbb{Q}^-_{\max}$, are typical $\mathbb{P}_{\max}$-variations.

Next, we formulate the relevant bounded and unbounded forcing axioms as general as possible.

Definition 4.39. Suppose $\psi(x)$ is a formula in the language $\{\in, \dot{I}, \dot{a}_0, \dots, \dot{a}_n\}$ and $\vec{A} = (A_0, \dots, A_n) \in H_{\omega_2}$.

(*i*) We define $R^{\psi}_{\vec{A}}$ via

$$R^{\psi}_{\vec{A}} := \{x \in H_{\omega_2} \mid (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A_0, \dots, A_n) \models \psi(x)\}.$$

(*ii*) For $x \in H_{\omega_2}$, we say that $C \subseteq \omega_1$ is a code for x if: Let $l\colon \omega_1 \to \omega_1 \times \omega_1$ denote Gödels pairing function and $E = l[C]$. Then $(\omega_1 \times \omega_1, E)$ is wellfounded and $(\mathrm{tc}(\{x\}), \in)$ is the transitive isomorph[35].

(*iii*) $C \subseteq \omega_1$ is a code for an element of $R^{\psi}_{\vec{A}}$ if C is a code for some $x \in R^{\psi}_{\vec{A}}$.

Definition 4.40. Suppose that

[35] tc denotes transitive closure.

- Γ is a class of forcings,
- $\vec{A} = (A_0, \ldots, A_n) \in H_{\omega_2}$ and
- Ψ is a set of formulae $\psi(x)$ in the language $\{\dot{I}, \dot{a}_0, \ldots, \dot{a}_n\}$.

(*i*) D-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma)$ states that $D \subseteq \mathbb{R}$ is ∞-universally Baire and whenever $\mathbb{P} \in \Gamma$ and g is $\mathbb{P}$-generic then

$$\left(H_{\omega_2}; \in, D, R^{\psi}_{\vec{A}} \mid \psi \in \Psi\right)^V \prec_{\Sigma_1} \left(H_{\omega_2}; \in, D^*, R^{\psi}_{\vec{A}} \mid \psi \in \Psi\right)^{V[g]}.$$

For $\Delta \subseteq \mathcal{P}(\mathbb{R})$, Δ-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma)$ means D-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma)$ for all $D \in \Delta$.

(*ii*) $\mathrm{FA}^{\Psi}_{\vec{A}}(\Gamma)$ states that whenever $\mathbb{P} \in \Gamma$ and

(FA.*i*) $\mathcal{D}$ is a set of at most ω_1-many dense subsets of $\mathbb{P}$,

(FA.*ii*) $\mathcal{N}_\psi$ is a set of at most ω_1-many $\mathbb{P}$-names for codes of elements of $(R^{\psi}_{\vec{A}})^{V^{\mathbb{P}}}$ for $\psi \in \Psi$

then there is a filter $g \subseteq \mathbb{P}$ so that

(g.*i*) $g \cap D \neq \emptyset$ for all $D \in \mathcal{D}$ and

(g.*ii*) $\dot{S}^g = \{\alpha < \omega_1 \mid \exists p \in g\ p \Vdash \check{\alpha} \in \dot{S}\}$ is a code for an element of $R^{\psi}_{\vec{A}}$ for all $\dot{S} \in \mathcal{N}_\psi$, $\psi \in \Psi$.

We note that the methods of Bagaria in [Bag00] readily yield the following.

Lemma 4.41. *Suppose that*

(*i*) Γ *is a class of forcings,*

(*ii*) $\vec{A} = (A_0, \ldots, A_n) \in H_{\omega_2}$ *and*

(*iii*) Ψ *is a set of formulae* $\psi(x)$ *in the language* $\{\dot{I}, \dot{a}_0, \ldots, \dot{a}_n\}$.

If $\mathrm{FA}^{\Psi}_{\vec{A}}(\Gamma)$ *holds then so does* uB-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma)$.

Definition 4.42. Let Ψ be a set of formulae in the language $\{\dot{I}, \dot{a}_0, \ldots, \dot{a}_n\}$ for some n. For $\vec{A} = (A_0, \ldots, A_n)$, we say that a forcing $\mathbb{P}$ is $(\Psi, \vec{A})$-*preserving* iff

$$R^{\psi}_{\vec{A}} = \left(R^{\psi}_{\vec{A}}\right)^{V^{\mathbb{P}}} \cap V$$

for all $\psi \in \Psi$. $\Gamma^{\Psi}_{\vec{A}}$ denotes the class of $(\Psi, \vec{A})$-preserving forcings.

Definition 4.43. A $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ *accepts $\diamondsuit$-iterations* if

$$\text{"If } p \in \mathbb{V}_{\max} \text{ and } p \to p^* = (M, I, a_0, \ldots, a_{n^{\mathbb{V}_{\max}}}) \\ \text{is a } \diamondsuit\text{-iteration then } \mathcal{H}_{(a_0,\ldots,a_{n^{\mathbb{V}_{\max}}})} \models \varphi^{\mathbb{V}_{\max}}(p^*)\text{"}$$

is provable in ZFC$^-$ + "ω_1 exists" (that is, from sufficiently much of ZFC).

First Blueprint Theorem 4.44. *Suppose that*

(i) $\mathbb{V}_{\max}$ *is a typical* $\mathbb{P}_{\max}$*-variation with typicality witnessed by* Ψ*,*

(ii) $\mathbb{V}_{\max}$ *has unique iterations and accepts* $\diamondsuit$*-iterations,*

(iii) $\vec{A} \in H_{\omega_2}$ *and* $\mathcal{H}_{\vec{A}}$ *is almost a* $\mathbb{V}_{\max}$*-condition,*

(iv) SRP *holds and*

(v) $\mathrm{FA}^{\Psi}_{\vec{A}}(\Gamma^{\Psi}_{\vec{A}})$ *holds.*

Then $\mathbb{V}_{\max}$-$(*)$ *holds as witnessed by* $g_{\vec{A}}$.

Proof. Let us assume $n^{\mathbb{V}_{\max}} = 0$, so $\vec{A} = A$. SRP entails "NS_{ω_1} is saturated" as well as $\forall\kappa \geqslant \omega_2 \neg\Box_\kappa$. Results of Steel [Ste05] show that the latter implies that V is closed under $X \mapsto M^\sharp_\omega(X)$. As a consequence

- $\mathrm{AD}^{L(\mathbb{R})}$,
- all sets of reals in $L(\mathbb{R})$ are ∞-universally Baire and
- $(L(\mathbb{R})^V; \in, D) \equiv (L(\mathbb{R})^{V[G]}; \in, D^*)$ for all sets $D \subseteq \mathbb{R}$ in $L(\mathbb{R})$ and any generic extension $V[G]$ of V.

Thus generic projective absoluteness holds in V and if $D \in L(\mathbb{R})$ is a dense subset of $\mathbb{V}_{\max}$, then D^* is a dense subset of $\mathbb{V}_{\max}$ in any generic extension. Thus $\mathbb{P}^\diamondsuit(\mathbb{V}_{\max}, A, D)$ exists for any such D.

Claim 4.45. *For any dense* $D \subseteq \mathbb{V}_{\max}$*,* $D \in L(\mathbb{R})$*,* $\mathbb{P}^\diamondsuit(\mathbb{V}_{\max}, A, D)$ *is* (Ψ, A)*-preserving.*

Proof. Let g be $\mathbb{P}^\diamondsuit(\mathbb{V}_{\max}, A, D)$-generic. By Theorem 4.20, in $V[g]$ we have

$$\begin{tikzcd}
& D^* & \\
& q_0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\sigma_{0,\omega_1}"] & q_{\omega_1} = (N^*, I^*, b^*) \\
p_0 \arrow[r, "\mu_{0,\omega_1^N}"] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & p_{\omega_1^N} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\mu_{\omega_1^N,\omega_1}"] & p_{\omega_1} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, phantom, "\parallel"] \\
\mathbb{V}_{\max} & & ((H_{\omega_2})^V, \mathrm{NS}^V_{\omega_1}, A) = \mathcal{H}_A
\end{tikzcd}$$

where

($\mathbb{P}^\diamondsuit.i$) $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ are generic iterations of p_0, q_0 respectively,

($\mathbb{P}^\diamondsuit.ii$) μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$,

($\mathbb{P}^\diamondsuit.iii$) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$ and

($\mathbb{P}^\diamondsuit.iv$) the generic iteration $\sigma_{0,\omega_1}: q_0 \to q_{\omega_1}$ is a $\diamondsuit$-iteration.

Note that

$$(N^*; \in, I^*, b^*) \models \varphi^{\mathbb{V}_{\max}}(\mathcal{H}_A).$$

As $\mathbb{V}_{\max}$ is typical, we must have $b^* = A$. As $\mathbb{V}_{\max}$ accepts $\diamondsuit$-iterations,

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A)^{V[g]} \models \varphi^{\mathbb{V}_{\max}}(q_{\omega_1})$$

and finally it follows from typicality that

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A)^{V[g]} \models \varphi^{\mathbb{V}_{\max}}(\mathcal{H}_A).$$

As Ψ witnesses the typicality of $\mathbb{V}_{\max}$, it follows that $\mathbb{P}^\diamondsuit(\mathbb{V}_{\max}, A, D)$ is (Ψ, A)-preserving. $\Box$

It follows from Theorem 4.20, Lemma 4.41 and Lemma 4.17 that

- $g_{\vec{A}} \cap D \neq \varnothing$ for all dense $D \subseteq \mathbb{V}_{\max}$, $D \in L(\mathbb{R})$ and
- $\mathcal{P}(\omega_1) = \bigcup\{\mathcal{P}(\omega_1) \cap p^* \mid p \in g_{\vec{A}} \wedge \mu: p \to p^* \text{ is guided by } g_{\vec{A}}\}$.

By Corollary 4.16, $g_{\vec{A}}$ witnesses $\mathbb{V}_{\max}$-$(*)$. $\Box$

Remark 4.46. If additionally there are a proper class of Woodin cardinals, then $g_{\vec{A}}$ meets all ∞-universally Baire dense subsets of $\mathbb{V}_{\max}$.

4.6 The second blueprint

From the right perspective, $\mathbb{V}_{\max}$-$(*)$ is a forcing axiom. As noted before, Asperó-Schindler show that if there is a proper class of Woodin cardinals, then $(*)$ is equivalent to $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}$. Some additional assumption like large cardinals is necessary as BMM implies closure of V under sharps while $(*)$ holds in the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$. We try to generalize this result roughly to all natural $\mathbb{P}_{\max}$-variations for which the $\mathbb{P}^\diamondsuit$-method can prove them from some forcing axiom. We will have to restrict to better behaved $\mathbb{P}_{\max}$-variations.

Definition 4.47. Let $\mathbb{V}_{\max}$ be a $\mathbb{P}_{\max}$-variation with unique iterations and g be $\mathbb{V}_{\max}$-generic over $L(\mathbb{R})$.

(i) We say that g *produces* $(A_0, \ldots, A_{n^{\mathbb{V}_{\max}}})$ if there is $p \in g$ so that if

$$\mu: p \to p^* = (M, I, a_0, \ldots, a_{n^{\mathbb{V}_{\max}}})$$

is the g-iteration of p then $a_i = A_i$ for all $i \leqslant n^{\mathbb{V}_{\max}}$.

(*ii*) If $\mathbb{V}_{\max}$ is typical, we set

$$\mathcal{H}_g := (H_{\omega_2}, \mathrm{NS}_{\omega_1}, A_0, \ldots, A_{n^{\mathbb{V}_{\max}}})^{L(\mathbb{R})[g]}$$

where $(A_0, \ldots, A_{n^{\mathbb{V}_{\max}}})$ is the unique sequence produced by g.

Definition 4.48. A $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ with unique iterations is *self-assembling* if: Whenever g is $\mathbb{V}_{\max}$-generic over $L(\mathbb{R})$ then

(*i*) $\mathcal{H}_g$ is almost a $\mathbb{V}_{\max}$-condition and

(*ii*) $(H_{\omega_2})^{L(\mathbb{R})[g]} = \bigcup\{p^* \mid p \in g, \mu\colon p \to p^* \text{ guided by } g\}$.

All $\mathbb{P}_{\max}$-variation we will work with are self-assembling (assuming AD in $L(\mathbb{R})$). For example, $\mathbb{P}_{\max}$ is self-assembling. The relevance of this property for us is partly explained by the following result.

Lemma 4.49. *Suppose $\mathbb{V}_{\max}$ is a self-assembling $\mathbb{P}_{\max}$-variation with unique iterations and typicality of $\mathbb{V}_{\max}$ is witnessed by a set Ψ of $(\Sigma_1 \cup \Pi_1)$-formulae. If $\mathbb{V}_{\max}$-$(*)$ holds as witnessed by g then*

(*i*) *$\mathcal{H}_{\vec{A}}$ is almost a $\mathbb{V}_{\max}$-condition and*

(*ii*) *$g = g_{\vec{A}}$*

where g produces $\vec{A}$.

Proof. As $\mathbb{V}_{\max}$ is self-assembling, $\mathcal{H}_g$ is almost a $\mathbb{V}_{\max}$-condition. Moreover, $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[g]$ as g witnesses $\mathbb{V}_{\max}$-$(*)$. It follows that $\mathcal{H}_g = \mathcal{H}_{\vec{A}}$ and thus (*i*) holds.
Let us now prove (*ii*), note that it suffices to show $g \subseteq g_{\vec{A}}$.

Claim 4.50. *If $q \in g$ and*

$$\sigma\colon q \to q^* = (M^*, I^*, a_0^*, \ldots, a^*_{n^{\mathbb{V}_{\max}}})$$

is the g-iteration of q then $I^* = \mathrm{NS}_{\omega_1} \cap M^*$ and $a_i^* = A_i$ for $i \leqslant n^{\mathbb{V}_{\max}}$.

Proof. $a_i^* = A_i$ for $i \leqslant n^{\mathbb{V}_{\max}}$ follows easily from typicality, we show $I^* = \mathrm{NS}_{\omega_1} \cap M^*$. It is clear that $I^* \subseteq \mathrm{NS}_{\omega_1}$ since if $S \in I^*$, then a tail of the iteration points of the iteration $\sigma\colon q \to q^*$ is missing from S. On the other hand, suppose $S \in \mathcal{P}(\omega_1)^{M^*} - I^*$. We may assume $S = \mu(\bar{S})$ for some $\bar{S} \in q$. If $C \subseteq \omega_1$ is club then as $\mathbb{V}_{\max}$ is self-assembling, there is $r \in g$, such that if $\nu\colon r \to r^*$ is the g-iteration of r, then $C \in \mathrm{ran}(\nu)$, say $C = \nu(\bar{C})$. Note that we may assume $r <_{\mathbb{V}_{\max}} q$, say this is witnessed by

$$\bar{\sigma}\colon q \to \bar{q} = (\bar{M}, \bar{I}, \bar{a}).$$

Write $r = (N, J, b)$. As $\mathbb{V}_{\max}$ is typical, $\bar{I} = J \cap \bar{M}$ and hence $\bar{\sigma}(\bar{S}) \cap \bar{C} \neq \emptyset$ which gives

$$\nu \circ \bar{\sigma}(\bar{S}) \cap C \neq \emptyset.$$

Clearly, $\nu(\bar{\sigma})$ is an iteration of q of length $\omega_1 + 1$ guided by g. Thus, by Lemma 4.11, $\nu(\bar{\sigma}) = \sigma$. $S \cap C \neq \emptyset$ follows. $\square$

Let $p \in g$ and let $\mu\colon p \to p^*$ be the g-iteration of p.

Claim 4.51. $\mathcal{H}_{\vec{A}} \models \varphi^{\mathbb{V}_{\max}}(p^*)$.

Proof. Let $\psi \in \Psi$ and assume ψ is Σ_1, so write $\psi(x) = \exists y\ \theta(x, y)$ where θ is Σ_0. So suppose for some $x \in p$ and $y \in H_{\omega_2}$ we have

$$\mathcal{H}_{\vec{A}} \models \exists y\ \theta(x, y).$$

As $\mathbb{V}_{\max}$ is self-assembling, we can find $q \in g$ with

$(q.i)$ $q <_{\mathbb{V}_{\max}} p$ as witnessed by $\bar{\mu}\colon p \to \bar{p}$ and

$(q.ii)$ $\mathcal{H}_{\vec{A}} \models \theta(x, \sigma(y))$ for some $y \in q$

where $\sigma\colon q \to q^*$ is the g-iteration of q. By Claim 4.50,

$$q^* \prec_{\Sigma_0} \mathcal{H}_{\vec{A}}$$

and as $\sigma(\bar{\mu}) = \mu$ by Lemma 4.11 as well as elementarity of σ we find

$$q \models \theta(\bar{\mu}(x), y).$$

Finally, $q \models (\varphi^{\mathbb{V}_{\max}}(\bar{p}))$ so that

$$\bar{p} \models \exists z\ \theta(\bar{\mu}(x), z)$$

and hence $p \models \exists z\ \theta(x, z)$ by elemntarity of $\bar{\mu}$.
The "dual argument" works if ψ is Π_1 instead. □

Now if G is $\mathrm{Col}(\omega, 2^{\omega_1})$-generic then the above shows that $\mu\colon p \to p^*$ witnesses $\mathcal{H}_{\vec{A}} <_{\mathbb{V}_{\max}} p$ in $V[G]$. Thus $p \in g_{\vec{A}}$. □

Theorem 4.44 gives a hint how the forcing axiom equivalent to $\mathbb{V}_{\max}$-$(*)$ should look like. However, $\Gamma^{\Psi}_{\vec{A}}$ is not the right class of forcings, for example one can construe two $\mathbb{P}_{\max}$-variations which are the same as forcings, but for which the resulting classes $\Gamma^{\Psi}_{\vec{A}}$ are fundamentally different for reasonable $\vec{A}$. Instead, we should look at the class of forcings which roughly lie on the way to the good extensions highlighted in the $\mathbb{V}_{\max}$-Multiverse View.

Definition 4.52. Suppose that

(i) $\mathbb{V}_{\max}$ is a typical $\mathbb{P}_{\max}$-variation,

(ii) typicality of $\mathbb{V}_{\max}$ is witnessed by Ψ and

(iii) $\vec{A} = (A_0, \ldots, A_{n^{\mathbb{V}_{\max}}}) \in H_{\omega_2}$.

The class $\Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi)$ consists of all $(\Psi, \vec{A})$-preserving forcings $\mathbb{P}$ so that if g is $\mathbb{P}$-generic, then there is a forcing $\mathbb{Q} \in V[g]$ with

$$V[g] \models \text{“}\mathbb{Q} \text{ is } (\Psi, \vec{A})\text{-preserving”}$$

and if further h is $\mathbb{Q}$-generic over $V[g]$, then in $V[g][h]$ both

$(h.i)$ $\mathcal{H}_{\vec{A}}$ is almost a $\mathbb{V}_{\max}$-condition and

$(h.ii)$ NS_{ω_1} is saturated.

It just so happens that, maybe by accident, for the $\mathbb{P}_{\max}$-variations we will look at explicitly, if there is a proper class of Woodin cardinals then one can choose Ψ so that $\Gamma^{\Psi}_{\vec{A}} = \Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi)$ in case that $\Gamma^{\mathbb{V}_{\max}}_{\vec{A}} \neq \emptyset$.

Definition 4.53. Suppose that (M, I) is a potentially iterable structure and $Y \subseteq \mathbb{R}$. We say that (M, I) is (generically) *Y-iterable* if for $X := Y \cap M$ we have

(i) $(M; \in, I, X)$ is a model of (sufficiently much of) ZFC where Y is allowed as a class parameter in the schemes and

(ii) whenever $\langle (M_\alpha, I_\alpha, X_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$ is a generic iteration of $(M_0, I_0, X_0) = (M, I, X)$, i.e.

$(\mu.i)$ $(M_{\alpha+1}; \in, I_{\alpha+1}, X_{\alpha+1})$ is an ultrapower of $(M_\alpha; \in, I_\alpha, X_\alpha)$ by a M_α-generic ultrafilter w.r.t. I_α for $\alpha < \gamma$,

$(\mu.ii)$ if $\alpha \leqslant \gamma$ is a limit then

$$\langle (M_\alpha, I_\alpha, X_\alpha), \mu_{\xi,\alpha} \mid \xi < \alpha \rangle = \varinjlim(\langle (M_\beta, I_\beta, X_\beta), \mu_{\beta,\xi} \mid \beta \leqslant \xi < \alpha \rangle)$$

then $X_\gamma = Y \cap M_\gamma$.

Proposition 4.54 (Folklore)**.** *Suppose that NS_{ω_1} is saturated and $X \subseteq \mathbb{R}$ is ∞-universally Baire. Then in any forcing extension $V[G]$ in which $H^V_{\omega_2}$ is countable, $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, X)^V$ is X^*-iterable.*

Proof. Let $\mathbb{P}$ be some forcing which collapses 2^{ω_1} to ω. Let $T, S \in V$ witness that X is $|\mathbb{P}|$-universally Baire with $p[T] = X, p[S] = \mathbb{R} - X$. Let G be $\mathbb{P}$-generic over V. Let

$$\langle (M_\alpha, I_\alpha, X_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$$

be any generic iteration of $(M_0, I_0, X_0) = (H_\kappa, \mathrm{NS}_{\omega_1}, X)^V$. Then as in Claim 4.30, this iteration can be lifted to a generic iteration

$$\langle (M^+_\alpha, I_\alpha, X_\alpha), \mu^+_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$$

of $(M^+_0, I_0, X_0) = (V, \mathrm{NS}^V_{\omega_1}, X)$. In particular, M_γ is wellfounded as M^+_γ is wellfounded. Let $\mu^+ = \mu^+_{0,\gamma}$, $M^+ = M^+_\gamma$.

Claim 4.55. *In $V[G]$, $p[\mu^+(T)] = X^*$.*

Proof. Work in $V[G]$. We have $X^* = p[T]$ and this implies $X^* \subseteq p[\mu^+(T)]$, likewise $\mathbb{R} - X^* \subseteq p[\mu^+(S)]$. In M^+, $\mu^+(T), \mu^+(S)$ project to complements and an absoluteness of wellfoundedness argument shows that this must be true in $V[G]$ as well, so that we indeed have $X^* = p[\mu^+(T)]$. □

We conclude

$$X_\gamma = \mu^+(X) = \mu^+(p[T]) = p[\mu^+(T)] \cap M^+ = X^* \cap M^+ = X^* \cap M_\gamma$$

which is what we had to show. □

Lemma 4.56. *Suppose that*

(i) $\mathbb{V}_{\max}$ *is a typical self-assembling* $\mathbb{P}_{\max}$*-variation with unique iterations,*

(ii) typicality of $\mathbb{V}_{\max}$ *is witnessed by a set of* $(\Sigma_1 \cup \Pi_1)$*-formulae* Ψ*,*

(iii) there is a proper class of Woodin cardinals,

(iv) $\mathbb{V}_{\max}$-$(*)$ *holds as witnessed by* g *and*

(v) g *produces* $\vec{A}$.

Then $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi))$ *holds true.*

Proof. We will assume $n^{\mathbb{V}_{\max}} = 0$. Let g witness $\mathbb{V}_{\max}$-$(*)$. Let $p \in g$ and $\mu\colon p \to p^* = (M, I, A)$ the generic iteration of p guided by g. We will show that

$$(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))\text{-}\mathrm{BFA}^{\Psi}_{A}(\Gamma^{\mathbb{V}_{\max}}_{A}(\Psi))$$

holds. By Lemma 4.49, $\mathcal{H}_g = \mathcal{H}_A$ is almost a $\mathbb{V}_{\max}$-condition. Now let $\mathbb{P} \in \Gamma^{\mathbb{V}_{\max}}_{A}(\Psi)$ and $X \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. Let G be $\mathbb{P}$-generic. We have to show that

$$(H_{\omega_2}; \in, X, R^{\psi}_{A} \mid \psi \in \Psi)^V \prec_{\Sigma_1} (H_{\omega_2}; \in, X^*, R^{\psi}_{A} \mid \psi \in \Psi)^{V[G]}.$$

So let $v \in H^V_{\omega_2}$, and θ a Σ_0-formula such that

$$(H_{\omega_2}; \in, X^*, R^{\psi}_{A} \mid \psi \in \Psi)^{V[G]} \models \exists u\ \theta(u, v).$$

As $\mathbb{V}_{\max}$ is self-assembling, we may assume without loss of generality that $v = \mu(\bar{v})$ for some $\bar{v} \in p$. Let $V[G][H]$ be a further generic extension by (Ψ, A)-preserving forcing so that in $V[G][H]$

(H.i) $\mathcal{H}^{V[G][H]}_{A}$ is almost a $\mathbb{V}_{\max}$-condition and

(H.ii) NS_{ω_1} is saturated.

Note that

$$(H_{\omega_2};\in,X^*,R_A^\psi\mid\psi\in\Psi)^{V[G]}\prec_{\Sigma_0}(H_{\omega_2};\in,X^{**},R_A^\psi\mid\psi\in\Psi)^{V[G][H]}$$

as the extension is (Ψ,A)-preserving. Here, X^{**} denotes the reevaluation of X^* in $V[G][H]$. Accordingly,

$$(H_{\omega_2};\in,X^{**},R_A^\psi\mid\psi\in\Psi)^{V[G][H]}\models\exists u\ \theta(u,v).$$

Let g be $\mathrm{Col}(\omega,2^{\omega_1})^{V[G][H]}$-generic over $V[G][H]$ and X^{***} the reevaluation of X^{**} in $V[G][H][g]$. Then in $V[G][H][g]$,

$$(H_{\omega_2},\mathrm{NS}_{\omega_1},X^{**})^{V[G][H]}$$

is X^{***}-iterable by Proposition 4.54.

Claim 4.57. $\mathcal{H}_A^{V[G][H]}<_{\mathbb{V}_{\max}} q$ *for all* $q\in g$.

Proof. Let $q\in g$ and $\sigma\colon q\to q^*$ the g-iteration of q. It follows from the proof of Lemma 4.49 that

$$(H_{\omega_2};\in,\mathrm{NS}_{\omega_1},A)^V\models\varphi^{\mathbb{V}_{\max}}(q^*)$$

and since the extension $V\subseteq V[G][H]$ is (Ψ,A)-preserving,

$$(H_{\omega_2};\in,\mathrm{NS}_{\omega_1},A)^{V[G][H]}\models\varphi^{\mathbb{V}_{\max}}(q^*)$$

follows. □

Let $q\in g$, $q<_{\mathbb{V}_{\max}}p$ as witnessed by $\bar\mu\colon p\to\bar p$. $\mathcal{H}_A^{V[G][H]}$ witnesses in $V[G][H][g]$ that there is $r=(M,I,a)<_{\mathbb{V}_{\max}}q$, as witnessed by $\sigma\colon q\to q^*$, so that

(r.i) (M,I,Y) is X^{***}-iterable,

(r.ii) $(M;\in,I)\models$ “$V=H_{\omega_2}\wedge I=\mathrm{NS}_{\omega_1}$” and

(r.iii) $(M;\in,Y,R_A^\psi\mid\psi\in\Psi)^M\models\exists u\ \theta(u,\sigma(\bar\mu(\bar v))$

where $Y=X^{***}\cap M$. As there is a proper class of Woodin cardinals,

$$(L(\mathbb{R})^V;\in,X)\equiv(L(\mathbb{R})^{V[G][H][g]};\in,X^{***})$$

and hence a density argument shows that there is $q=(N,J,b)\in g$, $q<_{\mathbb{V}_{\max}}p$, as witnessed by $\mu'\colon p\to p'$, such that

(q.i) $(N,J,X\cap N)$ is X-iterable,

(q.ii) $(N;\in,J)\models$ “$V=H_{\omega_2}\wedge J=\mathrm{NS}_{\omega_1}$” and

$(q.iii)$ for some $u \in N$, $(N; \in, X \cap N, R_A^{\psi} \mid \psi \in \Psi)^N \models \theta(u, \mu'(v))$.

Let $\sigma\colon q \to q^* = (N^*, J^*, a^*)$ be the g-iteration of q. By (the proof of) Lemma 4.49 (ii)

$$(H_{\omega_2}, \mathrm{NS}_{\omega_1}, A)^V \models \varphi^{\mathbb{V}_{\max}}(q^*)$$

and hence

$$(N^*; \in, X \cap N^*, R_A^{\psi} \mid \psi \in \Psi)^{N^*} \prec_{\Sigma_0} (H_{\omega_2}; \in, X, R_A^{\psi} \mid \psi \in \Psi)^V.$$

Moreover,

$$\sigma\colon (N, J, X \cap N) \to (N^*, J^*, X \cap N^*)$$

is fully elementary by $(q.i)$ so that

$$(N^*; \in, X \cap N^*, R_A^{\psi} \mid \psi \in \Psi)^{N^*} \models \theta(\sigma(u), \sigma(\mu'(v))).$$

By Lemma 4.11, $\sigma \circ \mu' = \mu$, so we can conclude

$$(H_{\omega_2}; \in, X, R_A^{\psi} \mid \psi \in \Psi)^V \models \theta(\sigma(u), v)$$

which is what we had to show. □

In fact, we get an equivalence in case we can apply the $\mathbb{P}^{\diamondsuit}$-method.

Second Blueprint Theorem 4.58. *Suppose that*

(i) *There are a proper class of Woodin cardinals,*

(ii) $\mathbb{V}_{\max}$ *is a self-assembling typical* $\mathbb{P}_{\max}$*-variation,*

(iii) $\mathbb{V}_{\max}$ *has unique iterations and accepts* $\diamondsuit$*-iterations,*

(iv) *typicality of* $\mathbb{V}_{\max}$ *is witnessed by a set* Ψ *of* $(\Sigma_1 \cup \Pi_1)$*-formulae,*

(v) $\vec{A} = (A_0, \ldots, A_{n^{\mathbb{V}_{\max}}}) \in H_{\omega_2}$ *and*

(vi) $\Gamma^{\Psi}_{\vec{A}} = \Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi)$.

The following are equivalent:

$(\ast.i)$ *There is a filter* $g \subseteq \mathbb{V}_{\max}$ *which witnesses* $\mathbb{V}_{\max}$*-*$(*)$ *and produces* $\vec{A}$.

$(\ast.ii)$ $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BFA}^{\Psi}_{\vec{A}}(\Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi))$.

Proof. "$(\ast.i) \Rightarrow (\ast.ii)$" follows from Theorem 4.56. "$(\ast.ii) \Rightarrow (\ast.i)$" can be proven similar to the First Blueprint Theorem 4.44. We use the existence of a proper class of Woodin cardinals instead of SRP to justify $\mathrm{AD}^{L(\mathbb{R})}$, that all sets of reals in $L(\mathbb{R})$ are ∞-universally Baire and generic $L(\mathbb{R})$-absoluteness. It is not immediate that $\mathcal{H}_{\vec{A}}$ is almost a $\mathbb{V}_{\max}$-condition, nor did we assume that NS_{ω_1} is saturated, however as $\Gamma^{\mathbb{V}_{\max}}_{\vec{A}}(\Psi) = \Gamma^{\Psi}_{\vec{A}}$, we can pass to a $(\Psi, \vec{A})$-preserving forcing extension in which both of this is true. It follows that

$$g = \{p \in \mathbb{V}_{\max} \mid \exists \mu\colon p \to p^* \text{ a generic iteration of length } \omega_1 + 1 \text{ with } \mathcal{H}_{\vec{A}} \models \varphi^{\mathbb{V}_{\max}}(p^*)\}$$

witnesses $\mathbb{V}_{\max}$-$(*)$ and produces $\vec{A}$. □

5 Instances of $\mathrm{MM}^{++}(f)$

We take closer look at the theory of $\diamondsuit(\mathbb{B})$ and $\mathrm{MM}^{++}(f)$ for several instances of $\mathbb{B}$.

5.1 $\mathbb{B} = \{\mathbb{1}\}$ is the trivial forcing and $(*)$

If we plug in the trivial forcing into the machinery developed here, we completely recover the picture of $\mathrm{MM}^{++} \Rightarrow (*)$. Observe that $\diamondsuit(\{0\})^*$ is canonically witnessed by

$$f : \omega_1 \to \omega_1,\ f(\alpha) = \begin{cases} \varnothing & \text{if } \alpha = 0 \\ \{0\} & \text{else.} \end{cases}$$

Now observe that

- f-complete forcing is the same as complete forcing,
- f-proper forcing is the same as proper forcing,
- f-semiproper forcing is the same as semiproper forcing,
- f-stationary is the same as stationary,
- $\mathrm{PFA}(f)$ is equivalent to PFA and
- $\mathrm{MM}^{++}(f)$ is equivalent to MM^{++}.

In particular, we recover versions of Shelah's iteration theorems for proper and semiproper forcings.

Corollary 5.1 (Shelah)**.** *Countable support iterations of proper forcings are proper.*

Corollary 5.2 (Miyamoto)**.** *Nice iterations of semiproper forcings are semiproper.*

Shelah has proven the above for iterations with revised countable support.

Corollary 5.3 (Foreman-Magidor-Shelah)**.** *If there is a supercompact cardinal then* MM^{++} *holds in a generic extension by semiproper forcing.*

We also mention that Shelah's version of S-properness and S-semiproperness for a stationary set $S \subseteq \omega_1$ can be expressed naturally in our context: Let f_S be defined by

$$f(\alpha) = \begin{cases} \{0\} & \text{if } \alpha \in S - \{0\} \\ \varnothing & \text{else} \end{cases}$$

and note that f_S-(semi)properness is equivalent to S-(semi)properness. We mention without proof that we can prove two of the main results of [AS21] from the Blueprint Theorems and $\mathbb{P}_{\max}$-theory.

Theorem 5.4 (Asperó-Schindler). $\mathrm{MM}^{++} \Rightarrow (*)$.

Theorem 5.5 (Asperó-Schindler). *If there is a proper class of Woodin cardinals, then the following are equivalent:*

(*i*) $(*)$.

(*ii*) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}$.

5.2 $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$ and $\mathbb{C}_{\max}$-$(*)$

Technically, $\mathrm{Col}(\omega, \omega_1)$ is not a subset of ω_1, but we can easily find a partial order $\leqslant$ on ω_1 so that

$$\mathrm{Col}(\omega, \omega_1) \cong (\omega_1, \leqslant).$$

There will then be a club $C \subseteq \omega_1$ so that

$$\mathrm{Col}(\omega, \alpha) \cong (\alpha, \leqslant\restriction \alpha)$$

for all $\alpha \in C$. Hence we can replace $\mathbb{B}$ by $\mathrm{Col}(\omega, \omega_1)$ and $\mathbb{B} \cap \alpha$ by $\mathrm{Col}(\omega, \alpha)$ in the definition of $\diamondsuit(\mathbb{B})$ (and its variants) for all our intents and purposes.

We denote $\diamondsuit(\mathrm{Col}(\omega, \omega_1))$ by $\diamondsuit(\omega_1^{<\omega})$ and $\diamondsuit(\mathrm{Col}(\omega, \omega_1))^+$ by $\diamondsuit^+(\omega_1^{<\omega})$. As mentioned already in Section 2, these are slight strengthenings of principles studied in [Woo10, Section 6.2]. From the results of the previous sections, it is consistent relative to a Woodin cardinal that NS_{ω_1} is saturated and there is a regular embedding of $\mathrm{Col}(\omega, \omega_1)$ into $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$. This is a step closer towards "NS_{ω_1} is ω_1-dense", as that is equivalent to a dense embedding of $\mathrm{Col}(\omega, \omega_1)$ into $(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$ and implies saturation. Moreover, "NS_{ω_1} saturated" does not imply even $\diamondsuit(\omega_1^{<\omega})$. If MA_{ω_1}, holds then $\diamondsuit(\omega_1^{<\omega})$ fails. For example there is a nonmeager set of reals of size $\aleph_1$ under $\diamondsuit(\omega_1^{<\omega})$, see Proposition 5.27. Observe that this implies that random forcing kills all ground model witnesses of $\diamondsuit(\omega_1^{<\omega})$. Moreover, if MM holds then NS_{ω_1} is saturated while $\diamondsuit(\omega_1^{<\omega})$ fails.

Fact 5.6 (Woodin,[Woo10, Theorem 6.49]). *If $\diamondsuit(\omega_1^{<\omega})$ holds then there is a Suslin tree*[36].

From a witness f of $\diamondsuit(\omega_1^{<\omega})$, Woodin describes a particular construction similar to the construction of a Suslin tree from $\diamondsuit$. Forcing with this tree does not preserve f. Given that such simple forcings can destroy "f witnesses $\diamondsuit(\omega_1^{<\omega})$", it seems quite miraculous how much of the MM^{++}-theory carries through to $\mathrm{MM}^{++}(f)$.

Corollary 5.7. *Any model of* ZFC *has a forcing extension in which $\diamondsuit^+(\mathsf{C})$ holds and $\diamondsuit(\omega_1^{<\omega})$ fails.*

[36]Recall that a Suslin tree is a tree T of height ω_1 so that $(T, \geqslant_T)$ has no uncountable antichains.

We will now apply the Blueprint Theorems in the context $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$. First we define the corresponding $\mathbb{P}_{\max}$-variation. Although not exactly necessary, it seems natural in this context to generalize $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$ to different ideals then the nonstationary ideal, i.e. we replace stationary by "is in I^+" and "contains a club" by "the complement is in I" for the formulations of $\diamondsuit(\mathbb{B})$, $\diamondsuit^+(\mathbb{B})$ for which this makes sense.

Definition 5.8. Let $\mathbb{B} \subseteq \omega_1$ be a forcing and I a normal uniform ideal on ω_1. $\diamondsuit_I(\mathbb{B})$ states that there is a function f so that

(*i*) f guesses $\mathbb{B}$-filters and

(*ii*) if $\langle D_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{B}$ and $b \in \mathbb{B}$ then

$$\{\alpha < \omega_1 \mid b \in f(\alpha) \wedge \forall \beta < \alpha\ f(\alpha) \cap D_\beta \neq \emptyset\} \in I^+.$$

$\diamondsuit_I^+(\mathbb{B})$ results from $\diamondsuit_I(\mathbb{B})$ by replacing (*ii*) above with:

$(ii)^+$ For any dense $D \subseteq \mathbb{B}$,

$$\{\alpha < \omega_1 \mid f(\alpha) \cap D = \emptyset\} \in I$$

and moreover for any $b \in \mathbb{B}$, we have

$$S_b^f = \{\alpha < \omega_1 \mid b \in f(\alpha)\} \in I^+.$$

Observe that $\diamondsuit(\mathbb{B}) = \diamondsuit_{\mathrm{NS}_{\omega_1}}(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B}) = \diamondsuit^+_{\mathrm{NS}_{\omega_1}}(\mathbb{B})$. We also define ψ_{AC} relativized to a normal uniform ideal.

Definition 5.9. Suppose I is a normal uniform ideal on ω_1. Then $\psi_{\mathrm{AC}}(I)$ holds iff for any $S, T \in I^+$ with $\omega_1 - S, \omega_1 - T \in I^+$ there is a canonical function η_ξ for some $\xi < \omega_2$ with

$$S = \eta_\xi^{-1}[T] \mod I.$$

Definition 5.10. $\mathbb{C}_{\max}$ is the following $\mathbb{P}_{\max}$-variation: Conditions are generically iterable structures

$$p = (M^p, I^p, f^p) = (M, I, f)$$

so that

$(\mathbb{C}_{\max}.i)$ $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$ and

$(\mathbb{C}_{\max}.ii)$ $(M; \in, I) \models$ "f witnesses $\diamondsuit_I^+(\omega_1^{<\omega})$".

The order on $\mathbb{C}_{\max}$ is defined as follows: We let

$$q = (N, J, g) < (M, I, f) = p$$

iff there is a generic iteration

$$\mu : p \to p^* = (M^*, I^*, f^*)$$

of p in q so that $I^* = J \cap p^*$ and $f^* = g^*$.

The point of ($\mathbb{C}_{\max}.i$) is to make sure that $\mathbb{C}_{\max}$ has unique iterations.

Fact 5.11 (Woodin,[Woo10, Lemma 5.15])**.** *Suppose (M, I) is a potentially iterable structure and $a \in \mathcal{P}(\omega_1)$, so that*

(*i*) $(M; \in, I) \models$ "*I is a normal uniform ideal on ω_1*",

(*ii*) $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$ *and*

(*iii*) $M \models \omega_1^{L[a]} = \omega_1$.

If $\mu_i : M \to M_i^*$ are generic iterations of (M, I) with M_i^* transitive for $i < 2$ and $\mu_0(a) = \mu_1(a)$ then

$$M_0^* = M_1^* \text{ and } \mu_0 = \mu_1.$$

Note that if f witnesses $\diamondsuit_I^+(\omega_1^{<\omega})$ then f is "essentially a subset of ω_1" and $\omega_1^{L(f)} = \omega_1$.

We remark that $\mathbb{P}_{\max}$ makes use of MA_{ω_1} for this purpose. The relevant consequence of MA_{ω_1} which makes this work is:

Definition 5.12. Coding holds if for any sequence $\langle a_\beta \mid \beta < \omega_1\rangle$ of pairwise almost disjoint sets in $[\omega]^\omega$ and any subset A of ω_1 there is $c \subseteq \omega$ with

$$\beta \in A \Leftrightarrow c \cap a_\beta \text{ is finite}$$

for all $\beta < \omega_1$.

Unfortunately, Coding is inconsistent with $\diamondsuit(\omega_1^{<\omega})$ and we will prove this later, see Lemma 5.45.

Let us define

- $\psi_0^{\mathbb{C}_{\max}} =$ "$x = \dot{f}$" and
- $\psi_1^{\mathbb{C}_{\max}} =$ "$x \in \dot{I}_{\dot{f}}$".

Then, for all our intents and purposes, $\Psi^{\mathbb{C}_{\max}} = \{\psi_i^{\mathbb{C}_{\max}} \mid i < 2\}$ witnesses the typicality of $\mathbb{C}_{\max}$: "$x \in \dot{I}$" is not an element of $\Psi^{\mathbb{C}_{\max}}$, but is implied by $\psi_1^{\mathbb{C}_{\max}}$ in the context it is used in. Moreover, if f witnesses $\diamondsuit(\omega_1^{<\omega})$ then Γ_f^Ψ is exactly the class of f-stationary set preserving forcings. Also $\psi_0^{\mathbb{C}_{\max}}$ is atomic while $\psi_1^{\mathbb{C}_{\max}}$ is (equivalent to) a Σ_1-formula. Note that ω_1 can be defined by a Σ_1 formula as the Mostowski collapse of a set with complement in $\dot{I}$. Moreover, $\mathbb{C}_{\max}$ accepts $\diamondsuit$-iterations by Lemma 4.19.

Theorem 5.13. *If f witnesses $\diamondsuit(\omega_1^{<\omega})$ then $\mathrm{MM}^{++}(f) \Rightarrow \mathbb{C}_{\max}$-$(*)$.*

Proof. Let f witness $\diamondsuit(\omega_1^{<\omega})$ so that $\mathrm{MM}^{++}(f)$ holds. SRP holds by Lemma 3.69 and f is a witness of $\diamondsuit^+(\omega_1^{<\omega})$ by Theorem 3.75 so that $\mathcal{H}_f$ is almost a $\mathbb{C}_{\max}$-condition. $\mathbb{C}_{\max}$-$(*)$ follows from the First Blueprint Theorem 4.44. □

These axioms are MM^{++} and $(*)$ respectively conditioned on the existence of a complete embedding

$$\eta : \mathrm{Col}(\omega, \omega_1) \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+.$$

However, NS_{ω_1} is not ω_1-dense under either of these axioms. The reason for this can be seen from two different perspectives: On the forcing axiom side, the "++"-part of $\mathrm{MM}^{++}(f)$ implies that for any set $\mathcal{S}$ of ω_1-many stationary sets, there is a (f-)stationary set T so that $T \cap S$, $S - T$ is (f-)stationary for all $S \in \mathcal{S}$, as the generic set added by $\mathrm{Add}(\omega_1, 1)$ has this property. We will see later that the weaker $\mathrm{MM}(f)$ is indeed consistent with "NS_{ω_1} is ω_1-dense", assuming appropriate large cardinals of course. If one takes a look at the standard iteration to produce a model of $\mathrm{MM}^{++}(f)$, the "++"-part comes from the fact that f-stationary sets are preserved along the iteration. When producing a model of a forcing axiom which implies "NS_{ω_1} is ω_1-dense" later, we will and must often kill f-stationary sets during the iteration.

On the other side, the fact that NS_{ω_1} is not ω_1-dense under $\mathbb{C}_{\max}$-$(*)$ is due to the ideals being preserved along the order, that is: If $q < p$ in $\mathbb{C}_{\max}$, this is witnessed by a generic iteration $\mu : p \to p^* = (M^*, I^*, f^*)$ with, crucially,

$$I^q \cap p^* = I^*.$$

If $M^p \models \mathcal{S} \in [(I^+)^p]^{\omega_1}$, then there are dense-below-p many $q < p$ so that if $\mu : p \to p^*$ witnesses $q < p$, then $\mu(\mathcal{S})$ is not dense in $(I^+)^q$, as witnessed by some T. This can then not be removed anymore: If further $r < q$ as witnessed by

$$\eta : q \to q^*$$

then, as $I^r \cap q^* = I^{q^*}$, $\eta(T)$ will witness in r that $\eta \circ \mu(\mathcal{S})$ is not dense in $(I^+)^r$. This suggests to drop the compatibility condition on the ideals if one wants to force "NS_{ω_1} is ω_1-dense". This will later lead us to $\mathbb{Q}^-_{\max}$.We will later see that $\mathbb{C}_{\max}$ is self-assembling assuming $\mathrm{AD}^{L(\mathbb{R})}$, see Corollary 8.12.

Theorem 5.14. *If there is a proper class of Woodin cardinals then the following are equivalent:*

(*i*) $\mathbb{C}_{\max}$-$(*)$.

(*ii*) *There is a witness f of $\diamondsuit(\omega_1^{<\omega})$ so that $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(f)$ holds.*

Proof. Note that as there is a proper class of Woodin cardinals, if f witnesses $\diamondsuit(\omega_1^{<\omega})$ then $\Gamma_f^{\Psi^{\mathbb{C}_{\max}}} = \Gamma_f^{\mathbb{C}_{\max}}(\Psi^{\mathbb{C}_{\max}})$ by Theorem 3.60. Now if g witnesses $\mathbb{C}_{\max}$-$(*)$ and g produces f then f witnesses $\diamondsuit(\omega_1^{<\omega})$ by Lemma 4.49. The desired equivalence follows from the Second Blueprint Theorem 4.58. $\square$

Finally, we will separate $\diamondsuit(\omega_1^{<\omega})$ from $\diamondsuit^+(\omega_1^{<\omega})$ to highlight the differences between the two principles.

Proposition 5.15. *No function $f \in V$ witnesses $\diamondsuit^+(\omega_1^{<\omega})$ in $V^{\mathrm{Add}(\omega_1,1)}$.*

Proof. We may assume that f witnesses $\diamondsuit(\omega_1^{<\omega})$ in V, otherwise f certainly does not witness $\diamondsuit^+(\omega_1^{<\omega})$ in any forcing extension. Furthermore, we may replace the role of $\mathrm{Col}(\omega,\alpha)$ in $\diamondsuit^+(\omega_1^{<\omega})$ by the poset $\mathrm{Col}_{\mathrm{inc}}(\omega,\alpha)$ of increasing conditions in $\mathrm{Col}(\omega,\alpha)$ as these two forcings are isomorphic for additively closed α (and uniformly so in α). We may further assume that

$$f : \omega_1 \to H_{\omega_1}$$

is a function with

$$f(\alpha) \text{ is a maximal filter in } \mathrm{Col}_{\mathrm{inc}}(\omega,\alpha)$$

for all nonzero $\alpha < \omega_1$. We will find a dense $D \subseteq \mathrm{Col}_{\mathrm{inc}}(\omega,\omega_1)$ in $V^{\mathrm{Add}(\omega_1,1)}$ so that

$$\{\alpha < \omega_1 \mid f(\alpha) \cap D = \varnothing\}$$

is stationary. Let G be $\mathrm{Add}(\omega_1,1)$-generic and $g = \bigcup G$. We define D as

$$D := \{p \in \mathrm{Col}_{\mathrm{inc}}(\omega,\omega_1) \mid \exists n \in \mathrm{dom}(p)\ g(p(n)) = 1\}.$$

Let $\dot{D}$ be a name for this set and let $\dot{C}$ be a name for a club in ω_1 and $p \in \mathrm{Add}(\omega_1,1)$. Let $\xi = (\sup\mathrm{dom}(p)) + 1$ and define $b \in \mathrm{Col}_{\mathrm{inc}}(\omega,\omega_1)$ by $b\colon 1 \to \omega_1$, $b(0) = \xi$. Let θ be sufficiently large and regular. As f witnesses $\diamondsuit(\omega_1^{<\omega})$, we can find some f-slim $X \prec H_\theta$ with $p, \dot{C}, \dot{D} \in X$ and $\delta^X \in S_b^f$. We may consider $f(\delta^X)$ as an increasing function from ω into α. We will find $q \leqslant p$ so that

$$q \Vdash \delta^{\check{X}} \in \dot{C} \wedge \check{f}(\delta^{\check{X}}) \cap \dot{D} = \varnothing.$$

Claim 5.16. *There is a filter H generic over M_X for* $\mathrm{Add}(\delta^X, 1)^{M_X}$ *with $p \in H$ so that for all $n < \omega$, $h(f(\delta^X)(n)) = 0$, where $h = \bigcup H$.*

Proof. Let us write α_n for $f(\delta^X)(n)$. Let $(D_n)_{n<\omega}$ be an enumeration of all dense open subsets of $\mathrm{Add}(\delta^X, 1)^{M_X}$ in M_X. Let us define a decreasing sequence $\langle p_n \mid n < \omega\rangle$ by induction satisfying

($\vec{p}.i$) $p_0 = p$,

($\vec{p}.ii$) $p_{n+1} \in D_n$ and

($\vec{p}.iii$) if $\alpha_m \in \mathrm{dom}(p_n)$ for some m then $p_n(\alpha_m) = 0$

for all $n < \omega$. Note that $\alpha_m > \sup \mathrm{dom}(p)$ for all $m < \omega$ as $b \in f(\delta^X)$ and thus $p_0 = p$ satisfies ($\vec{p}.iii$). Now suppose p_n is already defined. A simple density argument shows that there is some $\alpha_m > \mathrm{dom}(p_n)$ so that if p_n' is the extension of p_n to a condition of length α_m+1 by padding with 0's, then there is a further extension p_{n+1} of p_n' with $p_{n+1} \in D_n$ and $\mathrm{dom}(p_{n+1}) < \alpha_{m+1}$. This finishes the construction and the filter H generated by $(p_n)_{n<\omega}$ does the job. □

Let $q = \bigcup H$ is a condition in $\mathrm{Add}(\omega_1, 1)$ below p. The properties of H imply that q is $(X, \mathrm{Add}(\omega_1, 1))$-generic and thus

$$q \Vdash \check{\delta^X} \in \dot{C}.$$

But also by design

$$q \Vdash \dot{D} \cap \check{f}(\check{\delta^X}) = \varnothing.$$

□

Remark 5.17. Note that the q constructed above is $(X, \mathrm{Add}(\omega_1, 1))$-generic but not $(X, \mathrm{Add}(\omega_1, 1), f)$-generic, even though X is f-slim. Thus being $(X, \mathbb{P}, f)$-generic is in some cases strictly stronger than being $(X, \mathbb{P})$-generic, even for f-proper forcings.

Corollary 5.18. *If* GCH *holds then* $V^{\mathrm{Col}(\omega_1, <\omega_2)} \models \diamondsuit(\omega_1^{<\omega}) \wedge \neg\diamondsuit^+(\omega_1^{<\omega})$.

Proof. We can factor $\mathrm{Col}(\omega_1, <\omega_2)$ as $\mathrm{Col}(\omega_1, \omega_1) \times \mathrm{Col}(\omega_1, <\omega_2)$ and by Corollary 2.16, $\diamondsuit(\omega_1^{<\omega})$ holds in $V^{\mathrm{Col}(\omega_1, \omega_1)}$ as well as CH, and this is preserved in the further extension by the σ-closed forcing $\mathrm{Col}(\omega_1, <\omega_2)$ by Lemma 3.5.

Let G be generic for $\mathrm{Col}(\omega_1, <\omega_2)$ and work in $V[G]$. To see that $\diamondsuit^+(\omega_1^{<\omega})$ fails, note that any function

$$f : \omega_1 \to H_{\omega_1}$$

is already in $V[G \restriction \alpha]$ for some $\alpha < \omega_2$, where $G \restriction \alpha = G \cap \mathrm{Col}(\omega_1, <\alpha)$. This follows from GCH in V. If g is the slice of G at α, then f does not witness $\diamondsuit^+(\omega_1^{<\omega})$ in $V[G \restriction \alpha, g]$ by Proposition 5.15. The extension from $V[G \restriction \alpha, g]$ to $V[G]$ is stationary set preserving and hence f does not witness $\diamondsuit^+(\omega_1^{<\omega})$ in $V[G]$ either. □

5.3 Split witnesses

Instead of considering only one witness f of $\diamondsuit(\mathbb{B})$, one can split ω_1 into two stationary sets S, T and consider one witness f_0 of $\diamondsuit(\mathbb{B}_0)$ "on S" and another witness f_1 of $\diamondsuit(\mathbb{B}_1)$ "on T".

Definition 5.19. Suppose $\mathbb{B}_0, \mathbb{B}_1 \subseteq \omega_1$ are forcings. $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$ holds if there is pair (f_0, f_1) so that

(i) f_i witnesses $\diamondsuit(\mathbb{B}_i)$ for $i < 2$ and

(ii) for all $\alpha < \omega_1$, $f_0(\alpha) = \varnothing$ or $f_1(\alpha) = \varnothing$.

$\diamondsuit^+(\mathbb{B}_0, \mathbb{B}_1)$ if additionally

(iii) for any dense $D_0 \subseteq \mathbb{B}_0$, $D_1 \subseteq \mathbb{B}_1$

$$\{\alpha < \omega_1 \mid f_0(\alpha) \cap D_0 \neq \varnothing \vee f_1(\alpha) \cap D_1 \neq \varnothing\}$$

contains a club.

We call (f_0, f_1) a *split witness of* $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$, $\diamondsuit^+(\mathbb{B}_0, \mathbb{B}_1)$ respectively.

Note that if (f_0, f_1) is a split witness of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$ then condition (iii) is equivalent to "f_i witnesses $\diamondsuit^+_{\mathrm{NS}_{\omega_1}\upharpoonright S_i}(\mathbb{B}_i)$" where

$$S_i = \mathrm{supp}(f_i) := \{\alpha < \omega_1 \mid f(\alpha) \neq \varnothing\}$$

for $i < 2$.
All of what we have done so far for usual witnesses of $\diamondsuit(\mathbb{B})$ generalizes naturally to split witnesses. The reason for this is that any split witness (f_0, f_1) of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$ naturally corresponds to a usual witness f of $\diamondsuit(\mathbb{B})$ for some $\mathbb{B}$ as follows: Let $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_1 \subseteq \omega_1$ be the disjoint union of $\mathbb{B}_0$, $\mathbb{B}_1$ coded as a subset of ω_1. Then for a club C and $\alpha \in S_i \cap C$, $i < 2$, we have

$$\tau_i[\mathbb{B}_i \cap \alpha] = \mathrm{ran}(\tau_i) \cap \alpha$$

where τ_i is the canonical regular embedding $\mathbb{B}_i \to \mathbb{B}$. Define f by

$$f\colon \omega_1 \to H_{\omega_1},\ \alpha \mapsto \begin{cases} \tau_0[f_0(\alpha)] & \text{if } \alpha \in S_0 \cap C \\ \tau_1[f_1(\alpha)] & \text{if } \alpha \in S_1 \cap C \\ \varnothing & \text{else.} \end{cases}$$

We call (f_0, f_1) *a split of* f. Clearly, this translation procedure has an inverse: If f witnesses $\diamondsuit(\mathbb{B}_0 \oplus \mathbb{B}_1)$ and we define f_i by $f_i(\alpha) = \tau_i^{-1}[f(\alpha)]$ if $\alpha \in \mathrm{ran}(\tau_i) \cap C$ and $f_i(\alpha) = \varnothing$ otherwise, then (f_0, f_1) is a split witness of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$ and is a split of f. On the club C, (f_0, f_1) "behaves the same as f". In particular, we can express properties of (f_0, f_1) that only depend on f_0, f_1 modulo NS_{ω_1} in terms of properties of f and vice versa. For example, (f_0, f_1) is a split witness of $\diamondsuit^+(\mathbb{B}_0, \mathbb{B}_1)$ iff f witnesses $\diamondsuit^+(\mathbb{B})$. We can also translate all the notions related to usual witnesses into the split witness context.

Definition 5.20. Suppose (f_0, f_1) is a split witness of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$. A forcing $\mathbb{P}$ is (f_0, f_1)-complete/proper/semiproper iff $\mathbb{P}$ is f_i-complete/proper/semiproper for both $i = 0, 1$.

Thus the iteration theorems related to a split witness naturally follow from the iteration theorems related to usual witnesses of $\diamondsuit(\mathbb{B})$. The only results we will make use of for split witnesses is the existence of witnesses under $\diamondsuit$ and the translation of (a weak version of) Theorem 3.60.

Proposition 5.21. *Suppose $\diamondsuit$ holds. For any forcings $\mathbb{B}_0, \mathbb{B}_1 \subseteq \omega_1$ there is a split witness (f_0, f_1) of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$.*

Theorem 5.22. *Suppose there is a Woodin cardinal and (f_0, f_1) is a split witness of $\diamondsuit(\mathbb{B}_0, \mathbb{B}_1)$. Then there is a δ-c.c. (f_0, f_1)-semiproper forcing $\mathbb{P}$ so that NS_{ω_1} is saturated in $V^{\mathbb{P}}$.*

We note that we could have defined split witnesses of any length $\leqslant \omega_1$, but will not make use of this.

5.4 $\mathbb{B}$ is Cohen forcing and weakly Lusin sequences

The case $\mathbb{B}$ is Cohen forcing is related to the following concept which was introduced by Shelah-Zapletal [SZ99].

Definition 5.23. A sequence $\langle x_\alpha \mid \alpha < \omega_1\rangle$ of reals is *weakly Lusin* if

$$\{\alpha < \omega_1 \mid x_\alpha \in Y\} \in \mathrm{NS}_{\omega_1}$$

for all meager sets $Y \subseteq \mathbb{R}$.

Question 5.24 (Shelah-Zapletal, [SZ99])**.** Does the saturation of NS_{ω_1} plus the existence of a nonmeager set of reals of size $\aleph_1$ imply the existence of a weakly Lusin sequence?

This has been answered in the negative by Paul Larson by forcing with a $\mathbb{P}_{\max}$-style partial order over $L(\mathbb{R})$ under determinacy.

Fact 5.25 (Larson, [Lar05])**.** *Suppose $V = L(\mathbb{R}) \models \mathrm{AD}$. Then there is a forcing extension $V[G]$ which is a model of* ZFC *and in $V[G]$*

- NS_{ω_1} *is saturated,*
- *there is a nonmeager set of reals of size $\aleph_1$ and*
- *there is no weakly Lusin sequence.*

We improve this result by reducing the assumption $\mathrm{AD}^{L(\mathbb{R})}$ to the optimal one in terms of consistency strength. Let C denote Cohen forcing. We will do so by forcing "NS_{ω_1} is saturated" while simultaneously separating

$\diamondsuit(\mathsf{C})$ from $\diamondsuit^+(\mathsf{C})$. In order for these principles to make sense, we consider C as being recursively coded as subset of ω. We will now explain the connection between these two principles and the question of Shelah-Zapletal. The following standard fact is crucial.

Fact 5.26. *Suppose θ is regular uncountable and $X \prec H_\theta$ is countable. Let c be a real. The following are equivalent:*

(i) c is a Cohen real over M_X.

(ii) $c \notin Y$ whenever Y is a meager Borel set of reals definable over X.

See e.g. [BJ95].

Proposition 5.27. *If $\diamondsuit(\mathsf{C})$ holds then there is a nonmeager set of reals of size $\aleph_1$.*

This is essentially (1) of Theorem 6.49 in [Woo10].

Proof. Suppose f witnesses $\diamondsuit(\mathsf{C})$ and let $Z = \mathbb{R} \cap L(f)$. We have $L(f) = L[A]$ for some set $A \subseteq \omega_1$ and hence Z is of size $\aleph_1$. Let Y be any Borel meager set of reals and find some f-slim $X \prec H_{\omega_2}$ with Y definable over X. As $f(\delta^X)$ is C-generic over M_X, we have $\bigcup f(\delta^X) \in Z - Y$. Thus Z is nonmeager. $\square$

The existence of a weakly Lusin sequence is a familiar face in disguise.

Lemma 5.28. *The following are equivalent:*

(i) There is a weakly Lusin sequence.

(ii) $\diamondsuit^+(\mathsf{C})$.

Proof. $(i) \Rightarrow (ii)$: Suppose $\vec{x} = \langle x_\alpha \mid \alpha < \omega_1 \rangle$ is a weakly Lusin sequence. We identify $\mathbb{R}$ with ${}^\omega\omega$ and consider $\vec{x}$ to be a sequence in ${}^\omega\omega$. Define

$$f\colon \omega_1 \to \mathcal{P}(\mathsf{C})$$

via $f(\alpha)$ is the C-filter of all initial segments of x_α for $\omega \leqslant \alpha < \omega_1$ and $f(n) = \varnothing$ for $n < \omega$.

Claim 5.29. *Any countable $X \prec H_{\omega_2}$ with $f \in X$ is f-slim.*

Proof. $f(\delta^X)$ is C-generic over M_X iff $x_{\delta^X} \notin Y$ for any meager Borel $Y \subseteq \mathbb{R}$ definable over X by Fact 5.26. The latter holds as $\vec{x}$ is weakly Lusin and $\vec{x} \in X$. $\square$

We do not necessarily have S^f_p stationary for all $p \in \mathsf{C}$, we deal with this issue now. Let $\mathsf{C}_+ = \{p \in \mathsf{C} \mid S^f_p \in \mathrm{NS}^+_{\omega_1}\}$.

Claim 5.30. C_+ *is a nonatomic suborder of* C.

Proof. Suppose $p \in \mathsf{C}_+$. Find some increasing continuous chain

$$\langle X_\alpha \mid \alpha < \omega_1 \rangle$$

of countable elementary substructures of H_{ω_2} with $f \in X_0$. Note that

$$S := \{\alpha < \omega_1 \mid \delta^{X_\alpha} \in S_p^f\}$$

is stationary as $p \in \mathsf{C}_+$. For $n < \omega$ define

$$r_n \colon S \to \mathsf{C}$$

by $r_n(\alpha) = \bigcup f(\delta^{X_\alpha}) \upharpoonright n = x_{\delta^{X_\alpha}} \upharpoonright n$. By iterated applications of Fodor's lemma, we can find a real $x \in {}^\omega\omega$ so that

$$\{\alpha \in S \mid r_n(\alpha) = x \upharpoonright n\}$$

is stationary for all $n < \omega$. It follows that $x \upharpoonright n \in \mathsf{C}_+$ for all $n < \omega$. If there were no incompatible conditions in C_+ below p, then $x = \bigcup f(\delta^{X_\alpha})$ for almost all $\alpha \in S$ and hence x is definable from f and a countable ordinal but on the other hand, x is generic over M_{X_α} for almost all $\alpha \in S$, contradiction. □

It follows that there is a dense embedding $\sigma \colon \mathsf{C} \to \mathsf{C}_+$. For $\alpha < \omega_1$, let

$$f'(\alpha) = \sigma^{-1}[f(\alpha)].$$

It is easy to see that f' witnesses $\diamondsuit^+(\mathsf{C})$.
To see $(ii) \Rightarrow (i)$, suppose f witnesses $\diamondsuit^+(\mathsf{C})$. It follows that $f(\alpha)$ is a maximal C-filter on a club C. Let $\langle \xi_\alpha \mid \alpha < \omega_1 \rangle$ be the monotone enumeration of C. It easily follows from Fact 5.26 that

$$\left\langle \bigcup f(\xi_\alpha) \mid \alpha < \omega_1 \right\rangle$$

is a weakly Lusin sequence. □

Remark 5.31. If NS_{ω_1} is saturated then the existence of a weakly Lusin sequence is additionally equivalent to "$(\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$ adds a Cohen real", however this can fail in general.

We now state the theorem that reduces the assumption of Larson from $\mathrm{AD}^{L(\mathbb{R})}$ to just one Woodin cardinal.

Theorem 5.32. *The theories*

(*i*) ZFC + *"There is a Woodin cardinal" and* (*ii*) ZFC + "NS_{ω_1} *is saturated*" + $\diamondsuit(\mathsf{C}) + \neg\diamondsuit^+(\mathsf{C})$

are equiconsistent.

Note that in any model of the theory in (*ii*), there is a nonmeager set of size $\aleph_1$ by Proposition 5.27, but no weakly Lusin sequence by Lemma 5.28. An important ingredient of the argument will be *Miller forcing.*

Definition 5.33. Suppose $T \subseteq \omega^{<\omega}$ is a tree.

(*i*) For $t \in T$ the set of *immediate successors of* $t \in T$ is

$$\mathrm{succ}_T(t) = \{n < \omega \mid t^\frown n \in T\}.$$

(*ii*) A node $t \in T$ *splits in* T if $\mathrm{succ}_T(t)$ is infinite.

(*iii*) For $n < \omega$ and $t \in T$, t is a *splitting node of order* n, $t \in \mathrm{split}_n(T)$, if t splits in T and there are exactly n proper initial segments of t that split in T.

(*iv*) T is *superperfect* if $T \neq \varnothing$ and for any $t \in T$ there is $t \leqslant_T s$ so that s splits in T.

Definition 5.34. Conditions in Miller forcing $\mathbb{M}$ are superperfect trees $T \subseteq \omega^{<\omega}$. $\mathbb{M}$ is ordered by inclusion.

Lemma 5.35. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Miller forcing is f-proper.*

We adapt the usual argument which shows that $\mathbb{M}$ is proper.

Proof. Let θ be sufficiently large regular and $X \prec H_\theta$ f-slim. Suppose $p \in \mathbb{M} \cap X$, it is our task to find some $(X, \mathbb{M}, f)$-generic condition below p. For $q, r \in \mathbb{M}$, let $q \leqslant_n r$ iff $q \leqslant r$ and $\mathrm{split}_n(q) = \mathrm{split}_n(r)$. Let $\langle D_n \mid n < \omega\rangle$ be an enumeration of all dense open $D \subseteq \mathbb{M}^{M_X}$, $D \in M_X[f(\delta^X)]$. We will construct a descending sequence

$$\vec{p} = \langle p_n \mid n < \omega\rangle$$

satisfying

($\vec{p}$.*i*) $p_0 = p$,

($\vec{p}$.*ii*) if $t \in \mathrm{split}_n(p_n)$ then $p_n \restriction t \in X$,

($\vec{p}$.*iii*) $p_{n+1} \leqslant_n p_n$ and

($\vec{p}$.*iv*) $p_{n+1} \restriction t \in \pi_X[D_n]$ for every $t \in \mathrm{split}_{n+1}(p_n)$

for all $n < \omega$. Define p_n by induction on $n < \omega$. Set $p_0 = p$ and now assume that p_n is defined. For $t \in \mathrm{split}_{n+1}(p_n)$, find $p_n^t \leqslant p_n \upharpoonright t$ so that $p_n^t \in \pi_X[D_n]$. This is possible as $p_n \upharpoonright t \in X$. Define p_{n+1} by gluing all p_n^t for $t \in \mathrm{split}_{n+1}(p_n)$ together, that is

$$p_{n+1} = \bigcup \{p_n^t \mid t \in \mathrm{split}_{n+1}(p_n)\}.$$

It follows that $p_{n+1} \in \mathbb{M}$ and that $(\vec{p}.iii)$, $(\vec{p}.iv)$ hold as well as that $p_{n+1} \upharpoonright t \in X$ for all $t \in \mathrm{split}_{n+1}(p_{n+1})$.
We define $q := \bigcap_{n<\omega} p_n$, the fusion of the sequence $\vec{p}$. We have $q \in \mathbb{M}$ by $(\vec{p}.iii)$, $q \leqslant p$ by $(\vec{p}.i)$. Condition $(\vec{p}.iv)$ implies that q is $(X, \mathbb{M}, f)$-generic, see Proposition 3.15. □

The following was shown in [SZ99] and attributed to Baroszyński.

Fact 5.36. *No sequence $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ of reals in V is weakly Lusin in $V^{\mathbb{M}}$.*

Remark 5.37. In fact, their proof shows that for any stationary set $S \in V$ and any sequence $\langle x_\alpha \mid \alpha < \omega_1 \rangle \in V$ of reals, in $V^{\mathbb{M}}$ there is a meager set $Y \subseteq \mathbb{R}$ so that $\{\alpha \in S \mid x_\alpha \in Y\} \in \mathrm{NS}_{\omega_1}^+$.

Proof of Theorem 5.32. Suppose δ is a Woodin cardinal. By Proposition 5.21, we may assume that there is a split witness (f_0, f_1) of $\diamondsuit(\mathsf{C}, \{0\})$. Further, we may assume that δ is Woodin with $\diamondsuit$. By Theorem 5.22, there is a δ-c.c. (f_0, f_1)-semiproper forcing $\mathbb{P}$ so that NS_{ω_1} is saturated in $V^{\mathbb{P}}$. $\mathbb{P}$ is a nice iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \delta, \beta < \delta \rangle$$

of (f_0, f_1)-semiproper forcings. We will assume that for unboundedly many $\alpha < \delta$, we have $V^{\mathbb{P}_\alpha} \models \dot{\mathbb{Q}}_\alpha = \mathbb{M}$. The proof of Theorem 3.60 clearly allows for this by Lemma 5.35. Let G be $\mathbb{P}$-generic over V. As $\mathbb{P}$ is f_0-semiproper, f_0 witnesses $\diamondsuit(\mathsf{C})$ in $V[G]$.

Claim 5.38. *$\diamondsuit^+(\mathsf{C})$ fails in $V[G]$.*

Proof. Suppose $f \in V[G]$ guesses C-filters. As $\mathbb{P}$ is δ-c.c. in V, there is some $\alpha < \delta$ with $f \in V[G_\alpha]$ so that

$$V[G_\alpha] \models \dot{\mathbb{Q}}_\alpha^{G_\alpha} = \mathbb{M}.$$

By Fact 5.36 and the subsequent remark as well as (the proof of) Lemma 5.28, f does not witness $\diamondsuit^+(\mathsf{C})$ in $V[G_{\alpha+1}]$. In fact, in $V[G_{\alpha+1}]$ there is a dense $D \subseteq \mathsf{C}$ such that

$$T := \{\alpha \in \mathrm{supp}(f_1) \mid f(\alpha) \cap D = \varnothing\}$$

is stationary. Note that $\mathrm{supp}(f_1)$ is stationary in $V[G_\alpha]$ as $\mathbb{P}$ is f_1-semiproper. By Corollary 3.53, the extension $V[G_{\alpha+1}] \subseteq V[G]$ is f_1-stationary set preserving (i.e. preserves stationary subsets of $\mathrm{supp}(f_1)$) hence $T \notin \mathrm{NS}_{\omega_1}^{V[G]}$. □

The other direction holds as Jensen-Steel [JS13] have shown that the consistency of

$$\mathrm{ZFC} + \text{“There is a Woodin cardinal”}$$

follows from the consistency of ZFC + “NS_{ω_1} is saturated”. □

We separate $\diamondsuit(\mathsf{C})$ from $\diamondsuit(\omega_1^{<\omega})$ now.

Proposition 5.39. *Suppose f witnesses $\diamondsuit(\mathsf{C})$. If T is a Suslin tree then the forcing $(T, \geqslant_T)$ is f-proper.*

Proof. The reason for this is that Cohen forcing preserves all Suslin trees. If θ is sufficiently large, regular and $X \prec H_\theta$ is f-slim with $T \in X$ then in fact any $q \in T$ is $(X, \mathbb{P}, f)$-generic: If

$$M_X[f(\delta^X)] \models \text{“}A \text{ is a maximal antichain in } \pi_X^{-1}(T)\text{”}$$

then $M_X[f(\delta^X)] \models$ “A is countable” and thus $A \subseteq T_\xi$ for some $\xi < \delta^X$. It follows that A really is a maximal antichain in T and thus

$$q \Vdash \dot{G} \cap \check{A} \cap \check{X} \neq \varnothing.$$

□

It follows that $\diamondsuit^+(\mathsf{C})$ together with “there are no Suslin trees” is consistent and implies $\neg\diamondsuit(\omega_1^{<\omega})$ by Fact 5.6. We also note that $\diamondsuit^+(\mathsf{C})$ follows from CH by Lemma 5.28 as it is straightforward to construct a weakly Lusin sequence from CH. Jensen (cf. [DJ74]) has shown that CH is consistent with “there are no Suslin trees”, so CH does not even imply $\diamondsuit(\omega_1^{<\omega})$.

For several Σ_2-sentences ϕ, Shelah-Zapletal have investigated numerous $\mathbb{P}_{\max}$-style forcings that maximize the Π_2-theory of

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1})$$

conditioned to the existence of a witness for ϕ. Shelah-Zapletal make this precise and investigate what they call Π_2-compact sentences: Consider the following scheme depending on ϕ.

Theorem Scheme. *If $\mathrm{AD}^{L(\mathbb{R})}$ holds, then there is a σ-closed forcing P_ϕ definable in $L(\mathbb{R})$ such that in $L(\mathbb{R})^{P_\phi}$*

(P_ϕ.*i*) ZFC *holds,* $2^\omega = \omega_2 = \delta^1_2$, NS_{ω_1} *is saturated and*

(P_ϕ.*ii*) *ϕ holds in $L(\mathbb{R})^{P_\phi}$.*

Moreover, if ψ is any Π_2-sentence over $(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1})$ and

(*i*) *there is a Woodin with a measurable above it,* *(ii)* ϕ *holds and*

(iii) $(H_{\omega_2};\in,\mathrm{NS}_{\omega_1})\models\psi$

then $(H_{\omega_2};\in,\mathrm{NS}_{\omega_1})^{L(\mathbb{R})^{P_\phi}}\models\psi$.

Definition 5.40. A Σ_2-sentence ϕ is Π_2-*compact* if the instance of the scheme above with ϕ can be proven in ZFC.

For example, it follows from Woodin's Theorem 4.64 in [Woo10] that $\phi=$ "$\forall x\ x=x$" is Π_2-compact via the $\mathbb{P}_{\max}$-method and that "NS_{ω_1} is ω_1-dense" is Π_2-compact using $\mathbb{Q}_{\max}$, see [Woo10, Theorem 6.33].
The instances of ϕ that Shelah-Zapletal were interested in were mostly of the form $\mathfrak{r}=\aleph_1$ for a cardinal characteristic $\mathfrak{r}$ of the continuum, but also of the form "There exists a Suslin tree with θ" for some Σ_1-properties θ. We will take a look at sentences ϕ of the former form in Section 9 and 10, of the latter form in Subsection 5.6. Right now, we consider the sentence

$$\phi^{\mathrm{wL}}=\text{“There is a weakly Lusin sequence”}$$

which Shelah-Zapletal have proven Π_2-compact. The forcing $P_{\phi^{\mathrm{wL}}}$ they defined to accomplish this is not a $\mathbb{P}_{\max}$-variation according to our definition, however their methods together with results of Woodin in [Woo10, Section 5.4] show that, for all our intents and purposes, $P_{\phi^{\mathrm{wL}}}$ can be replaced by the $\mathbb{P}_{\max}$-variation $\mathbb{P}_{\max}^{\mathrm{wL}}$ we are about to define.

Definition 5.41. Suppose I is a normal uniform ideal on ω_1. A sequence $\langle x_\alpha\mid\alpha<\omega_1\rangle$ of reals is I-*weakly Lusin* if

$$\{\alpha<\omega_1\mid x_\alpha\in Y\}\in I$$

whenever $Y\subseteq\mathbb{R}$ is meager.

Definition 5.42. $\mathbb{P}_{\max}^{\mathrm{wL}}$ conditions are generically iterable structures $p=(M,I,\vec{x},a)$ with

($\mathbb{P}_{\max}^{\mathrm{wL}}$.*i*) $(M;\in,I)\models\psi_{\mathrm{AC}}(I)$,

($\mathbb{P}_{\max}^{\mathrm{wL}}$.*ii*) $(M;\in,I)\models$ "$\vec{x}$ is an I-weakly Lusin sequence" and

($\mathbb{P}_{\max}^{\mathrm{wL}}$.*iii*) $M\models a\subseteq\omega_1\wedge\omega_1^{L[a]}=\omega_1$.

The order on $\mathbb{P}_{\max}^{\mathrm{wL}}$ is given by

$$q=(N,J,\vec{y},b)<_{\mathbb{P}_{\max}^{\mathrm{wL}}}p$$

if there is a generic iteration

$$\mu\colon p\to p^*=(M^*,I^*,\vec{x}^*,a^*)$$

in N of length ω_1^N+1 so that

($<_{\mathbb{P}^{\mathrm{wL}}_{\max}}.i$) $I^* = J \cap M^*$,

($<_{\mathbb{P}^{\mathrm{wL}}_{\max}}.ii$) $\vec{x}^* = \vec{y}$ and

($<_{\mathbb{P}^{\mathrm{wL}}_{\max}}.iii$) $a^* = b$.

Let Ψ^{wL} consist of

- $\psi_0^{\mathrm{wL}}(x) =$ "$x = \dot{\vec{x}}$",
- $\psi_1^{\mathrm{wL}}(x) =$ "$x = \dot{a}$" and
- $\psi_2^{\mathrm{wL}}(x) =$ "$x \in \dot{I}_{f_{\vec{x}}}$"

where $f_{\vec{x}}$ denotes the C-filter guessing function f constructed from a sequence of reals $\vec{x}$ in the proof of Lemma 5.28. As in the case of $\mathbb{C}_{\max}$, Ψ^{wL} "witnesses typicality for all intents and purposes". $\mathbb{P}^{\mathrm{wL}}_{\max}$ accepts $\diamondsuit$-iterations by Lemma 4.19. $\mathbb{P}^{\mathrm{wL}}_{\max}$ has unique iterations by Fact 5.11.

Theorem 5.43. *Suppose f witnesses $\diamondsuit(\mathsf{C})$. Then $\mathrm{MM}^{++}(f)$ implies $\mathbb{P}^{\mathrm{wL}}_{\max}$-$(*)$.*

Proof. Suppose $\mathrm{MM}^{++}(f)$ holds. It follows that f witnesses $\diamondsuit^+(\mathsf{C})$. Let $\vec{x}$ be the weakly Lusin sequence constructed from f as in Lemma 5.28. Note that any f-preserving forcing preserves $\vec{x}$ as a weakly Lusin sequence. We have that SRP holds by Corollary 3.76, so that NS_{ω_1} is saturated. Hence for any $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$, we have that

$$(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \vec{x}, A)$$

is almost a $\mathbb{P}^{\mathrm{wL}}_{\max}$-condition. $\mathbb{P}^{\mathrm{wL}}_{\max}$ accepts $\diamondsuit$-iterations by Lemma 4.19 and (the proof of) Lemma 5.28. This implies $\mathbb{P}^{\mathrm{wL}}_{\max}$-$(*)$ by the First Blueprint Theorem 4.44. $\square$

Theorem 5.44. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(i) $\mathbb{P}^{\mathrm{wL}}_{\max}$-$(*)$.

(ii) *There is a witness f of $\diamondsuit(\mathsf{C})$ so that $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(f)$ holds.*

Proof. It follows from results in [SZ99] (also from results in Section 8) that $\mathbb{P}^{\mathrm{wL}}_{\max}$ is self-assembling. Suppose that f witnesses $\diamondsuit^+(\mathsf{C})$ and $\vec{x}$ is the associated weakly Lusin sequence. Let $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$. We have that $\Gamma^{\Psi^{\mathrm{wL}}}$ is exactly the class of f-stationary set preserving forcings. It is a consequence of the existence of a proper class of Woodin cardinals and of Theorem 3.60 that

$$\Gamma^{\Psi^{\mathrm{wL}}}_{(\vec{x},A)} = \Gamma^{\mathbb{P}^{\mathrm{wL}}_{\max}}_{(\vec{x},A)}(\Psi^{\mathrm{wL}}).$$

The desired equivalence now follows from the Second Blueprint Theorem 4.58. $\square$

We finally fulfill our earlier promise and explain the tension between Coding and $\diamondsuit(\omega_1^{<\omega})$ and even $\diamondsuit(\mathsf{C})$ which forced our hands to use $\psi_{\mathrm{AC}}(I)$ to ensure that $\mathbb{C}_{\max}$ and $\mathbb{P}_{\max}^{\mathrm{wL}}$ have unique iterations.

Lemma 5.45. *If* $\diamondsuit(\mathsf{C})$ *holds then* Coding *fails.*

Proof. Suppose f witnesses $\diamondsuit(\mathsf{C})$. Let

$$C = \{\alpha < \omega_1 \mid \exists X \prec H_{\omega_2}\ X \text{ is } f\text{-slim} \wedge f \in X \wedge \delta^X = \alpha\}$$

and observe that $\omega_1 - C \in \mathrm{NS}_f$. Let $\langle \xi_\beta \mid \beta < \omega_1\rangle$ be the monotone enumeration of C and define

$$a_\beta := h\left[\left\{\bigcup f(\xi_\beta) \upharpoonright n \mid n < \omega\right\}\right]$$

where $h\colon \omega^{<\omega} \to \omega$ is some fixed recursive bijection. It is straightforward to see that $\vec{a} := \langle a_\beta \mid \beta < \omega_1\rangle$ consists of pairwise almost disjoint sets in $[\omega]^\omega$. Suppose for a contradiction that Coding holds for $\vec{a}$. Consider the embedding

$$\eta_f\colon \mathsf{C} \to (\mathcal{P}(\omega_1)/\mathrm{NS}_f)^+$$

and note that η_f is a complete embedding with this codomain.

Claim 5.46. η_f *is a dense embedding.*

Proof. Let $S \in \mathrm{NS}_f^+$ and let $G \subseteq (\mathcal{P}(\omega_1)/\mathrm{NS}_f)^+$ be a generic filter with $[S]_{\mathrm{NS}_f} \in G$. Let U_G be the corresponding V-ultrafilter and

$$j\colon V \to \mathrm{Ult}(V, U_G) =: M$$

the induced ultrapower. We assume that the wellfounded part of M is transitive. Find $x \in [\omega]^\omega$ which codes S via $\vec{a}$. NS_f is a uniform normal ideal by Lemma 2.20 so $\omega_1 \in j(S)$ and we have

$$|x \cap a_{\omega_1}| < \omega$$

by elementarity, where a_{ω_1} denotes the ω_1-st entry in $j(\vec{a})$. However, this only depends on $g := \eta_f^{-1}[G]$: By construction we have

$$a_{\omega_1} = h\left[\left\{\bigcup j(f)(\omega_1) \upharpoonright n \mid n < \omega\right\}\right] = h\left[\left\{\bigcup g \upharpoonright n \mid n < \omega\right\}\right].$$

It follows that there is some $p \in \mathsf{C}$ so that

$$\eta_f(p) \Vdash |\dot{a}_{\omega_1} \cap \check{x}| < \omega$$

where $\dot{a}_{\omega_1}$ is a name for a_{ω_1}. This is only possible if $\eta_f(p) \leqslant [S]_{\mathrm{NS}_f}$. □

This is clearly absurd, C is c.c.c. while $(\mathcal{P}(\omega_1)/\mathrm{NS}_f)^+$ collapses ω_1. □

5.5 Uniform sequences of witnesses

In some way, shape or form, everything from Section 2 through Section 3.3 generalizes to *uniform sequences of witnesses* for $\diamondsuit(\mathbb{B})$, resp. $\diamondsuit^+(\mathbb{B})$. We immediately introduce them relative to arbitrary ideals on ω_1.

Definition 5.47. Suppose I is a normal uniform ideal on ω_1 and $\mathbb{B} \subseteq \omega_1$ a forcing. A sequence $\mathbf{f} = (f_n)_{n<\omega}$ *uniformly witnesses* $\diamondsuit_I(\mathbb{B})$ if

(*i*) f_n guesses $\mathbb{B}$-filters for all $n < \omega$ and

(*ii*) if $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{B}$, $n < \omega$ and $\langle b_i \mid i < n \rangle$ is a finite sequence of conditions in BP then

$$\{\alpha < \omega_1 \mid b \in f_n(\alpha) \wedge \forall m < \omega\ \forall \beta < \alpha\ f_m(\alpha) \cap D_\beta \neq \emptyset\} \in I^+.$$

$\mathbf{f}$ *uniformly witnesses* $\diamondsuit_I^+(\mathbb{B})$ if (*i*) above holds and (*ii*) is replaced by:

(*ii*)′ For any dense subset $D \subseteq \mathbb{B}$,

$$\{\alpha < \omega_1 \mid \exists m < \omega\ f_m(\alpha) \cap D = \emptyset\} \in I$$

and moreover, for for all $n < \omega$ and $b \in \mathbb{B}$

$$S^{\mathbf{f}}_{n,b} := \{\alpha < \omega_1 \mid b \in f_n(\alpha)\} \in I^+.$$

Note that $\mathbf{f}$ uniformly witnesses $\diamondsuit_I^+(\mathbb{B})$ iff all f_n, $n < \omega$ witness $\diamondsuit^+(\mathbb{B})$ and thus we may and will drop the adverb "uniformly" in this case and only say $\mathbf{f}$ witnesses $\diamondsuit_I^+(\mathbb{B})$. The same is however not true in general for $\diamondsuit_I(\mathbb{B})$ instead of $\diamondsuit_I^+(\mathbb{B})$.

Definition 5.48. If $\mathbf{f}$ is a sequence of functions guessing $\mathbb{B}$-filters and θ is an uncountable cardinal, we say that $X \prec H_\theta$ is $\mathbf{f}$-*slim* iff X is countable, $\mathbf{f} \in X$ and X is f_n-slim for all $n < \omega$.

This leads also leads to $\mathbf{f}$-stationary sets, simply replace f-slim in the definition of f-stationary by $\mathbf{f}$-slim. We leave it to the reader to generalize more concepts, e.g. f-proper and f-semiproper forcings to $\mathbf{f}$-proper and $\mathbf{f}$-semiproper forcings.

Convention 5.49. Assume and $\mathcal{G}$ is a countable set of filters. Then $\mathcal{G}$ is generic over M if all $g \in \mathcal{G}$ are generic over M. In this case we set

$$M[\mathcal{G}] = \bigcup_{g \in \mathcal{G}} M[g].$$

If $\mathbf{f}$ uniformly witnesses $\diamondsuit_I(\mathbb{B})$, we let

$$\mathbf{f}(\alpha) = \{f_n(\alpha) \mid n < \omega\}.$$

With this convention, many results hold and plenty of arguments work verbatim when f is replaced by $\mathbf{f}$ and "witnesses $\diamondsuit(\mathbb{B})$" is replaced by "uniformly witnesses $\diamondsuit(\mathbb{B})$" (and similar for $\diamondsuit^+(\mathbb{B})$ and other variants). The "meta-reason" for this is that usually we care about dense subsets in $M_X[f(\delta^X)]$ of some forcing in M_X and since there are only countable such sets, we can diagonalize against them in some way. But in $M_X[\mathbf{f}(\delta^X)]$ there are also only countably many dense sets we care about and the same diagonalizing works. In some arguments we had f-slim $X, Y \prec H_\theta$ with $X \sqsubseteq Y$ and needed to lift the natural elementary embedding

$$\mu\colon M_X \to M_Y$$

to

$$\mu^+ : M_X[f(\delta)] \to M_Y[f(\delta)]$$

where $\delta = \delta^X = \delta^Y$. See for example the proofs of Theorem 3.48 and Theorem 11.41. In these cases one would instead do countably many liftings of the from

$$\mu_n^+ : M_X[f_n(\delta)] \to M_Y[f_n(\delta)]$$

for $n < \omega$ if X, Y are $\mathbf{f}$-slim instead. Some additional, but boring, bookkeeping may then be required but essentially the same arguments work nonetheless. Since doing this only makes the proof more notationally exhausting but not mathematically inspiring, we opted to give the arguments in the less general case. We leave the details to the reader. In the next section, it will for the first time be interesting and necessary to work with uniform witnesses.

5.6 $\mathbb{B}$ is a Suslin tree and $\mathbb{S}^T_{\max}$-(*)

In this section, we will assume that $\mathbb{B} = T$ is a Suslin tree. We may indeed suppose $T \subseteq \omega_1$, but then on a club of $\alpha < \omega_1$, we will have

$$T_{<\alpha} = T \cap \alpha.$$

It is thus more natural for witnesses f of $\diamondsuit(T)$ to require that any $f(\alpha)$ is a filter in $T_{<\alpha}$. In fact, in this section we will mostly only care about uniform sequences of witnesses of $\diamondsuit(T)$.

The following observation is key. It is probably either folklore or due to Jensen.

Proposition 5.50. *Suppose T is a ω_1-tree and let θ be a sufficiently large regular cardinal. The following are equivalent:*

(i) T is Suslin.

(*ii*) *For all countable $X \prec H_\theta$ with $T \in X$ and all $s \in T_{\delta^X}$,*

$$\pi_X^{-1}[s] \text{ is generic for } \pi_X^{-1}(T) \text{ over } M_X.$$

(*iii*) *There is a countable $X \prec H_\theta$ with $T \in X$ so that for all $s \in T_{\delta^X}$,*

$$\pi_X^{-1}[s] \text{ is generic for } \pi_X^{-1}(T) \text{ over } M_X.$$

Proof. We assume $T \subseteq \omega_1$.
(*i*)⇒(*ii*): If $\mathcal{A} \in X$ is an antichain of T, then it is countable and thus there is some $\alpha < \delta^X$ with $\mathcal{A} \subseteq T_\alpha$. Hence any $s \in T_\delta$ is above some node in $\mathcal{A}$.
(*ii*)⇒(*iii*) is trivial, so let us prove (*iii*)⇒(*i*). Let $X \prec H_\theta$ witness (*iii*). Suppose for a contradiction that there is a maximal antichain of T of size ω_1. By elementarity, there is some such antichain $\mathcal{A} \in X$. Let $\bar{\mathcal{A}} = \pi_X^{-1}(\mathcal{A}) = \mathcal{A} \cap T_{<\delta^X}$ and $\bar{T} = \pi_X^{-1}(T) = T \cap \delta^X$. If $s \in T_{\delta^X}$, then s is generic over M_X and hence above some condition in $\bar{\mathcal{A}}$. But then $\bar{\mathcal{A}}$ is already a maximal antichain in T, so that $\mathcal{A} = \bar{\mathcal{A}}$ is countable, contradiction. □

Definition 5.51. An ω_1-tree[37] T is *strongly homogeneous* if there is a collection

$$\{\mu_{s,t} \mid s,t \in T \text{ are on the same level of } T\}$$

with the following properties:

(μ.*i*) For $s,t \in T$ on the same level, $\mu_{s,t}\colon T \upharpoonright s \to T \upharpoonright t$ is a level preserving isomorphism and if $s = t$ then $\mu_{s,t}$ is the identity.

(μ.*ii*) If s,t,u are all on the same level of T then $\mu_{t,u} \circ \mu_{s,t} = \mu_{s,u}$.

(μ.*iii*) If s,t are on the same level of T and u,v are on the same level of T so that $\mu_{s,t}(u) = v$ then $\mu_{u,v} = \mu_{s,t} \upharpoonright (T \upharpoonright u)$.

(μ.*iv*) If u,v are on the same limit level of T then there are s,t on the same level of T with $s <_T u$ and $\mu_{s,t}(u) = v$.

Definition 5.52. Suppose T is an ω_1-tree.

(*i*) A *canonical witness* of $\diamondsuit(T)$ is a witness f of $\diamondsuit(T)$ so that for all $\alpha < \omega_1$, $f(\alpha) = \mathrm{pred}_T(t)$ for some $t \in T$.

(*ii*) A sequence $\mathbf{f} = (f_n)_{n<\omega}$ is a *canonical witness sequence* (cws) for T if

- ($\mathbf{f}$.*i*) $\mathbf{f}$ uniformly witnesses $\diamondsuit(T)$,
- ($\mathbf{f}$.*ii*) f_n is a canonical witness of $\diamondsuit(T)$ for all $n < \omega$ and
- ($\mathbf{f}$.*iii*) for any limit $\alpha < \omega_1$ and any $t \in T_\alpha$ there is $n < \omega$ so that

$$f_n(\alpha) = \mathrm{pred}_T(t).$$

[37] An ω_1-tree is a tree of height ω_1 with all levels countable.

Lemma 5.53. *Suppose T is an ω_1-tree. The following are equivalent:*

(*i*) *T is Suslin.*

(*ii*) *There is a cws for T.*

Proof. $(i)\Rightarrow(ii)$: Let $\mathcal{S}$ be a partition of ω_1 into ω_1-many stationary sets and enumerate $\mathcal{S}$ as

$$\langle S^{\beta}_{n,i} \mid n, i < \omega,\ \beta < \omega_1\rangle.$$

Also enumerate T_β as $\langle t^{\beta}_i \mid i < \omega\rangle$ for any countable ordinal β. Now, for $\alpha < \omega_1$ and $n < \omega$ find $s^{\alpha}_n \in T_\alpha$ so that

(*s.i*) if $\alpha \in S^{\beta}_{n,i}$ and $\beta \leqslant \alpha$ then $t^{\beta}_i \leqslant_T s^{\alpha}_n$ and

(*s.ii*) $T_\alpha = \{s^{\alpha}_n \mid n < \omega\}$.

Define $f_n(\alpha)$ as $\mathrm{pred}_T(s^{\alpha}_n)$. It follows that

$$S^{\mathbf{f}}_{n,t^{\beta}_i} \supseteq S^{\beta}_{n,i} - (\beta + 1)$$

is stationary for all $\beta < \omega_1$ and $i, n < \omega$. By Proposition 5.50, whenever θ is sufficiently large regular and $X \prec H_\theta$ is countable with $T \in X$ then $f_n(\delta^X)$ is generic over X for all $n < \omega$. Thus $\mathbf{f}$ is a cws.
$(ii)\Rightarrow(i)$: Suppose that $\vec{f}$ is a cws for T. Let θ be regular with $T \in H_\theta$ and let $X \prec H_\theta$ be $\mathbf{f}$-slim with $T \in X$. As $\mathbf{f}$ is a cws, we have that $\pi_X^{-1}[t]$ is generic for $\pi_X^{-1}(T)$ over M_X for all $t \in T_{\delta^X}$. Thus T is Suslin by Proposition 5.50. □

Note that if $\mathbf{f}$ is a cws for T then in fact $\mathbf{f}$ witnesses $\diamondsuit^+(T)$.

Definition 5.54. Suppose T is a Suslin tree.

(*i*) $\mathrm{PFA}(T)$ is FA(proper and T-preserving[38]).

(*ii*) $\mathrm{MM}(T)$ is FA(stationary-set- and T-preserving).

(*iii*) $\mathrm{MM}^{++}(T)$ is FA^{++}(stationary-set- and T-preserving).

For a coherent Suslin tree T, $\mathrm{PFA}(T)$ and its implications about the forcing extension V^T have been investigated by Todorcevic e.g. in [Tod11]. $\mathrm{MM}(T)$ has been considered in, e.g. [Tal17] and [DT18].

Observation 5.55. Suppose $\mathbf{f}$ is a cws for a Suslin tree T. We have that

(*i*) $\mathbf{f}$-stationary sets are exactly the stationary sets,

(*ii*) $\mathbf{f}$-complete forcings are exactly the complete forcings,

[38] A forcing $\mathbb{P}$ is T-preserving if T is Suslin in $V^{\mathbb{P}}$.

(*iii*) **f**-proper forcings are exactly the proper forcings preserving T,

(*iv*) **f**-semiproper forcings are exactly the semiproper forcings preserving T

(*v*) PFA(T) is PFA(**f**) and

(*vi*) MM(T) is MM(**f**), $\mathrm{MM}^{++}(T)$ is $\mathrm{MM}^{++}(\mathbf{f})$.

Our iteration theorems in this context yield theorems due to Miyamoto.

Corollary 5.56 (Miyamoto, [Miy93])**.** *Suppose T is Suslin. If $\mathbb{P}$ is a countable support iteration of proper forcings which preserve T, then $\mathbb{P}$ is proper and preserves T.*

Corollary 5.57 (Miyamoto,[Miy02])**.** *Suppose T is Suslin. If $\mathbb{P}$ is nice iteration of semiproper forcings which preserve T, then $\mathbb{P}$ is semiproper and preserves T.*

The following is a consequence of Lemma 5.53 and generalizing Lemma 4.19 to uniform sequences of witnesses.

Corollary 5.58. *Suppose that*

(*i*) *$p = (M, I)$ is generically iterable,*

(*ii*) *$\mu\colon p \to p^* = (M^*, I^*)$ is a $\diamondsuit$-iteration and*

(*iii*) *$M^* \models$ "T is a Suslin tree".*

Then T is a Suslin tree.

Larson has investigated a $\mathbb{P}_{\max}$-variation $\mathbb{S}^T_{\max}$ which is conditioned to the existence of a distinguished strongly homogeneous Suslin tree. The existence of such trees follows, for example, from $\diamondsuit$. We refer the reader to [Lar99].

Definition 5.59. Conditions in $\mathbb{S}^T_{\max}$ are generically iterable structures $p = (M, I, s, a)$ so that

($\mathbb{S}^T_{\max}$.*i*) $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$,

($\mathbb{S}^T_{\max}$.*ii*) $M \models$ "s is a strongly homogeneous Suslin tree" and

($\mathbb{S}^T_{\max}$.*iii*) $M \models a \subseteq \omega_1 \wedge \omega_1^{L[a]} = \omega_1$.

The order on $\mathbb{S}^T_{\max}$ is defined by

$$q = (N, J, t, b) <_{\mathbb{S}^T_{\max}} p$$

iff there is a generic iteration

$$\mu\colon p \to p^* = (M^*, I^*, s^*, a^*)$$

in q of length $\omega_1^q + 1$ so that

($<_{\mathbb{S}^T_{\max}}.i$) $I^* = J \cap M^*$,

($<_{\mathbb{S}^T_{\max}}.ii$) $s^* = t$ and

($<_{\mathbb{S}^T_{\max}}.iii$) $a^* = b$.

Remark 5.60. This is not how $\mathbb{S}^T_{\max}$ is presented by Paul Larson in [Lar99], however it follows from the results of that paper that both presentations are equivalent assuming $\mathrm{AD}^{L(\mathbb{R})}$.

$\mathbb{S}^T_{\max}$ is clearly a typical $\mathbb{P}_{\max}$-variation and we will pick a specific witness for this. Let $\Psi^{\mathbb{S}^T_{\max}}$ consist of

- $\psi_0^{\mathbb{S}^T_{\max}}(x) =$ "$x \in \dot{I}$",
- $\psi_1^{\mathbb{S}^T_{\max}}(x) =$ "$x = \dot{s}$",
- $\psi_2^{\mathbb{S}^T_{\max}}(x) =$ "$x = \dot{a}$" and
- $\psi_3^{\mathbb{S}^T_{\max}}(x) =$ "$x = \dot{s} \wedge x$ is a Suslin tree".

Note that $\psi_3^{\mathbb{S}^T_{\max}}(x)$ can be expressed by a Π_1-formula.

Definition 5.61. Suppose that S is a Suslin tree.

(i) $\Gamma_S = \{\mathbb{P} \mid \mathbb{P}$ is S-preserving and preserves stationary sets$\}$.

(ii) Δ-$\mathrm{BMM}^{++}(S)$ is Δ-$\mathrm{BFA}^{++}(\Gamma_S)$ for $\Delta \subseteq \mathcal{P}(\mathbb{R})$.

Note that for a Suslin tree S, $\Gamma^{\Psi^{\mathbb{S}^T_{\max}}}_{(S,A)}$ is exactly Γ_S for any $A \subseteq \omega_1$.

Theorem 5.62. *Suppose that S is a strongly homogeneous Suslin tree. Then* $\mathrm{MM}^{++}(S) \Rightarrow \mathbb{S}^T_{\max}$-$(*)$.

Proof. Let $\mathbf{f}$ be a cws for S. Then $\mathrm{MM}^{++}(S)$ is equivalent to $\mathrm{MM}^{++}(\mathbf{f})$ and hence implies SRP by Lemma 3.69 so that NS_{ω_1} is saturated and it follows that $\mathcal{H}_{(S,A)}$ is almost a $\mathbb{S}^T_{\max}$-condition for any set $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$. $\mathbb{S}^T_{\max}$ accepts $\diamondsuit$-iterations by Corollary 5.58. Hence $\mathbb{S}^T_{\max}$-$(*)$ follows from $\mathrm{MM}^{++}(S)$ by the First Blueprint Theorem 4.44. $\square$

Theorem 5.63. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(i) $\mathbb{S}^T_{\max}$-$(*)$.

(ii) *There is a strongly homogeneous Suslin tree S so that* $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(S)$ *holds.*

Proof. The results in [Lar99] imply that $\mathbb{S}^T_{\max}$ is a self-assembling $\mathbb{P}_{\max}$-variation under $\mathrm{AD}^{L(\mathbb{R})}$ which holds as there is a proper class of Woodin cardinals. Let $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$. We have $\Gamma^{\mathbb{S}^T_{\max}}_{(S,A)}(\Psi^{\mathbb{S}^T_{\max}}) = \Gamma_S$ from Theorem 3.60 (generalized to uniform sequences of witnesses). The desired equivalence now follows from the Second Blueprint Theorem 4.58. □

We have some final remarks on canonical witnesses. The implementation of uniform sequences of witnesses is necessary to deduce the iteration theorems of (semi)proper forcings preserving a distinguished Suslin tree.

Definition 5.64. A Suslin tree T is 2-*free* if whenever $s, t \in T$ are two different nodes on the same level of T then $T \restriction s \times T \restriction t$ is a Suslin tree.

Lemma 5.65. *It is consistent that there is a Suslin tree T and a canonical witness f of $\diamondsuit^+(T)$ so that T considered as a forcing is f-proper.*

Proof. We may assume that $\diamondsuit$ holds. Jensen showed that this implies the existence of a 2-free (and even more) Suslin tree T, see [DJ74]. Lemma 5.53 tells us that there is a canonical witness f of $\diamondsuit(T)$ and it follows from Proposition 5.50 that f witnesses even $\diamondsuit^+(T)$. Now suppose θ is sufficiently large and $X \prec H_\theta$ is countable with $f, T \in X$. If $t \in X \cap T$ then there is some $s \in T$ above t so that $s \not\leq_T x$ where $x \in T_{\delta^X}$ is so that $f(\delta^X) = \mathrm{pred}_T(x)$. As T is 2-free, $\bar{T} \restriction \bar{s}$ is a Suslin tree in $M_X[f(\delta^X)]$ and hence if $\mathcal{A} \in M_X[f(\delta^X)]$ is a maximal antichain of conditions below $\bar{s}$ in $\bar{T}$ then $\mathcal{A}$ is countable in $M_X[f(\delta^X)]$, thus contained in $\bar{T}_{<\alpha}$ for some $\alpha < \delta^X$. It follows that $\pi_X[\mathcal{A}]$ is a maximal antichain in $T_{<\alpha}$ so that

$$s \Vdash \dot{G} \cap \pi_{\check{X}}[\check{\mathcal{A}}] \neq \emptyset.$$

This shows that s is (X, T, f)-generic. □

However a single canonical witness is enough for a strongly homogeneous Suslin tree.

Proposition 5.66. *If T is a strongly homogeneous tree and f is a canonical witness of $\diamondsuit(T)$ then the following is equivalent.*

(i) *T is a Suslin tree.*

(ii) *f witnesses $\diamondsuit(T)$.*

(iii) *f witnesses $\diamondsuit^+(T)$.*

Proof. Let θ be sufficiently large, regular and $X \prec H_\theta$ countable with $f, T \in X$. Condition $(\mu.iv)$ in the definition of strong homogeneity guarantees the following: For $t \in T_{\delta^X}$ let $G(t)$ hold iff $\pi_X^{-1}[\mathrm{pred}_T(t)]$ is generic over M_X. Then $\exists t \in T_{\delta^X} G(t)$ is equivalent to $\forall t \in T_{\delta^X} G(t)$. It follows immediately from Proposition 5.50 that (i)-(iii) are equivalent. □

Note that "T is a strongly homogeneous tree" is $\Sigma_1(T, \omega_1)$ and thus holds in all ω_1-preserving extensions.

6 Suslin's Minimum

We introduce Suslin's Minimum^{++}, a variant of Martin's Maximum^{++} which implies the existence of many Suslin trees and prove that it has $\mathbb{S}_{\max}$-$(*)$ as a consequence.

Definition 6.1. (*i*) We say that a Suslin tree is *on* ω_1 if its underlying set is ω_1. Note that every Suslin tree is isomorphic to one on ω_1.

(*ii*) If $\mathbb{P}$ is a forcing and $\dot{T}$ is a $\mathbb{P}$-name for a Suslin tree on ω_1, then for a filter $g \subseteq \mathbb{P}$, we set
$$\dot{T}^g := (\omega_1, \leqslant^g_{\dot{T}})$$
where
$$\leqslant^g_{\dot{T}} = \{(\alpha, \beta) \in \omega_1 \times \omega_1 \mid \exists p \in g \; p \Vdash \check{\alpha} \leqslant_{\dot{T}} \check{\beta}\}.$$

The idea behind Suslin's Minimum^{++} is that it is MM^{++} restricted to forcings additionally preserving all Suslin trees, but we can also evaluate ω_1-many names for Suslin trees to Suslin trees in V. We have to restrict to names for Suslin trees on ω_1 for this to be consistent.

Definition 6.2. *Suslin's Minimum*$^{++}$, denoted SM^{++}, is the following statement: Assume that

(*i*) $\mathbb{P}$ is a forcing preserving stationary sets and all Suslin trees,

(*ii*) $\mathcal{D}$ is a set of at most ω_1-many dense subsets of $\mathbb{P}$,

(*iii*) $\mathcal{S}$ is a set of at most ω_1-many $\mathbb{P}$-names for stationary subsets of ω_1 and

(*iv*) $\mathcal{T}$ is a set of at most ω_1-many $\mathbb{P}$-names for Suslin trees on ω_1.

Then there is a filter $g \subseteq \mathbb{P}$ with

(SM^{++}.*i*) $g \cap D \neq \varnothing$ for any $D \in \mathcal{D}$,

(SM^{++}.*ii*) $\dot{S}^g = \{\alpha < \omega_1 \mid \exists p \in g \; p \Vdash \check{\alpha} \in \dot{S}\}$ is stationary for any $\dot{S} \in \mathcal{S}$ and

(SM^{++}.*iii*) $\dot{T}^g = (\omega_1, \leqslant^g_{\dot{T}})$ is a Suslin tree for any $\dot{T} \in \mathcal{T}$.

Suslin's hypothesis, the nonexistence of a Suslin line, has been shown equivalent to the nonexistence of a Suslin tree. In some sense, SM^{++} postulates a maximal failure of Suslin's hypothesis which justifies the name Suslin's Minimum.

We note that Miyamoto has investigated the axiom MM(Suslin) which is the weakening of SM^{++} in which (SM^{++}.*ii*) and (SM^{++}.*iii*) are not required. In contrast to SM^{++}, MM(Suslin) does not imply the existence of a Suslin tree on its own.

Proposition 6.3. *If* SM^{++} *holds then there are (many) Suslin trees.*

Proof. There are a number of well known forcings that add a Suslin tree while preserving all Suslin trees. Among them Cohen forcing, due to Shelah [She84], or Jech's forcing to add a Suslin tree [Jec67]. We note as a further example that by choosing the forcing and suitable dense sets correctly, for any set $\mathcal{T}$ of at most ω_1-many Suslin trees, an application of SM^{++} yields a Suslin tree S so that $S \times T$ is Suslin for any $T \in \mathcal{T}$ (this is more smoothly achieved by applying BSM^{++}, see the next definition). It follows that there is a set of ω_2-many (which turns out to be 2^{ω_1}, so the maximal amount) trees which are pairwise mutually Suslin. This can easily be improved. $\square$

Nonetheless, Miyamoto showed MM(Suslin) to be consistent with the existence of a Suslin tree from a supercompact cardinal. We note that SM^{++} is a maximal forcing axiom in the sense that the class of forcings it applies to cannot be increased. To see this, we introduce the natural bounded version of SM^{++}, which will also be helpful later on.

Definition 6.4. For $X \subseteq \mathbb{R}$, BSM^{++} holds if

($\mathrm{BSM}^{++}.i$) X is ∞-universally Baire and

($\mathrm{BSM}^{++}.ii$) for any forcing $\mathbb{P}$ which preserves stationary sets as well as all Suslin trees we have

$$(H_{\omega_2}; \mathrm{NS}_{\omega_1}, \mathrm{ST})^V \prec_{\Sigma_1} (H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, \mathrm{ST})^{V^{\mathbb{P}}}.$$

For $\Delta \subseteq \mathcal{P}(\mathbb{R})$, Δ-BSM^{++} means X-BSM^{++} for all $X \in \Delta$.

We note that Lemma 4.41 implies that uB-BSM^{++} is a consequence of SM^{++}.

We justify the maximality of SM^{++} now.

Proposition 6.5. *Suppose* $\mathbb{P}$ *is a forcing that kills a Suslin tree. Then* $\mathrm{BFA}(\{\mathbb{P}\})$ *fails.*

Proof. Suppose T is a Suslin tree on ω_1 killed in an extension by $\mathbb{P}$. The statement "T is not Suslin" is Σ_1 over H_{ω_2} and would thus reflect down to V if $\mathrm{BFA}(\{\mathbb{P}\})$ were to hold. $\square$

The two main results of this section are that SM^{++} can be forced from a supercompact cardinal and that SM^{++} implies $\mathbb{S}_{\max}$-$(*)$.

6.1 Forcing SM^{++}

Theorem 6.6. *Suppose* ZFC+ *"there is a supercompact cardinal" is consistent. Then so is* $\mathrm{ZFC} + \mathrm{SM}^{++}$.

We will prove this by iterating semiproper forcings which preserve all Suslin trees.

Lemma 6.7. *Suppose κ is a supercompact cardinal. Then there is a nice iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$ of semiproper forcings that forces* SM^{++}.

Proof. By Fact 6.9, it is enough to force the version of SM^{++} that only applies to semiproper forcings preserving all Suslin trees. We follow the lines of the Foreman-Magidor-Shelah argument that gives a model of MM^{++}. This time, we opt to give details. Let $f : \kappa \to V_\kappa$ be a Laver function, i.e. for any $x \in V$ and λ we can find a elementary embedding

$$j : V \to M$$

witnessing that κ is λ-supercompact so that

$$j(f)(\kappa) = x.$$

The iteration $\mathbb{P}$ is defined inductively so that if $\alpha < \kappa$ then $\dot{\mathbb{Q}}_\alpha = f(\alpha)$ if $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a forcing with

$$\Vdash_{\mathbb{P}_\alpha} \text{“}f(\alpha) \text{ is semiproper and preserves all Suslin trees”}$$

and

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\beta \text{ is the trivial forcing”}$$

otherwise. We will show that $V^{\mathbb{P}} \models \mathrm{SM}^{++}$. First note that $\mathbb{P}$ is semiproper (and preserves all Suslin trees) by Corollary 5.57. Let G be $\mathbb{P}$-generic over V. Also observe that if all trees $T \in V[G_\alpha]$ that are Suslin in $V[G_\alpha]$ are Suslin in $V[G]$ for any $\alpha \leqslant \kappa$. This is a consequence of Corollary 3.53 (generalized to uniform sequences of witnesses). Assume that in $V[G]$

(*i*) $\mathbb{Q}$ is a semiproper forcing that preserves all Suslin trees,

(*ii*) $\mathcal{D} = \langle D_\alpha \mid \alpha < \omega_1 \rangle$ are ω_1-many dense subsets of $\mathbb{Q}$,

(*iii*) $\mathcal{S} = \langle \dot{S}_\alpha \mid \alpha < \omega_1 \rangle$ are ω_1-many $\mathbb{Q}$-names for stationary subsets of ω_1 and

(*iv*) $\mathcal{T} = \langle \dot{T}_\alpha \mid \alpha < \omega_1 \rangle$ are ω_1-many $\mathbb{Q}$-names for Suslin trees on ω_1.

In V, we may find a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ that is forced to be a semiproper forcing preserving all Suslin trees. We may further find an elementary embedding

$$j : V \to M$$

with $j(f)(\kappa) = \dot{\mathbb{Q}}$ and ${}^{\lambda}M \subseteq M$ for some large $\lambda > 2^{|\mathbb{Q}|}$. Consider the iteration

$$j(\mathbb{P}) = \langle \hat{\mathbb{P}}_\alpha, \dot{\hat{\mathbb{Q}}}_\beta \mid \alpha \leqslant j(\kappa), \beta < \kappa \rangle$$

and note that

$$\hat{\mathbb{P}}_\kappa = \mathbb{P} \text{ and } \hat{\mathbb{P}}_{\kappa+1} = \mathbb{P} * \dot{\mathbb{Q}}.$$

Note that as κ is inaccessible and $|\mathbb{P}_\alpha| < \kappa$ for any $\alpha < \kappa$, $\mathbb{P}$ is κ-c.c. by Fact 3.47. It follows that $\mathbb{P}$ preserves κ and thus $\mathbb{P}$ is the direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \kappa \rangle$. Thus in a further extension of $V[G]$, we may lift j to

$$j^+ : V[G] \to M[H]$$

with $G = H_\kappa$ which gives $\dot{\mathbb{Q}}^{H_\kappa} = \mathbb{Q}$. The closure of M guarantees that $j \upharpoonright \mathbb{Q} : \mathbb{Q} \to j^+(\mathbb{Q})$ is in $M[H]$, as well as $\mathcal{D}, \mathcal{S}, \mathcal{T} \in M[H]$. Let h be the slice of H at κ that is generic for $\mathbb{Q}$ and let $g = j^+[h] \in M[H]$. Also note that

$$j^+(\langle D_\alpha \mid \alpha < \omega_1 \rangle) = \langle j^+(D_\alpha) \mid \alpha < \omega_1 \rangle,$$
$$j^+(\langle \dot{S}_\alpha \mid \alpha < \omega_1 \rangle) = \langle j^+(\dot{S}_\alpha) \mid \alpha < \omega_1 \rangle$$

and

$$j^+(\langle \dot{T}_\alpha \mid \alpha < \omega_1 \rangle) = \langle j^+(\dot{T}_\alpha) \mid \alpha < \omega_1 \rangle.$$

Now we have

$$j^+[h \cap D_\alpha] \subseteq g \cap j(D_\alpha) \neq \varnothing,$$
$$S_\alpha := \dot{S}^h_\beta = \{\xi < \omega_1 \mid \exists p \in g\ p \Vdash_{j^+(\mathbb{Q})} \check{\xi} \in j^+(\dot{S}_\alpha)\} \in (\mathrm{NS}^+_{\omega_1})^{M[H_{\kappa+1}]},$$

$$\begin{aligned} \leqslant^h_{\dot{T}_\alpha} &= \{(\beta, \gamma) \in \omega_1 \times \omega_1 \mid \exists p \in h\ h \Vdash^{M[H_\kappa]}_{\mathbb{Q}} \check{\beta} \leqslant_{\dot{T}_\alpha} \check{\gamma}\} \\ &= \{(\beta, \gamma) \in \omega_1 \times \omega_1 \mid \exists p \in g\ p \Vdash^{M[H]}_{j^+(\mathbb{Q})} \check{\beta} \leqslant_{j^+(\dot{T}_\alpha)} \check{\gamma}\} \end{aligned}$$

and

$$(\omega_1, \leqslant^g_{\dot{T}_\alpha}) \text{ is a Suslin tree in } M[H_{\kappa+1}]$$

for all $\alpha < \omega_1$. It follows that, for all $\alpha < \omega_1$, S_α is still stationary in $M[H]$ and $(\omega_1, \leqslant^g_{\dot{T}_\alpha})$ is still Suslin in $M[H]$. We may now pull the relevant statement back via j^+ to $V[G]$ and find that there must be a filter $g' \subseteq \mathbb{Q}$ so that for all $\alpha < \omega_1$

($g'.i$) $g' \cap D_\alpha \neq \varnothing$,

($g'.ii$) $\dot{S}^{g'}_\alpha \in \mathrm{NS}^+_{\omega_1}$ and

($g'.iii$) $\dot{T}^{g'}$ is Suslin,

which is what we had to show. □

6.2 SM^{++} implies $\mathbb{S}_{\max}$-$(*)$

We now introduce the $\mathbb{P}_{\max}$-variation $\mathbb{S}_{\max}$.

Definition 6.8 (Woodin, [Woo10, Definition 7.4])**.** $\mathbb{S}_{\max}$-conditions are generically iterable structures (M, I, a) so that

($\mathbb{S}_{\max}.i$) $M \models \mathrm{FA}(\sigma\text{-centered})$ and

($\mathbb{S}_{\max}.ii$) $M \models a \subseteq \omega_1 \wedge \omega_1^{L[a]} = \omega_1$. The order on $\mathbb{S}_{\max}$ is given by

$$q = (N, J, b) < (M, I, a) = p$$

iff there is an iteration

$$j : p \to p^* = (M^*, I^*, a^*)$$

in N of length $\omega_1^N + 1$ with

($<_{\mathbb{S}_{\max}}.i$) $a^* = b$,

($<_{\mathbb{S}_{\max}}.ii$) $I^* = J \cap M^*$ and

($<_{\mathbb{S}_{\max}}.iii$) whenever $T \in M^*$ and $M^* \models$ "T is a Suslin tree" then $N \models$ "T is a Suslin tree".

$\mathrm{FA}(\sigma\text{-centered})$ is large enough of a fragment of MA_{ω_1} to still imply Coding, ensuring generic iterations of $\mathbb{S}_{\max}$-conditions are uniquely determined by the image of their third component. That is, $\mathbb{S}_{\max}$ has unique iterations. The point is that this fragment is consistent with the existence of Suslin trees, which MA_{ω_1} is not. Other than that, the difference between $\mathbb{P}_{\max}$ and $\mathbb{S}_{\max}$ conditions lies solely in the more restrictive order on $\mathbb{S}_{\max}$, namely condition ($<_{\mathbb{S}_{\max}}.iii$). It follows that $\mathbb{S}_{\max}$ is a typical $\mathbb{P}_{\max}$-variation. Note that typicality of $\varphi^{\mathbb{S}_{\max}}$ is witnessed by $\Psi^{\mathbb{S}_{\max}}$ consisting of

- $\psi_0^{\mathbb{S}_{\max}}(x) =$ "$x \in \dot{I}$",
- $\psi_1^{\mathbb{S}_{\max}}(x) =$ "$x = \dot{a}$" and
- $\psi_2^{\mathbb{S}_{\max}}(x) =$ "x is a Suslin tree".

Let SPFA(Suslin) denote the version of MM(Suslin) that only applies to semiproper Suslin tree preserving forcings. We will make use of the following fact.

Fact 6.9 (Miyamoto, [Miy02])**.** SPFA(Suslin) *implies* SRP.

So the same is true under the stronger SM^{++}. We are in good shape to apply the Blueprint Theorems.

Theorem 6.10. SM^{++} *implies* $\mathbb{S}_{\max}$-$(*)$.

Proof. Assume SM^{++}. As all σ-centered forcings preserve all Suslin trees and preserve stationary sets, $\mathrm{FA}(\sigma\text{-centered})$ holds. Also SRP holds so NS_{ω_1} is saturated. If A is any subset of ω_1 with $\omega_1^{L[A]} = \omega_1$ then $\mathcal{H}_A$ is almost a $\mathbb{S}_{\max}$-condition. $\mathbb{S}_{\max}$ accepts $\diamondsuit$-iterations by Corollary 5.58. Also $\Gamma_A^{\Psi^{\mathbb{S}_{\max}}}$ is exactly the class of stationary set preserving forcings preserving all Suslin trees. Finally, $\mathrm{FA}_A^{\Psi^{\mathbb{S}_{\max}}}(\Gamma_A^{\Psi^{\mathbb{S}_{\max}}})$ follows from SM^{++}, we leave the details there to the reader. $\mathbb{S}_{\max}$-$(*)$ holds by the First Blueprint Theorem 4.44. $\square$

We remark on an interesting consequence of this. Woodin has introduced the following axiom.

Definition 6.11 (Woodin, [Woo10, Definition 7.2])**.** Φ_{S}^+ is the statement that for any $A \subseteq \omega_1$, there is $B \subseteq \omega_1$ so that[39]

$(B.i)$ $A \in L[B]$ and

$(B.ii)$ all trees $T \in L[B]$ Suslin in $L[B]$ are Suslin in V.

Woodin proved that Φ_{S}^+ holds after forcing with $\mathbb{S}_{\max}$ over canonical models of determinacy.

Fact 6.12 (Woodin, [Woo10, Theorem 7.12])**.** *Assume* AD *in* $L(\mathbb{R})$*. Then* $L(\mathbb{R})^{\mathbb{S}_{\max}} \models \Phi_{\mathrm{S}}^+$.

Note that Φ_{S}^+ is a statement purely about $\mathcal{P}(\omega_1)$ so that Φ_{S}^+ is also a consequence of $\mathbb{S}_{\max}$-$(*)$. Woodin [Woo10, Remark 7.13 (1)] writes that "Φ_{S}^+ is not obviously consistent with any large cardinals (above $0^\sharp$)" and notes that by forcing over stronger models of determinacy, Φ_{S}^+ can be seen consistent with a bit more than measurable cardinals, i.e. Woodin cardinals and somewhat more. It follows from the results here that Φ_{S}^+ is consistent with all natural large cardinals, assuming consistency of appropriate large cardinals.

Theorem 6.13. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(i) $\mathbb{S}_{\max}$-$(*)$.

(ii) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BSM}^{++}$.

Proof. Theorem 7.12 in [Woo10] implies that $\mathbb{S}_{\max}$ is self-assembling.

Claim 6.14. $\Gamma_A^{\Psi^{\mathbb{S}_{\max}}} = \Gamma_A^{\mathbb{S}_{\max}}(\Psi^{\mathbb{S}_{\max}})$ *for any* $A \subseteq \omega_1$ *with* $\omega_1^{L[A]} = \omega_1$.

[39] Observe that if $\omega_1^{L[B]} = \omega_1$, as is true for a cone of $B \subseteq \omega_1$ in constructibility degree, then $\diamondsuit$ holds in $L[B]$, so that there are many Suslin trees in $L[B]$.

Proof. Clearly $\Gamma_A^{\Psi^{\mathbb{S}_{\max}}}$ is the class Γ of all stationary set preserving forcings preserving all Suslin trees. Let $V[G]$ be a generic extension by a forcing in Γ and work in $V[G]$. It is not difficult to see that antichain sealing forcings preserve all Suslin trees. Let $\mathbb{P}$ be the Shelah forcing at the least Woodin cardinal δ to force "NS_{ω_1} is saturated", but we use nice supports instead of RCS-support, see [Sch11]. $\mathbb{P}$ is a nice iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \delta, \beta < \delta \rangle$$

of semiproper forcings of size $<\delta$. By Corollary 5.57, $\mathbb{P}$ is semiproper and preserves all Suslin trees. Moreover, $\mathbb{P}_\alpha$ forces $\mathrm{SCC}_{\mathrm{cof}}$ for cofinally many $\alpha < \delta$ by Fact 3.12. It follows that Chang's Conjecture holds in $V[G]^{\mathbb{P}}$ and hence NS_{ω_1} is saturated in any further c.c.c.-extension of $V[G]^{\mathbb{P}}$. See Theorem 2.4 in [BT82] for a proof of this. There is thus an extension $V[G][H]$ of $V[G]$ that preserves stationary sets as well as all Suslin trees and in which NS_{ω_1} is saturated and $\mathrm{FA}(\sigma - \text{centered})$ holds. □

The equivalence now follows from the Second Blueprint Theorem 4.58. □

7 A Forcing Axiom That Implies "NS_{ω_1} Is ω_1-Dense"

We formulate a forcing axiom that implies $\mathbb{Q}_{\max}$-$(*)$. We go on and show that it can be forced from a supercompact limit of supercompact cardinals.

7.1 Q-Maximum

Definition 7.1. Q-Maximum, denoted QM, holds if there is a witness f of $\diamondsuit(\omega_1^{<\omega})$ and $\mathrm{FA}(\Gamma)$ holds where

$$\Gamma = \{\mathbb{P} \mid \mathbb{P} \text{ preserves } f\} = \{\mathbb{P} \mid \forall p \in \mathrm{Col}(\omega, \omega_1)\ S_p^f \in (\mathrm{NS}_f^+)^{V^{\mathbb{P}}}\}.$$

So QM is the strengthening of $\mathrm{MM}(f)$ for f a witness of $\diamondsuit(\omega_1^{<\omega})$, which applies to all forcing preserving the f-stationarity only of all S_p^f instead of all f-stationary sets. We remark that the consistency of QM is a subtle matter: If the role of $\mathrm{Col}(\omega, \omega_1)$ is replaced by, for example, the trivial forcing then the result is the forcing axiom for all ω_1-preserving forcings which is clearly inconsistent. Indeed, $\mathrm{Col}(\omega, \omega_1)$ is, up to forcing equivalence, the only forcing $\mathbb{B}$ for which

$$\mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves } f\})$$

is consistent for a witness f of $\diamondsuit(\mathbb{B})$ (assuming large cardinals).

Lemma 7.2. *If f witnesses* QM *then η_f is a dense embedding. In particular, NS_{ω_1} is ω_1-dense.*

Proof. Suppose $S \subseteq \omega_1$ is so that

$$S_p^f \nsubseteq S \mod \mathrm{NS}_{\omega_1}$$

for all $p \in \mathrm{Col}(\omega, \omega_1)$. Let $\mathbb{P}$ be the canonical forcing that shoots a club through $T := \omega_1 - S$. That is $p \in \mathbb{P}$ iff $p \subseteq T$ is closed and bounded and $p \leqslant q$ iff q is an initial segment of p.

Claim 7.3. $\mathbb{P}$ *preserves* f*.*

Proof. Let $b \in \mathrm{Col}(\omega, \omega_1)$, we have to show that S_b^f is f-stationary in $V^{\mathbb{P}}$. Let $p \in \mathbb{P}$, $\dot{C}$ a $\mathbb{P}$-name for a club and $\langle \dot{D}_i \mid i < \omega_1 \rangle$ a sequence of $\mathbb{P}$-names for dense subsets of $\mathrm{Col}(\omega, \omega_1)$. We will find $q \leqslant p$ with

$$q \Vdash \exists \xi \in \dot{C} \cap S_{\check{b}}^{\check{f}} \ \forall i < \xi \ \check{f}(\xi) \cap \dot{D}_i \neq \varnothing. \tag{q}$$

Let θ be large and regular. Note that $\mathrm{MM}(f)$ holds and hence f witnesses $\diamondsuit^+(\omega_1^{<\omega})$. As $T \cap S_b^f$ is stationary, $T \cap S_b^f$ is f-stationary and we can find some $X \prec H_\theta$ with

($X.i$) X is f-slim,

($X.ii$) $\mathbb{P}, p, \dot{C}, \langle \dot{D}_i \mid i < \omega_1 \rangle \in X$ and

($X.iii$) $\delta^X \in T \cap S_b^f$.

Now find a decreasing sequence $\langle p_n \mid n < \omega \rangle$ with

($\vec{p}.i$) $p_0 = p$,

($\vec{p}.ii$) $\forall n < \omega \ p_n \in \mathbb{P} \cap X$ and

($\vec{p}.iii$) for all $D \in M_X[f(\delta^X)]$ dense in $\pi_X^{-1}(\mathbb{P})$, there is $n < \omega$ with $p_n \in \pi_X[D]$.

Set $q = \bigcup_{n<\omega} p_n \cup \{\delta^X\}$ and note that $q \in \mathbb{P}$ as $\delta^X \in T$. It is clear that q is $(X, \mathbb{P}, f)$-semigeneric so that if G is $\mathbb{P}$-generic with $q \in G$ then

$$\forall i < \delta^X = \delta^{X[G]} \ f(\delta^X) \cap \dot{D}_i^G \neq \varnothing$$

as well as $\delta^X \in \dot{C}^G \cap S_b^f$. Thus q indeed satisfies (q). $\square$

Thus $\mathrm{FA}(\{\mathbb{P}\})$ holds. This implies that if G is $\mathbb{P}$-generic then

$$(H_{\omega_2}; \in)^V \prec_{\Sigma_1} (H_{\omega_2}; \in)^{V[G]}$$

and as T contains a club in $V[G]$, this must already be true in V. This means S is nonstationary which is what we had to show. $\square$

Remark 7.4. The argument here also proves the remark we made at the beginning of Section 3: For any witness f of $\diamondsuit(\mathbb{B})$, η_f is a dense embedding if and only if all f-preserving forcings are f-stationary set preserving.

This yields an equivalent formulation of QM involving an already familiar forcing axiom.

Lemma 7.5. *The following are equivalent:*

(*i*) QM.

(*ii*) NS_{ω_1} *is* ω_1*-dense and there is* f *a witness of* $\diamondsuit(\omega_1^{<\omega})$ *so that* $\mathrm{MM}(f)$ *holds.*

Proof. (*i*)⇒(*ii*): Clearly, if f witnesses QM then $\mathrm{MM}(f)$ holds. Also NS_{ω_1} is dense by Lemma 7.2.
(*ii*)⇒(*i*): Assume NS_{ω_1} is dense, f witnesses $\diamondsuit(\omega_1^{<\omega})$ and $\mathrm{MM}(f)$ holds. As NS_{ω_1} is dense, we can find a witness f' of $\diamondsuit(\omega_1^{<\omega})$ so that

$$\eta_{f'} \colon \mathrm{Col}(\omega, \omega_1) \to (\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+$$

is a dense embedding, see the proof of Lemma 2.12. We will show that f' witnesses QM, so let $\mathbb{P}$ be a forcing preserving f'. As $\mathrm{MM}(f)$ holds, it is enough to argue that $\mathbb{P}$ preserves f-stationary sets. Let S be f-stationary and G be $\mathbb{P}$-generic over V. Clearly, S is f'-stationary in $V[G]$, we have to show that S is also f-stationary. Work in $V[G]$ and let θ be some sufficiently large regular cardinal. We may now find some f'-slim $X \prec H_\theta$ with $f, f' \in X$. Let $\bar{\eta}_f = \pi_X^{-1}(\eta_f)$, $\bar{\eta}_{f'} = \pi_X^{-1}(\eta_{f'})$. Let g be the upwards closure of $\bar{\eta}_{f'}[f'(\delta^X)]$ in $((\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1})^+)^{M_X}$. Note that g is generic over M_X as $\eta_{f'}$ is a dense embedding.

Claim 7.6. $\bar{\eta}_f^{-1}[g] = f(\delta^X)$.

Proof. Note that it suffices to show "$\subseteq$". Suppose $p \in \bar{\eta}_f^{-1}[g]$. This means $[S_p^{\bar{f}}]_{\mathrm{NS}_{\omega_1}^{M_X}} \in g$ and thus there is $q \in \mathrm{Col}(\omega, \omega_1)$ with $q \in f'(\delta^X)$ and

$$M_X \models S_q^{\bar{f}'} \subseteq S_p^{f} \mod \mathrm{NS}_{\omega_1}$$

so that

$$X \models S_q^{f'} \subseteq S_p^{\bar{f}} \mod \mathrm{NS}_{\omega_1}.$$

As $\delta^X \in S_q^{f'}$, this implies $\delta^X \in S_p^{f}$, i.e. $p \in f(\delta^X)$. □

As $\mathrm{MM}(f)$ holds, f witnesses $\diamondsuit^+(\omega_1^{<\omega})$ and hence η_f is a complete embedding. It follows that $f(\delta^X)$ is generic over M_X, that is X is f-slim. This shows that S is f-stationary. □

7.2 $\mathbb{Q}$-iterations

Let us assume that $\diamondsuit(\omega_1^{<\omega})$ holds in V as witnessed by f.

Our goal is to force QM as witnessed by f. Our strategy for this will differ to the standard construction of models of MM^{++} for example. For MM, one iterates semiproper forcings and shows only a posteriori that in fact MM holds, despite seemingly not allowing arbitrary stationary set preserving forcings along the iteration. The reason why one does this is simple: An iteration of stationary set preserving forcing, even of short length, can collapse[40] ω_1. We do not know of a class of forcings that can play the role of semiproperness when trying to force QM. Our approach is more straightforward: Instead of using a precise tool like semiproperness, we take out the sledgehammer. We will cook up an iteration that directly allows essentially arbitrary f-preserving forcings at limit steps. The price we pay is a stronger large cardinal assumption, just one supercompact cardinal will not suffice. Instead, we need a supercompact limit of supercompact cardinals. All things considered, this price is cheap compared to the alternative of failure.
So what is the basic idea? As always, we want to imitate the argument of the mother of all iteration theorems, the iteration theorem for proper forcings. Suppose we have a full support iteration

$$\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$$

and for the moment assume only that

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{\mathbb{Q}}_n \text{ preserves } \omega_1\text{”}.$$

There are (at least) two type of counterexamples we have to avoid: First there is Shelah's iteration of stationary set preserving forcings of the above type that collapses ω_1, see [She98, VII §5]. This is dealt with a regularity condition we will impose. Further, if $\omega_1 = \bigcup_{n<\omega} S_n$ is a partition of ω_1 into stationary sets so that

$$\Vdash_{\mathbb{P}_n} S_n \in \mathrm{NS}_{\omega_1}$$

then necessarily $\mathbb{P}$ collapses ω_1. In our applications, the forcings we consider can and will destroy many stationary sets. We try to motivate some additional reasonable constraints imply $\mathbb{P}$ to be ω_1-preserving. For the moment, we try to consider Shelah's argument as a game: In the beginning there some countable $X \prec H_\theta$ as well as $p_0 \in X \cap \mathbb{P}$. The argument proceeds as follows: In round n, we have already constructed a $(X, \mathbb{P}_n)$-semigeneric condition $q_n \in \mathbb{P} \upharpoonright n$ and have

$$q_n \Vdash \dot{p}_n \upharpoonright n \in \dot{G}_n \cap \check{X}[\dot{G}_n].$$

[40] This is also true for iterations of f-stationary set preserving forcings for f some witness of $\diamondsuit(\mathbb{B})$.

Next, our adversary hits us with a dense subset $D \subseteq \mathbb{P}$ in X and we must find $\dot{p}_{n+1} \in V^{\mathbb{P}_n}$ and some $(X, \mathbb{P}_{n+1})$-semigeneric q_{n+1} with $q_{n+1} \upharpoonright n = q_n$ and[41]

$$q_{n+1} \Vdash \dot{p}_{n+1} \in \check{D} \wedge p_{n+1} \upharpoonright n+1 \in \dot{G}_{n+1} \cap \check{X}[\dot{G}_{n+1}].$$

Our job is to survive this game for ω-many steps. If we have a winning strategy then we can find a $(X, \mathbb{P})$-semigeneric condition, so in particular $\mathbb{P}$ preserves ω_1.

Destroying stationarity makes it significantly more difficult to survive the above game: Suppose for example that

$$p_0(0) \Vdash \check{S} \in \mathrm{NS}_{\omega_1}$$

for some $S \in X$ with $\delta^X \in S$. Then there is no hope of finding a $(X, \mathbb{P}_1)$-semigeneric condition q with $q \leqslant p_0 \upharpoonright 1$. Hence, we must already be careful with what X we start the game. This leads us to the following definitions.

Definition 7.7. Suppose θ is sufficiently large and regular, $X \prec H_\theta$ is countable. If I is an ideal on ω_1, we say that X *respects* I if for all $A \in I \cap X$ we have $\delta^X \notin A$.

Note that all countable $X \prec H_\theta$ respect NS_{ω_1} and countable $Y \prec H_\theta$ with $f \in Y$ respects NS_f if and only if Y is f-slim.

Definition 7.8. Suppose $\mathbb{P}$ is a forcing and $\dot{I} \in V^{\mathbb{P}}$ is a name for an ideal on ω_1. For p in $\mathbb{P}$, we denote the *partial evaluation of* $\dot{I}$ *by* p by

$$\dot{I}^p := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \dot{I}\}.$$

Back to the discussion, we need to start with an X so that X respects $\dot{I}^{p_0 \upharpoonright 1}$ where $\dot{I}$ is a name for the nonstationary ideal. This gives us a shot at getting past the first round. Luckily, there are enough of these X.

Definition 7.9. Let A be an uncountable set with $\omega_1 \subseteq A$ and I a normal uniform ideal on ω_1. Then $\mathcal{S} \subseteq [A]^\omega$ is *projective* I*-positive* if for any $S \in I^+$ the set

$$\{X \in \mathcal{S} \mid \delta^X \in S\}$$

is stationary in $[A]^\omega$.

Proposition 7.10. *Suppose θ is sufficiently large and regular. Let I be a normal uniform ideal on ω_1. Then*

$$\mathcal{S} = \{X \in [H_\theta]^\omega \mid X \prec H_\theta \text{ respects } I\}$$

is projective I-positive.

[41] Here, we consider $\dot{p}_n$ also as a $\mathbb{P}_{n+1}$-name.

Proof. Let $\mathcal{C}$ be a club in $[H_\theta]^\omega$ and assume that all elements of $\mathcal{C}$ are elementary substructures of H_θ and contain I as an element. Let

$$\vec{X} := \langle X_\alpha \mid \alpha < \omega_1 \rangle$$

be a continuous increasing chain of elements in $\mathcal{C}$. Let $X := \bigcup_{\alpha<\omega_1} X_\alpha$ and let

$$\vec{A} := \langle A_\alpha \mid \alpha < \omega_1 \rangle$$

be an enumeration of $X \cap I$. Let $C \subseteq \omega_1$ be the set of α so that

($C.i$) $\delta^{X_\alpha} = \alpha$ and

($C.ii$) $\vec{A} \upharpoonright \alpha$ is an enumeration of $X_\alpha \cap I$

and note that C is club. Let $A = \triangledown_{\alpha<\omega_1} I_\alpha$. As I is normal, $A \in I$. Then $C - A$ is a complement of a set in I and for any $\alpha \in C - A$ we have

$$\delta^{X_\alpha} = \alpha \notin I_\beta$$

for all $\beta < \alpha$. Hence $X_\alpha \in \mathcal{S} \cap \mathcal{C}$. □

Of course, the problem continues. What if we have found a suitable q_1 and now we work in $V[G_1]$ with $q_1 \in G_1$. At the very least, we need that $X[G_1]$ respects $\dot{I}^{p_0 \upharpoonright [1,2)}$, where $\dot{I}$ is now a $\mathbb{P}_{1,2}$-name for the nonstationary ideal. Ensuring this is a matter of being able to pick the right q_1 to begin with. This motivates the following class of forcings.

Definition 7.11. We say that a forcing $\mathbb{P}$ is *respectful* if $\mathbb{P}$ preserves ω_1 and the following is true: Whenever

- θ is sufficiently large and regular,
- $X \prec H_\theta$ is countable with $\mathbb{P} \in X$,
- $\dot{I} \in X$ is a $\mathbb{P}$-name for a normal uniform ideal and
- $p \in \mathbb{P} \cap X$

then exactly one of the following holds:

(Res.i) Either there is some $(X, \mathbb{P})$-semigeneric $q \leqslant p$ so that

$$q \Vdash \text{“}\check{X}[\dot{G}] \text{ respects } \dot{I}\text{”}$$

or

(Res.ii) X does not respect $\dot{I}^p$.

Roughly, this condition states that we can find a $\mathbb{P}$-generic filter G with $p \in G$ so that $X \sqsubseteq X[G]$ respects $\dot{I}^G$ as long as there is no obvious obstruction to it.

Remark 7.12. If $\mathbb{P}$ is respectful and preserves stationary sets then $\mathbb{P}$ is semiproper. However, the converse is not true in general. Similarly, a respectful f-stationary set preserving forcing is f-semiproper, which follows from plugging in a name for NS_f as $\dot{I}$ in the definition of respectfulness.

We require[42] now that

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{\mathbb{Q}}_n \text{ is respectful”}$$

for all $n < \omega$. We then aim to make sure (assuming $\dot{p}_{n+1}$ is already defined) to find q_{n+1} in round n so that in addition to the prior constraints,

$$q_{n+1} \Vdash \text{“}\check{X}[\dot{G}_{n+1}] \text{ respects } \dot{I}\text{”}$$

where $\dot{I}$ is a $\mathbb{P}_{n+1}$ name for the ideal of sets forced to be nonstationary by $\dot{p}_{n+1}(n+1)$. Consider $\dot{I}$ as a $\mathbb{P}_n$-name $\ddot{I}$ for a $\dot{\mathbb{Q}}_n$-name. By respectfulness, this reduces to avoiding an instance of the "bad case" (Res.*ii*), namely we should make sure that whenever G_n is $\mathbb{P}_n$-generic with $q_n \in G_n$ then

$$X[G_n] \text{ respects } \left(\ddot{I}^{G_n}\right)^{p_{n+1}(n+1)}$$

where $p_{n+1} = \dot{p}_{n+1}^{G_{n+1}}$. he next key insight is that this reduces to

$$\text{“}X[G_n] \text{ respects } J := \{S \subseteq \omega_1 \mid p_{n+1}(n) \Vdash \check{S} \in \mathrm{NS}_{\omega_1}\}\text{”}$$

which we have (almost)[43] already justified inductively, assuming $\dot{\mathbb{Q}}_{n+1}$ only kills new stationary sets: Our final requirement[44] is that

$$\Vdash_{\mathbb{P}_{n+1}} \text{“}\dot{\mathbb{Q}}_{n+1} \text{ preserves stationary sets which are in } V[\dot{G}_n]\text{”}$$

for all $n < \omega$. The point is that trivially $\left(\ddot{I}^{G_n}\right)^{p_{n+1}(n)}$ only contains sets in $V[G_n]$, so all such sets will be preserved by $\dot{\mathbb{Q}}_{n+1}$. The sets that are killed are then already killed in the extension by $\dot{\mathbb{Q}}_n^{G_n}$.
Modulo some details we have shown the following.

Theorem 7.13. *Suppose* $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ *is a full support iteration so that*

[42]This excludes the first counterexample due to Shelah, but not yet all the counterexamples of the second kind.
[43]We made sure of this if p_{n+1} is replaced by p_n in the definition of J, we ignore this small issue for now.
[44]It is readily seen that this eliminates the counterexamples of the second kind.

($\mathbb{P}.i$) $\Vdash_{\mathbb{P}_n} \dot{\mathbb{Q}}_n$ *is respectful and*

($\mathbb{P}.ii$) $\Vdash_{\mathbb{P}_{n+1}} \dot{\mathbb{Q}}_{n+1}$ *preserves stationary sets which are in* $V[\dot{G}_n]$"

for all $n < \omega$. *Then* $\mathbb{P}$ *does not collapse* ω_1.

Two issues arise when generalizing this to longer iterations. The first issue is the old problem that new relevant indices may appear along the iteration in the argument, which we deal with by using nice supports. The second problem is that it seemingly no longer suffices that each iterand individually is respectful. For longer iterations, say of length γ, the argument then requires that

$$\Vdash_\alpha \text{“}\dot{\mathbb{P}}_{\alpha,\beta} \text{ is respectful”}$$

for sufficiently many $\alpha < \beta < \gamma$. This is problematic as we will not prove an iteration theorem of any kind for respectful forcings[45]. This is where we take out the sledgehammer.

Definition 7.14. ($\ddagger$) holds if and only if all ω_1-preserving forcings are respectful.

Lemma 7.15. SRP *implies* ($\ddagger$).

Proof. Let $\mathbb{P}$, θ, $\dot{I}$, p be as Definition 7.11. It is easy to see that (Res.i) and (Res.ii) cannot hold simultaneously. It is thus enough to prove that one of them holds. Let λ be regular, $2^{|\mathbb{P}|} < \lambda < \theta$ and $\lambda \in X$ and consider the set

$$\begin{aligned}\mathcal{S} = \{Y \in [H_\lambda]^\omega \mid & Y \prec H_\lambda \wedge \neg(\exists q \leqslant p \ q \text{ is} \\ & (Y, \mathbb{P})\text{-semigeneric and } q \Vdash \text{“}\check{Y}[\dot{G}] \text{ respects } \dot{I}\text{”})\}.\end{aligned}$$

By SRP, there is a continuous increasing elementary chain

$$\vec{Y} = \langle Y_\alpha \mid \alpha < \omega_1 \rangle$$

so that

($\vec{Y}.i$) $\mathbb{P}, p, \dot{I} \in Y_0$ and

($\vec{Y}.ii$) for all $\alpha < \omega_1$, either $Y_\alpha \in \mathcal{S}$ or there is no $Y_\alpha \sqsubseteq Z \prec H_\theta$ with $Z \in \mathcal{S}$.

Let $S = \{\alpha < \omega_1 \mid Y_\alpha \in \mathcal{S}\}$.

Claim 7.16. $p \Vdash \check{S} \in \dot{I}$.

[45] Indeed it seems that no useful iteration theorem for respectful forcings is provable in ZFC, see Subsection 11.5.

Proof. Let G be generic with $p \in G$ and let $I = \dot{I}^G$. Assume toward a contradiction that S is I-positive. Note that $\langle Y_\alpha[G] \mid \alpha < \omega_1\rangle$ is a continuous increasing sequence of elementary substructure of $H_\theta^{V[G]}$. Hence there is a club C of α so that for $\alpha \in C$

$$\delta^{Y_\alpha} = \delta^{Y_\alpha[G]} = \alpha$$

and thus there is a $(Y_\alpha, \mathbb{P})$-semigeneric condition $q \leqslant p$, $q \in G$. Hence by definition of S, for any $\alpha \in S \cap C$, we may find some $N_\alpha \in I \cap Y_\alpha[G]$ so that $\delta^{Y_\alpha} \in N_\alpha$. By normality of I, there is some I-positive $T \subseteq S \cap C$ and some N so that $N = N_\alpha$ for all $\alpha \in T$. But then for $\alpha \in T$, we have

$$\alpha = \delta^Y \in N$$

so that $T \subseteq N$. But $N \in I$, contradiction. $\square$

Thus if $\delta^X \in S$, then S witnesses (Res.*ii*) to hold. Otherwise, $\delta^X \notin S$. Note that $\delta^{Y_{\delta^X}} = \delta^X$ as $\vec{Y} \in X$. We find that $Y_{\delta^X} \sqsubseteq X \cap H_\lambda \prec H_\lambda$. Thus, $X \cap H_\lambda \notin \mathcal{S}$, so that there must be some $q \leqslant p$ that is $(X \cap H_\lambda, \mathbb{P})$-semigeneric and

$$q \Vdash \text{“}(\widetilde{X \cap H_\lambda})[\dot{G}] \text{ respects } \dot{I}\text{”}.$$

This q witnesses that (Res.*i*) holds. $\square$

We will get around this second issue by forcing SRP often along the iteration. Remember that what we really care about is preserving a witness f of $\diamondsuit(\omega_1^{<\omega})$ along an iteration of f-preserving forcings, so fix such an f now. It will be quite convenient to introduce some short hand notation.

Definition 7.17. Suppose $\mathbb{P}$ is a forcing and $p \in \mathbb{P}$. Then we let $I_p^{\mathbb{P}}$ denote $\dot{I}^p$ where $\dot{I}$ is a $\mathbb{P}$-name for NS_f. That is

$$I_p^{\mathbb{P}} := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \mathrm{NS}_f\}.$$

Definition 7.18. Suppose f witnesses $\diamondsuit(\mathbb{B})$. An *f-ideal* is an ideal I on ω_1 so that

(*i*) whenever $S \in I^+$ and $\langle D_i \mid i < \omega_1\rangle$ is a sequence of dense subsets of $\mathrm{Col}(\omega, \omega_1)$, then

$$\{\alpha \in S \mid \forall \beta < \alpha \; f(\alpha) \cap D_\beta \neq \varnothing\} \in I^+$$

(*ii*) and $S_b^f \in I^+$ for all $b \in \mathbb{B}$.

Recall that NS_f is clearly an f-ideal and it is normal and uniform by Lemma 2.20.

Proposition 7.19. *Suppose $\mathbb{P}$ is a forcing that preserves f and $p \in \mathbb{P}$. Then $I_p^{\mathbb{P}}$ is a normal uniform f-ideal.*

Condition ($\mathbb{P}.ii$) should now be replaced by requiring the next forcing of the iteration to kill the f-stationarity only of new sets. This restricts the forcings we allow later in the iteration, but on the other hand we want to incorporate basically arbitrary f-preserving forcings into our iteration if we want to force QM. The solution is to often kill the f-stationarity of as many sets as possible while still preserving f.

Observation 7.20. The following are equivalent for any $S \subseteq \omega_1$:

1. There is an f-preserving forcing $\mathbb{P}$ with $\Vdash_{\mathbb{P}} \check{S} \in \mathrm{NS}_f$.
2. $S_p^f \nsubseteq S \mod \mathrm{NS}_f$ for all $p \in \mathrm{Col}(\omega, \omega_1)$.

The nontrivial direction follows from the proof of Lemma 7.2.

Definition 7.21. Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. We say that a forcing $\mathbb{P}$ *freezes* NS_{ω_1} along f if for any $\mathbb{P}$-generic G we have

(i) f witnesses $\diamondsuit(\omega_1^{<\omega})$ in $V[G]$ and

(ii) for any $S \in \mathcal{P}(\omega_1) \cap V$, we either have $S \in \mathrm{NS}_{\omega_1}^{V[G]}$ or there is $p \in \mathrm{Col}(\omega, \omega_1)$ with $S_p^f \subseteq S \mod \mathrm{NS}_{\omega_1}^{V[G]}$.

The point is that if $\mathbb{P}$ freezes NS_{ω_1} along f then no further f-preserving forcing $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ can kill the f-stationarity of sets in V. It is nontrivial to prove such forcings to exist from reasonable assumptions, we will do so in Lemma 7.41.

Proposition 7.22. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. Suppose $\mathbb{P}$ freezes NS_{ω_1} along f and*

$$\Vdash_{\mathbb{P}} \text{“}\dot{\mathbb{Q}} \textit{ preserves } \check{f}\text{”}.$$

Assume $G * H$ is generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Then

$$\mathrm{NS}_f^{V[G]} \cap \mathcal{P}(\omega_1)^V = \mathrm{NS}_f^{V[G][H]} \cap \mathcal{P}(\omega_1)^V$$

In particular, for any $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ we have

$$I_{(p,\dot{q})}^{\mathbb{P}*\dot{\mathbb{Q}}} = I_p^{\mathbb{P}}.$$

Proof. “ $\subseteq$ ” is trivial, so assume $S \in (\mathcal{P}(\omega_1) \cap V) - \mathrm{NS}_f^{V[G]}$. As $\mathbb{P}$ freezes NS_{ω_1} along f, there must be some $b \in \mathrm{Col}(\omega, \omega_1)$ with

$$S_b^f \subseteq S \mod \mathrm{NS}_{\omega_1}^{V[G]}$$

As $\dot{\mathbb{Q}}^G$ preserves f, $\dot{\mathbb{Q}}^G$ preserves f-stationarity of S_b^f, hence S is f-stationary in $V[G][H]$. □

We hope to have motivated the following definition.

Definition 7.23. Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. A *Q-iteration* (w.r.t. f) is a nice iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ which satisfies

(*i*) $\Vdash_{\mathbb{P}_\alpha}$ “$\dot{\mathbb{Q}}_\alpha$ preserves f”,

(*ii*) $\Vdash_{\mathbb{P}_{\alpha+1}}$ (‡) and

(*iii*) if $\alpha + 1 < \gamma$ then $\Vdash_{\mathbb{P}_{\alpha+1}}$ “$\dot{\mathbb{Q}}_{\alpha+1}$ freezes NS_{ω_1} along f”

for all $\alpha < \gamma$.

The main result of this section is the following “iteration theorem”.

Theorem 7.24. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. All Q-iterations (w.r.t. f) preserve f.*

Lemma 7.25. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$ and $\mathbb{P}$ is a forcing with the following property: For any sufficiently large regular θ and $p \in \mathbb{P}$ there is a normal uniform f-ideal I so that*

$$\{X \in [H_\theta]^\omega \mid X \prec H_\theta \wedge \mathbb{P}, p \in X \wedge \exists q \leqslant p \ q \textit{ is } (X, \mathbb{P}, f)\textit{-semigeneric}\}$$

is projective I-positive. Then $\mathbb{P}$ preserves f.

Proof. Assume $p \in \mathbb{P}$, θ is sufficiently large and regular. Let $b \in \mathrm{Col}(\omega, \omega_1)$,

$$\vec{\dot{D}} = \langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle$$

be a sequence of $\mathbb{P}$-names for dense subsets of $\mathrm{Col}(\omega, \omega_1)$ and $\dot{C}$ a $\mathbb{P}$-name for a club in ω_1. We will find $q \leqslant p$ so that

$$q \Vdash \exists \alpha \in S_b^{\check{f}} \cap \dot{C} \forall \beta < \alpha \ \check{f}(\alpha) \cap \dot{D}_\beta \neq \varnothing.$$

By our assumption, there is some normal uniform f-ideal I so that

$$\{X \in [H_\theta]^\omega \mid X \prec H_\theta \wedge \mathbb{P}, p \in X \wedge \exists q \leqslant p \ q \text{ is } (X, \mathbb{P}, f)\text{-semigeneric}\}$$

is projective I-positive. It follows that we can find some countable $X \prec H_\theta$ so that

($X.i$) $\mathbb{P}, p, \vec{\dot{D}}, \dot{C} \in X$ as well as

($X.ii$) $b \in f(\delta^X)$

and some $q \leqslant p$ that is $(X, \mathbb{P}, f)$-semigeneric. If G is then any $\mathbb{P}$-generic with $q \in G$, we have

$$X \sqsubseteq X[G] \text{ is } f\text{-slim}$$

and hence $\delta^X \in \dot{C}^G$ as well as

$$\forall \beta < \delta^X \ f(\delta^X) \cap \dot{D}_\beta^G \neq \varnothing.$$

□

We also need to resolve a small issue that we glossed over in the sketch of a proof of Theorem 7.13.

Lemma 7.26. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. Further assume that*

- *$\mathbb{P}$ is a respectful, f-preserving forcing and $p \in \mathbb{P}$,*
- *θ is sufficiently large and regular,*
- *$X \prec H_\theta$ is countable, respects $I_p^{\mathbb{P}}$ and $\mathbb{P}, p \in X$ and*
- *$M_X[f(\delta^X)] \models$ "D is dense below $\pi_X^{-1}(p)$ in $\pi_X^{-1}(\mathbb{P})$".*

Then there are Y, q with

(i) *$X \sqsubseteq Y \prec H_\theta$ is countable,*

(ii) *$q \leqslant p$,*

(iii) *Y respects $I_q^{\mathbb{P}}$, in particular Y is f-slim and*

(iv) *$q \in \pi_Y[\mu^+_{X,Y}(D)]$.*

Proof. We may assume that X is an elementary substructure of

$$\mathcal{H} := (H_\theta; \in, \trianglelefteq)$$

where $\trianglelefteq$ is a wellorder of H_θ. As $\mathbb{P}$ is respectful and X respects $I_p^{\mathbb{P}}$, there is a $(X, \mathbb{P})$-semigeneric condition $r \leqslant p$ so that

$$r \Vdash \text{“}\check{X}[\dot{G}] \text{ respects } \mathrm{NS}_{\check{f}}\text{”}$$

, i.e. r is $(X, \mathbb{P}, f)$-semigeneric. Let G be $\mathbb{P}$-generic with $r \in G$. Then $X[G]$ is f-slim. Let $Z = X[G] \cap V$, note that $\mu^+_{X,Z}$ exists by Proposition 2.18. By Proposition 3.33, there is thus some $q \leqslant p$, $q \in G$ with

$$q \in \pi_Z[\mu^+_{X,Z}(D)].$$

Finally, note that q and $Y := \mathrm{Hull}^{\mathcal{H}}(X \cup \{q\})$ have the desired properties. □

Proof of Theorem 7.24. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a Q-iteration. We may assume inductively that $\mathbb{P}_\alpha$ preserves f for all $\alpha < \gamma$. The successor step is trivial, so we may restrict to $\gamma \in \mathrm{Lim}$. Note that we may further assume that (‡) holds in V, otherwise we could work in $V^{\mathbb{P}_1}$. Let $p \in \mathbb{P}$ and let $I := I_{p(0)}^{\mathbb{Q}_0}$. I is a normal uniform f-ideal by Proposition 7.19. Now let θ be sufficiently large and regular, $X \prec H_\theta$ countable with

(X.i) $\mathbb{P}, p, f \in X$ and

(X.ii) X respects I.

By Proposition 7.10 and Lemma 7.25, it suffices to find $q \leqslant p$ that is $(X, \mathbb{P}, f)$-semigeneric. Note that X is f-slim as I is a f-ideal. Let

$$h\colon \omega \to \omega \times \omega$$

be a surjection with $i \leqslant n$ whenever $h(n) = (i, j)$.
Let δ denote δ^X. We will construct a fusion structure

$$T, \langle p^{(a,n)}, T^{(a,n)} \mid a \in T_n,\ n < \omega \rangle$$

in $\mathbb{P}$ as well as names

$$\left\langle \dot{X}^{(a,n)}, \dot{Z}^{(a,n)} \left(\dot{D}_j^{(a,n)}\right)_{j<\omega}, \dot{I}^{(a,n)} \mid a \in T_n, n < \omega \right\rangle$$

so that for any $n < \omega$ and $a \in T_n$

(*F.i*) $T_0 = \{\mathbb{1}\}$, $p^{(\mathbb{1},0)} = p$, $\dot{X}^{(\mathbb{1},0)} = \check{X}$, $\dot{I}^{(\mathbb{1},0)} = \check{I}$,

(*F.ii*) $T^{(\mathbb{1},0)} \in X$ is a nested antichain that p is a mixture of with $T_0^{(\mathbb{1},0)} = \{\mathbb{1}\}$,

(*F.iii*) $a \Vdash_{\mathrm{lh}(a)} \dot{Z}^{(a,n)} = \dot{X}^{(a,n)} \cap V$,

(*F.iv*) $\left(\dot{D}_j^{(a,n)}\right)_{j<\omega}$ is forced by a to be an enumeration of all dense subsets of $\pi^{-1}_{\dot{Z}^{(a,n)}}(\check{\mathbb{P}})$ in
$$M_{\dot{Z}^{(a,n)}}\left[\widetilde{f(\delta)}\right],$$

(*F.v*) $a \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(a)$,

(*F.vi*) $\mathrm{lh}(a)$ is not a limit ordinal,

(*F.vii*) $a \Vdash_{\mathrm{lh}(a)} \check{p}^{(a,n)}, \check{T}^{(a,n)}, \dot{G}_{\mathrm{lh}(a)} \in \dot{X}^{(a,n)}$,

(*F.viii*) $a \Vdash_{\mathrm{lh}(a)} \dot{I}^{(a,n)} = I^{\dot{\mathbb{Q}}_{\mathrm{lh}(a)}}_{\check{p}^{(a,n)}(\mathrm{lh}(a))}$ and

(*F.ix*) $a \Vdash$ "$\check{X} \sqsubseteq \dot{X}^{(a,n)} \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(a)}]}$ is countable and respects $\dot{I}^{(a,n)}$".

Moreover, for any $b \in \mathrm{suc}^n_T(a)$

(*F.x*) $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)}$ "$\check{p}^{(b,n+1)}, \check{T}^{(b,n+1)} \in \dot{X}^{(a,n)}$, in particular $\mathrm{lh}(\check{b}), \mathbb{P}_{\mathrm{lh}(\check{a}),\mathrm{lh}(\check{b})} \in \dot{X}^{(a,n)}$",

(*F.xi*) $b \Vdash_{\mathrm{lh}(b)} \dot{X}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}] \sqsubseteq \dot{X}^{(b,n+1)}$ and

(*F.xii*) if $h(n) = (i, j)$ and $c = \mathrm{pred}^i_T(b)$ then

$$b \Vdash_{\mathrm{lh}(n)} \check{p}^{(b,n+1)} \in \pi_{\dot{X}^{(a,n)}}[\dot{\mu}^+_{c,a}(\dot{D}_j^{(c,i)})].$$

Here, $\mu^+_{c,a}$ denotes[46]

$$\mu^+_{\dot{Z}^{(c,i)},\dot{Z}^{(a,n)}} : M_{\dot{Z}^{(c,i)}}[\check{f}(\check{\delta})] \to M_{\dot{Z}^{(a,n)}}[\check{f}(\check{\delta})].$$

We define all objects by induction on $n < \omega$.

$$T_0 = \{\mathbb{1}\}, p^{(\mathbb{1},0)}, T^{(\mathbb{1},0)}, \dot{X}^{(\mathbb{1},0)}, \dot{Z}^{(\mathbb{1},0)} \left(\dot{D}_j^{(\mathbb{1},0)}\right)_{j<\omega}, \dot{I}^{(\mathbb{1},0)}$$

are given by $(F.i)$-$(F.iv)$ and $(F.viii)$. Suppose we have already defined

$$T_n, \left\langle p^{(a,n)}, T^{(a,n)}, \dot{X}^{(a,n)}, \dot{Z}^{(a,n)}, \left(\dot{D}_j^{(a,n)}\right)_{j<\omega} \mid a \in T_n \right\rangle$$

and we will further construct

$$T_{n+1}, \left\langle p^{(b,n+1)}, T^{(b,n+1)}, \dot{X}^{(b,n+1)}, \dot{Z}^{(b,n+1)}, \left(\dot{D}_j^{(b,n+1)}\right)_{j<\omega} \mid b \in T_{n+1} \right\rangle.$$

Fix $a \in T_n$. Let E be the set of all b so that

$(E.i)$ $b \in \mathbb{P}_{\mathrm{lh}(b)}$ and $\mathrm{lh}(b) < \gamma$,

$(E.ii)$ $\mathrm{lh}(a) \leqslant \mathrm{lh}(b)$ and $b \upharpoonright \mathrm{lh}(a) \leqslant a$,

and there are a nested antichain S in $\mathbb{P}$, $s \in \mathbb{P}$ and names $\dot{X}$, $\dot{I}$ with

$(E.iii)$ $S \angle\, T^{(a,n)}$,

$(E.iv)$ $s \leqslant p^{(a,n)}$ is a mixture of S,

$(E.v)$ if $h(n) = (i,j)$ and $c = \mathrm{pred}^i_T(a)$ then

$$b \Vdash_{\mathrm{lh}(b)} \check{s} \in \pi_{\dot{Z}^{(a,n)}}[\dot{\mu}^+_{c,a}(\dot{D}_j^{(c,i)})],$$

$(E.vi)$ $\mathrm{lh}(b)$ is not a limit ordinal,

$(E.vii)$ $b \upharpoonright \mathrm{lh}(a) \Vdash_{\mathrm{lh}(a)} \check{s}, \check{S} \in \dot{X}$,

$(E.viii)$ $b \Vdash_{\mathrm{lh}(b)} \check{s} \upharpoonright \mathrm{lh}(b) \in \dot{G}_{\mathrm{lh}(b)}$,

$(E.ix)$ $b \Vdash_{\mathrm{lh}(b)} \dot{X}^{(a,n)} \sqsubseteq \dot{X}^{(a,n)}[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(b)}] \sqsubseteq \dot{X} \prec H_{\check{\theta}}^{V[\dot{G}_{\mathrm{lh}(\check{b})}]}$,

$(E.x)$ $b \Vdash_{\mathrm{lh}(b)}$ "$\dot{X}$ is countable and respects $\dot{I}$",

$(E.xi)$ $b \Vdash_{\mathrm{lh}(b)} \dot{I} = I^{\dot{\mathbb{Q}}_{\mathrm{lh}(b)}}_{\check{s}(\mathrm{lh}(b))}$ and

$(E.xii)$ if $S_0 = \{c_0\}$ then $\mathrm{lh}(b) = \mathrm{lh}(c_0)$ and $b \leqslant c_0$.

[46] There is some slight abuse of notation here in an effort to improve readability.

Claim 7.27. *$E \upharpoonright \mathrm{lh}(a) := \{b \upharpoonright \mathrm{lh}(a) \mid b \in E\}$ is dense in $\mathbb{P}_{\mathrm{lh}(a)}$.*

Proof. Let $a' \leqslant a$ and let G be $\mathbb{P}_{\mathrm{lh}(a)}$-generic with $a' \in G$. By $(F.v)$, $p^{(a,n)} \upharpoonright \mathrm{lh}(a) \in G$. Work in $V[G]$. Let $h(n) = (i, j)$ and $c = \mathrm{pred}^i_T(a)$. Let

$$X^{(c,i)} = \left(\dot{X}^{(c,i)}\right)^{G_{\mathrm{lh}(c)}} \text{ and } X^{(a,n)} = \left(\dot{X}^{(a,n)}\right)^{G}$$

as well as $Z^{(c,i)} = X^{(c,i)} \cap V$, $Z^{(a,n)} = X^{(a,n)} \cap V$. Find $r \in T^{(a,n)}_1$ with $r \upharpoonright \mathrm{lh}(a) \in G$. As $p^{(a,n)}$ is a mixture of $T^{(a,n)}$, we have

$$r \leqslant p^{(a,n)} \upharpoonright \mathrm{lh}(r).$$

Let $\hat{r} = r^\frown p^{(a,n)} \upharpoonright [\mathrm{lh}(r), \gamma)$. Note that $\hat{r} \in X^{(a,n)}$, as

$$p^{(a,n)}, T^{(a,n)}, G \in X^{(a,n)}$$

by $(F.vii)$. Moreover, $\hat{r} \upharpoonright \mathrm{lh}(a) \in G$. Let $\mathbb{Q} := \dot{\mathbb{Q}}^G_{\mathrm{lh}(a)}$ and

$$D := \mu^+_{c,a}((\dot{D}^i_j)^{G_{\mathrm{lh}(c)}}) \in M_{Z^{(a,n)}}[f(\delta)] \subseteq M_{X^{(a,n)}}[f(\delta)].$$

Subclaim 7.28. *There are s, Y with*

(i) $X^{(a,n)} \sqsubseteq Y \prec H^{V[G]}_\theta$,

(ii) $s \leqslant p^{(a,n)}$,

(iii) $s \upharpoonright \mathrm{lh}(a) \in G$,

(iv) $s \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D)]$ *and*

(v) Y *respects* $I^{\mathbb{Q}}_{s(\mathrm{lh}(a))}$.

Proof. Let

$$D_0 := \{t \in D \mid \pi_{X^{(a,n)}}(t) \leqslant p^{(a,n)} \wedge \pi_{X^{(a,n)}}(t) \upharpoonright \mathrm{lh}(a) \in G\}$$

and D_1 be the projection of D_0 onto $\pi^{-1}_{X^{(a,n)}}(\mathbb{Q})$. Observe that

$$M_{X^{(a,n)}}[f(\delta)] \models \text{“}D_1 \text{ is dense below } \pi^{-1}_{X^{(a,n)}}(p^{(a,n)}(\mathrm{lh}(a)) \text{ in } \pi^{-1}_{X^{(a,n)}}(\mathbb{Q})\text{”}.$$

Applying Lemma 7.26 with (making use of the notation there)

- $\mathbb{P} = \dot{\mathbb{Q}}$,
- $p = p^{(a,n)}(\mathrm{lh}(a))$,
- $X = X^{(a,n)}$ and
- $D = D_0$,

we find some countable Y and some s_0 with

(*i*) $X^{(a,n)} \sqsubseteq Y \prec H_\theta^{V[G]}$,

(*ii*) $s_0 \leqslant p^{(a,n)}(\mathrm{lh}(a))$,

(*iii*) $s_0 \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D_1)]$ and

(*iv*) Y respects $I^{\mathbb{Q}}_{s_0}$.

By definition of D_1, there is $s \leqslant p^{(a,n)}$ with

(*s.i*) $s \upharpoonright \mathrm{lh}(a) \in G$,

(*s.ii*) $s \in \pi_Y[\mu^+_{X^{(a,n)},Y}(D)]$ and

(*s.iii*) $s(\mathrm{lh}(a)) = s_0$.

Y, s have the desired properties.

□

We can now apply Fact 3.44 in Y and get a nested antichain $S \in X^{(a,n)}$ with

(*S.i*) s is a mixture of S,

(*S.ii*) if $S_0 = \{d\}$ then $\mathrm{lh}(r) \leqslant \mathrm{lh}(d)$, $d \upharpoonright \mathrm{lh}(r) \leqslant r$ and $\mathrm{lh}(d)$ is not a limit ordinal and

(*S.iii*) $S \angle\, T^{(a,n)}$.

Let $\dot{X}$ be a name for $Y[\dot{G}_{\mathrm{lh}(a),\mathrm{lh}(d)}]$ and $\dot{I}$ a name for $I^{\dot{\mathbb{Q}}_{\mathrm{lh}(d)}}_{s(\mathrm{lh}(d))}$.

Subclaim 7.29. *In $V[G]$, we have*

$$\dot{I}^{s\upharpoonright \mathrm{lh}(d)} = I^{\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)+1}}_{s\upharpoonright \mathrm{lh}(d)+1} = I^{\mathbb{Q}}_{s(\mathrm{lh}(a))}.$$

Proof. The first equality is simply by definition of $\dot{I}$. The second equality follows from Proposition 7.22. Here we use that $\mathbb{Q}$ freezes NS_{ω_1} along f and that $\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)+1}$ preserves f by our inductive hypothesis. □

It follows that

$$Y \text{ respects } \dot{I}^{s\upharpoonright \mathrm{lh}(d)}.$$

As $\mathrm{lh}(a)$ is not a limit ordinal, (‡) holds in $V[G]$, so that $\mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)}$ is respectful by Lemma 7.15. Thus there is $b \in \mathbb{P}_{\mathrm{lh}(a),\mathrm{lh}(d)}$, $b \leqslant s \upharpoonright \mathrm{lh}(d)$ so that

$$b \Vdash_{\mathrm{lh}(b)} \text{“}\check{Y} \sqsubseteq \check{Y}[\dot{G}] \text{ respects } \dot{I}\text{”}.$$

Since $b \upharpoonright \mathrm{lh}(a) \in G$, we may assume further that $b \upharpoonright \mathrm{lh}(a) \leqslant a'$. $s, S, \dot{X}, \dot{I}$ witness $b \in E$. □

To define T_{n+1}, fix a maximal antichain $A \subseteq E \upharpoonright \mathrm{lh}(a)$, and for any $e \in A$ choose $b_e \in E$ with $b_e \upharpoonright \mathrm{lh}(a) = e$. We set $\mathrm{suc}^n_T(a) = \{b_e \mid e \in A\}$. For any $b \in \mathrm{suc}^n_T(a)$, let $S, s, \dot{X}, \dot{I}$ witness $b \in E$. We then let

- $p^{(b,n+1)} = s,\ T^{(b,n+1)} = S,\ \dot{X}^{(b,n+1)} = \dot{X},\ \dot{I}^{(b,n+1)} = \dot{I}$,
- $\dot{Z}^{(b,n+1)}$ be a name for $\dot{X} \cap V$ and
- $\left(\dot{D}_j^{(b,n+1)}\right)_{j<\omega}$ be a sequence of names that are forced by b to enumerate all dense subsets of $\pi^{-1}_{\dot{Z}^{(b,n+1)}}(\mathbb{P})$ in $M_{\dot{Z}^{(b,n+1)}}\left[\widetilde{f(\delta)}\right]$.

This finishes the construction.

By Fact 3.44, there is a mixture q of T. Let G be $\mathbb{P}$-generic with $q \in T$. By Fact 3.46, in $V[G]$ there is a sequence $\langle a_n \mid n < \omega\rangle$ so that for all $n < \omega$

($\vec{a}.i$) $a_0 = q_0$,

($\vec{a}.ii$) $a_{n+1} \in \mathrm{suc}^n_T(a_n)$ and

($\vec{a}.iii$) $p^{(a_n,n)} \in G$.

For $n < \omega$, let $\alpha_n = \mathrm{lh}(a_n) < \gamma$. For $n < \omega$ we let

$$X_n := \left(\dot{X}^{(a_n,n)}\right)^{G_{\alpha_n}}$$

and also

$$X_\omega = \bigcup_{n<\omega} X_n[G_{\alpha_n,\gamma}].$$

Further, for $n \leqslant \omega$ let

$$Z_n := X_n \cap V \text{ and } \pi_n := \pi_{Z_n}.$$

We remark that

$$X_n[G_{\alpha_n,\gamma}] \sqsubseteq X_m[G_{\alpha_m,\gamma}] \prec H_\theta^{V[G]}$$

follows inductively from ($F.vii$) and ($F.ix$) for $n \leqslant m < \omega$ so that $X_\omega \prec H_\theta^{V[G]}$. We aim to prove that

$$X \sqsubseteq X_\omega \text{ is } f\text{-slim.}$$

In fact, we will show

($Z_\omega.i$) $X \sqsubseteq Z_\omega$,

($Z_\omega.ii$) Z_ω is f-slim and

($Z_\omega.iii$) $\pi_\omega^{-1}[G]$ is generic over $M_\omega[f(\delta)]$, which implies the above.

Claim 7.30. $Z_\omega = \bigcup_{n<\omega} Z_n$.

Proof. " $\supseteq$ " is trivial, so we show " $\subseteq$ ". Let $x \in Z_\omega$ and find $i < \omega$ with $x \in X_i[G_{\alpha_i,\gamma}]$. Note that there is $\dot{x} \in Z_i$ a $\mathbb{P}$-name for a set in V with $x = \dot{x}^G$. Let $D \in M_i$ be the dense set of conditions in $\pi_n^{-1}(\mathbb{P})$ deciding $\pi_i^{-1}(\dot{x})$. There must be some $j < \omega$ so that

$$\left(\dot{D}_j^{(a_i,i)}\right)^G = D.$$

Now find n with $h(n) = (i,j)$. We then have

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{a_i,a_{n+1}}(D)]$$

by $(F.xii)$. We have that $p^{(a_{n+1},n+1)}$ decides $\dot{x}$ to be some $z \in X_n$, and as $p^{(a_{n+1},n+1)} \in G$,

$$x = \dot{x}^G = z \in X_n \cap V = Z_n.$$

□

As $X \sqsubseteq X_n$ is f-slim by $(F.ix)$ for $n < \omega$, $(Z_\omega.i)$ and $(Z_\omega.ii)$ follow at once. It remains to show $(Z_\omega.iii)$.
As Z_ω is f-slim and by Claim 7.30, we have that

$$\langle M_\omega[f(\delta)], \mu^+_{n,\omega} \mid n < \omega\rangle = \varinjlim \langle M_n[f(\delta)], \mu^+_{n,m} \mid n \leqslant m < \omega\rangle$$

for some $(\mu^+_{n,\omega})_{n<\omega}$. Let $E \in M_\omega[f(\delta)]$ be dense in $\pi_\omega^{-1}(\mathbb{P})$. Then for some $i,j < \omega$, $E = \mu^+_{i,\omega}(D)$ for

$$D := \left(\dot{D}_j^{(a_i,i)}\right)^G.$$

Find n with $h(n) = (i,j)$. By $(F.xii)$,

$$p^{(a_{n+1},n+1)} \in \pi_n[\mu^+_{i,n}(D)] \subseteq \pi_\omega[\mu^+_{i,\omega}(D)] = \pi_\omega[E].$$

As $p^{(a_{n+1},n+1)} \in G$, we have $E \cap \pi_\omega^{-1}[G] \neq \varnothing$, which is what we had to show. □

7.3 The $\mathbb{Q}_{\max}$-variation $\mathbb{Q}^-_{\max}$

We will have to do some work in order to find a forcing which freezes NS_{ω_1} along a witness f of $\diamondsuit(\omega_1^{<\omega})$. The main idea is to find the correct $\mathbb{P}_{\max}$-variation to throw into the $\diamondsuit$-$(*)$-forcing. Let us first introduce Woodin's $\mathbb{Q}_{\max}$.

Definition 7.31. A condition $p \in \mathbb{Q}_{\max}$ is a generically iterable structure $p = (N, I, f)$ with

($\mathbb{Q}_{\max}.i$) $N \models$ “f guesses $\mathrm{Col}(\omega, \omega_1)$-filters” and

($\mathbb{Q}_{\max}.ii$) $N \models$ “$\eta_f : \mathrm{Col}(\omega, \omega_1) \to (\mathcal{P}(\omega_1)/I)^+$ is a dense embedding”, where η_f is the embedding associated to f.

The order on $\mathbb{Q}_{\max}$ is given by

$$q = (M, J, h) <_{\mathbb{Q}_{\max}} p$$

iff there is an iteration

$$j : p \to p^* = (N^*, I^*, f^*)$$

in q with $f^* = h$.

We mention that it follows from Lemma 2.12 that if (N, I, f) is a $\mathbb{Q}_{\max}$-condition then $N \models$ “f witnesses $\diamondsuit^+(\omega_1^{<\omega})$”.
Forcing that $\mathcal{H}_f$ is almost a $\mathbb{Q}_{\max}$-condition for some f essentially amounts to forcing “NS_{ω_1} is ω_1-dense”. We replace $\mathbb{Q}_{\max}$ by an equivalent forcing for which this is easier to achieve.

Definition 7.32. A condition $p \in \mathbb{Q}^-_{\max}$ is a generically iterable structure of the form $p = (N, I, f)$ so that

$$(N; \in, I) \models \text{“}f \text{ witnesses } \diamondsuit^+_I(\omega_1^{<\omega})\text{”}.$$

The order on $\mathbb{Q}^-_{\max}$ is given by $q := (M, J, h) <_{\mathbb{Q}^-_{\max}} (N, I, f) =: p$ iff there is an iteration

$$j : p \to p^* = (N^*, I^*, f^*)$$

in q so that

($<_{\mathbb{Q}^-_{\max}}.i$) $f^* = h$ and

($<_{\mathbb{Q}^-_{\max}}.ii$) if $S \in J^+ \cap p^*$ then there is $b \in \mathrm{Col}(\omega, \omega_1^q)$ with $S^h_b \subseteq S \mod J$.

We note that $\mathbb{Q}^-_{\max}$ is essentially unchanged if condition ($<_{\mathbb{Q}^-_{\max}}.ii$) is dropped, but demanding it is convenient for us.

Proposition 7.33 (Woodin, [Woo10, Definition 6.20])**.** *Suppose $\mathcal{P}(\omega_1)$ is closed under $A \mapsto A^\sharp$ and I is a normal uniform ideal. Suppose f guesses $\mathrm{Col}(\omega, \omega_1)$-filters. The following are equivalent:*

(*i*) *f witnesses $\diamondsuit^+_I(\omega_1^{<\omega})$.*

(*ii*) *For any $A \subseteq \omega_1$,*

$$\{\alpha < \omega_1 \mid f(\alpha) \text{ is not generic over } L[A \cap \alpha]\} \in I$$

and for all $b \in \mathbb{B}$, $S^f_b \in I^+$.

The argument is due to Woodin. We provide it for convenience of the reader.

Proof. We only have to show $(i)\Rightarrow(ii)$. Let $\theta \geqslant \omega_2$ be regular, $S \in I^+$ and $X \prec H_\theta$ countable with

(i) $A, I, f \in X$ and

(ii) $\delta^X \in S$.

If follows that $f(\delta^X)$ is generic over M_X. By elementarity $A^\sharp \in X$ and moreover,

$$\pi_X^{-1}(A^\sharp) = (A \cap \delta^X)^\sharp.$$

Thus $((\delta^X)^+)^{L[A\cap\delta^X]} \in M_X$ so that

$$\mathcal{P}(\delta^X)^{L[A\cap\delta^X]} \subseteq M_X$$

Thus $f(\delta^X)$ is generic over $L[A \cap \delta^X]$ as well. This shows

$$\{\alpha < \omega_1 \mid f(\alpha) \text{ is generic over } L[A \cap \alpha]\} \cap S \neq \varnothing.$$

As S was an arbitrary I-positive set, we conclude

$$\{\alpha < \omega_1 \mid f(\alpha) \text{ is not generic over } L[A \cap \alpha]\} \in I.$$

□

Lemma 7.34. *Suppose J is a normal uniform ideal, h witnesses $\diamondsuit_J^+(\omega_1^{<\omega})$, and $\mathcal{P}(\omega_1)$ is closed under $A \mapsto A^\sharp$. For any $p = (N, I, f) \in \mathbb{Q}^-_{\max}$ there is an iteration*

$$j\colon p \to p^* = (N^*, I^*, f^*)$$

so that

(i) *$f^* = h \mod J$ (so in particular f^* witnesses $\diamondsuit_J^+(\omega_1^{<\omega})$) and*

(ii) *if $S \in J^+ \cap N^*$ then there is $b \in \mathrm{Col}(\omega, \omega_1)$ with $S_b^{f^*} \subseteq S \mod J$.*

Proof. Let x be a real coding p and let D be the club of x-indiscernibles below ω_1. By induction along ω_1 we will define a filter $g \subseteq \mathrm{Col}(\omega, {<}\omega_1)$. Let

$$\vec{\alpha} := \langle \alpha_i \mid i < \omega_1 \rangle$$

be the increasing enumeration of D. Assume that $g \upharpoonright \alpha_i$ is already defined. First we define $g(\alpha_i)$:
<u>Case 1:</u> $h(\alpha_i)$ is generic over $L[x, g \upharpoonright \alpha_i]$. Then let $g(\alpha_i) = h(\alpha_i)$.
<u>Case 2:</u> Case 1 fails. Then let $g(\alpha_i)$ be some generic for $\mathrm{Col}(\omega, \alpha_i)$ over $L[x, g \upharpoonright \alpha_i]$.
Next, we choose $g \upharpoonright (\alpha_i, \alpha_{i+1})$ to be any generic for $\mathrm{Col}(\omega, (\alpha_i, \alpha_{i+1}))$ over $L[x, g \upharpoonright \alpha_i + 1]$.

Claim 7.35. *g is generic over $L[x]$.*

Proof. $\vec{\alpha}$ enumerates a club of $L[x]$-regular ordinals. Thus for any $i < \omega_1$, $\mathrm{Col}(\omega, {<}\alpha_i)$ has the α_i-c.c. in $L[x]$. It follows by induction that $g \upharpoonright \alpha_i$ is $\mathrm{Col}(\omega, {<}\alpha_i)$-generic over $L[x]$ and finally that g is $\mathrm{Col}(\omega, {<}\omega_1)$-generic over $L[x]$. □

By induction on $\alpha < \omega_1$, we now define a generic iteration

$$\langle p_i, \sigma_{i,j}, U_i \mid i \leqslant j \leqslant \alpha \rangle$$

of $p_0 = p$. Here, U_i denotes the generic filter that produces the ultrapower $\sigma_{i,i+1}$.
Let η_α denote the map

$$(\eta_{\sigma_{0,\alpha}(f)})^{p_\alpha} \colon \mathrm{Col}(\omega, \omega_1^{p_\alpha}) \to ((\mathcal{P}(\omega_1)/\sigma_{0,\alpha}(I))^+)^{p_\alpha}.$$

Simply pick U_α least, according to the canonical global wellorder in

$$L[x, g \upharpoonright \omega_1^{p_\alpha} + 1]$$

so that

(*U.i*) U_α is $((\mathcal{P}(\omega_1)/\sigma_{0,\alpha}(I))^+))^{p_\alpha}$-generic over p_α and

(*U.ii*) $\eta_\alpha[g(\omega_1^{p_\alpha})] \subseteq U_\alpha$.

This is possible as $g(\omega_1^{p_\alpha})$ is $\mathrm{Col}(\omega, \omega_1^{p_\alpha})$-generic over p_α, as

$$p_\alpha \models \text{"}\eta^{p_\alpha} \text{ is a regular embedding"}$$

and as p_α is countable in $L[x, g \upharpoonright \omega_1^{p_\alpha} + 1]$. U_α induces the generic ultrapower $\sigma_{\alpha,\alpha+1} \colon p_\alpha \to \mathrm{Ult}(p_\alpha, U_\alpha) =: p_{\alpha+1}$.

Finally we get a generic iteration map

$$\sigma := \sigma_{0,\omega_1} \colon p \to p^* := p_{\omega_1} = (N^*, I^*, f^*).$$

Claim 7.36. *$f^* = h \mod J$.*

Proof. f^* and g agree on the club of iteration points, i.e. we have $f^*(\omega_1^{p_\alpha}) = g(\omega_1^{p_\alpha})$ for any $\alpha < \omega_1$ by the argument of Proposition 2.11. Here we use that U_α extends $\pi^{p_\alpha}[g(\alpha)]$.
Moreover,

$$\{\alpha < \omega_1 \mid h(\alpha) \text{ is not generic over } L[x, g \upharpoonright \alpha]\} \in J$$

by Proposition 7.33 as h witnesses $\diamondsuit_J^+(\omega_1^{<\omega})$. By construction of g, it follows that $\{\alpha < \omega_1 \mid h(\alpha) \neq g(\alpha)\} \in J$. As J is a normal uniform ideal, we can conclude

$$\{\alpha < \omega_1 \mid f^*(\alpha) \neq h(\alpha)\} \in J.$$

□

It follows that f^* witnesses $\diamondsuit_J^+(\omega_1^{<\omega})$. Now let $S \in J^+ \cap N^*$. We have to show the following.

Claim 7.37. *$S_b^{f^*} \subseteq S \mod J$ for some $b \in \mathrm{Col}(\omega, \omega_1)$.*

Proof. We will prove that the intersection of D with $S_b^{f^*} - S$ is bounded below ω_1 for some b. Find $\alpha \in D$ so that

($\alpha.i$) there is $\bar{S} \in p_\alpha$ with $\sigma_{\alpha,\omega_1}(\bar{S}) = S$ and

($\alpha.ii$) $\alpha \in S$.

By ($\alpha.ii$), there must be some $b \in g(\alpha)$ with

$$b \Vdash^{L[x,g\restriction\alpha]}_{\mathrm{Col}(\omega,\alpha)} \bar{S} \in \dot{U}_\alpha$$

where $\dot{U}_\alpha$ is a name for the least filter U that is generic over p_α and contains $\eta_\alpha[\dot{g}]$, where $\dot{g}$ is now the canonical name for the generic. Now suppose $\alpha < \beta \in S_b^{f^*} \cap D$. There is then an elementary embedding

$$j \colon L[x] \to L[x]$$

with

($j.i$) $j(\alpha) = \beta$ and

($j.ii$) $\mathrm{crit}(j) = \alpha$.

We have that j lifts to an elementary embedding

$$j^+ \colon L[x, g \restriction \alpha] \to L[x, g \restriction \beta]$$

so that

$$b = j(b) \Vdash^{L[x,g\restriction\beta]}_{\mathrm{Col}(\omega,\beta)} j^+\left(\bar{S}\right) \in j^+\left(\dot{U}_\alpha\right).$$

Clearly, $j^+\left(\dot{U}_\alpha\right)^{g(\beta)} = U_\beta$ and thus

$$\beta \in \sigma_{\beta,\omega_1}\left(j^+\left(\bar{S}\right)\right)$$

as $b \in f^*(\beta) = g(\beta)$. Note that all points in D are iteration points and recall that f^* and g agree on iteration points.

Subclaim 7.38. $j^+\left(\bar{S}\right) = \sigma_{\alpha,\beta}\left(\bar{S}\right)$.

Proof. The reason is that, since α is a limit ordinal, p_α is the direct limit along $\langle p_i, \sigma_{i,k} \mid i \leqslant k < \alpha\rangle$ and thus there is some $\gamma < \alpha$ and $\bar{\bar{S}} \in p_\gamma$ with $\sigma_{\gamma\alpha}\left(\bar{\bar{S}}\right) = \bar{S}$. Hence

$$\begin{aligned} j^+\left(\bar{S}\right) &= j^+\left(\sigma_{\gamma,\alpha}\left(\bar{\bar{S}}\right)\right) = j^+(\sigma_{\gamma,\alpha})\left(j^+\left(\bar{\bar{S}}\right)\right) \\ &= \sigma_{\gamma,\beta}\left(\bar{\bar{S}}\right) = \sigma_{\alpha,\beta}\left(\sigma_{\gamma,\alpha}\left(\bar{\bar{S}}\right)\right) = \sigma_{\alpha,\beta}\left(\bar{S}\right). \end{aligned}$$

Here, we use $j^+(\sigma_{\gamma,\alpha}) = \sigma_{\gamma,\beta}$ in the third equation. This holds as our lift j^+ satisfies $j^+(g \restriction \alpha) = g \restriction \beta$ and so it is easy to see that $j^+(\langle U_i \mid i < \alpha\rangle) = \langle U_i \mid i < \beta\rangle$ so that

$$j^+(\langle p_i, \sigma_{i,k} \mid i \leqslant k < \alpha\rangle) = \langle p_i, \sigma_{i,k} \mid i \leqslant k < \beta\rangle.$$

□

All in all, $\beta \in \sigma_{\beta,\omega_1}\left(\sigma_{\alpha,\beta}\left(\bar{S}\right)\right) = S$. Thus

$$\left(S_b^{f*} - S\right) \cap D \subseteq \alpha$$

so that $S_b^{f*} \subseteq S \mod J$.

□

□

Proposition 7.39 (Folklore?)**.** *Suppose there is a precipitous ideal on ω_1. Then $\mathcal{P}(\omega_1)$ is closed under $A \mapsto A^\sharp$.*

Proof. It is easy to see that $\mathbb{R}$ is closed under $x \mapsto x^\sharp$. Let I be a precipitous ideal and let $j\colon V \to M = \mathrm{Ult}(V, g)$ be the generic ultrapower of V in the extension $V[g]$, g generic for I^+. Then $A = j(A) \cap \omega_1^V \in M$ and is coded by a real in M. By elementarity, $\mathbb{R} \cap M$ is closed under $x \mapsto x^\sharp$. Thus $A^\sharp$ exists in $M \subseteq V[g]$. As forcing cannot add a sharp, $A^\sharp \in V$. □

Lemma 7.40. *Assume* AD *in* $L(\mathbb{R})$*. The inclusion* $\mathbb{Q}_{\max} \hookrightarrow \mathbb{Q}^-_{\max}$ *is a dense embedding.*

Proof. It is easy to see that if $p, q \in \mathbb{Q}_{\max}$ then

$$q <_{\mathbb{Q}_{\max}} p \Leftrightarrow q <_{\mathbb{Q}^-_{\max}} p.$$

Now let $p \in \mathbb{Q}^-_{\max}$ and find x a real coding p. Our assumptions imply by Woodin's analysis of $\mathbb{Q}_{\max}$ under $\mathrm{AD}^{L(\mathbb{R})}$ that there is $q = (M, J, h) \in \mathbb{Q}_{\max}$ with $x^\sharp \in M$. By Proposition 7.39,

$$M \models \text{“}\mathcal{P}(\omega_1) \text{ is closed under } A \mapsto A^\sharp\text{”}.$$

Thus we may apply Lemma 7.33 inside M and find an iteration

$$j\colon p \to p^* = (N^*, I^*, f^*)$$

so that

$$q' := (M, J, f^*) \in \mathbb{Q}_{\max}$$

and j witnesses $q' <_{\mathbb{Q}^-_{\max}} p$. $\square$

It is not obvious how to even prove construct a single $\mathbb{Q}_{\max}$-condition assuming only $\mathrm{AD}^{L(\mathbb{R})}$. Woodin worked with a variant $\mathbb{Q}^*_{\max}$ of $\mathbb{Q}_{\max}$ instead to analyze the $\mathbb{Q}_{\max}$-extension of $L(\mathbb{R})$. We remark that this can be done with $\mathbb{Q}^-_{\max}$ as well. The arguments are, modulo Lemma 7.34, quite similar to the arguments in the $\mathbb{Q}^*_{\max}$ analysis.

7.4 Consistency of QM and forcing "NS_{ω_1} is ω_1-dense"

We are now in position to force QM and answer Woodin's Question. We repeat it here.

Question (Woodin)**.** Assume the existence of some large cardinal. Is there a semiproper partial order $\mathbb{P}$ with

$$V^{\mathbb{P}} \models \text{"}\mathrm{NS}_{\omega_1} \text{ is } \omega_1\text{-dense" ?}$$

The first step this to find a forcing which freezes NS_{ω_1} along f assuming large cardinals and that f witnesses $\diamondsuit(\omega_1^{<\omega})$.

We will finally reap what we have sown by replacing $\mathbb{Q}_{\max}$ with $\mathbb{Q}^-_{\max}$.

Lemma 7.41. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$, there is a Woodin cardinal and V is closed under $X \mapsto M_1^\sharp(X)$. Then there is a f-preserving forcing which freezes NS_{ω_1} along f.*

Proof. Use the Woodin cardinal to make NS_{ω_1} saturated while turning f into a witness of $\diamondsuit^+(\omega_1^{<\omega})$ by f-semiproper forcing in a generic extension $V[g]$ using Theorem 3.60. Observe that

$$(H_{\omega_2}, \mathrm{NS}_{\omega_1}, f)^{V[g]}$$

is a almost a $\mathbb{Q}^-_{\max}$-condition in $V[g]$. Work in $V[g]$. Next we want to apply Theorem 4.20 with $\mathbb{V}_{\max} = \mathbb{Q}^-_{\max}$ for the dense set $D = \mathbb{Q}^-_{\max}$. Note that the universe is closed under $X \mapsto X^\sharp$ and as D is Π^1_2, D is ∞-universally Baire. We cannot guarantee full generic absoluteness for small forcings, however we actually only need that for any forcing $\mathbb{P}$ of size $\leqslant 2^{\omega_2}$ we have that

(*i*) $(\mathbb{Q}^-_{\max})^{V[g]^{\mathbb{P}}} \cap V[g] = (\mathbb{Q}^-_{\max})^{V[g]}$ and

(*ii*) $(\mathbb{Q}^-_{\max})^{V[g]^{\mathbb{P}}}$ is a $\mathbb{P}_{\max}$-variation in $V[g]^{\mathbb{P}}$

(i) is again guaranteed by the closure under $X \mapsto X^\sharp$. The only nontrivial thing one has to verify for (ii) is that $\mathbb{Q}^-_{\max}$ has no minimal conditions in $V[g]^{\mathbb{P}}$. This follows from the closure of $\mathbb{R}$ under $x \mapsto M_1^\sharp(x)$, the argument is similar to Corollary 8.8.
Thus $\mathbb{P}^\diamondsuit = \mathbb{P}^\diamondsuit(\mathbb{Q}^-_{\max}, f, \mathbb{Q}^-_{\max})$ exists and in a further extension $V[g][h]$ by $\mathbb{P}^\diamondsuit$ we have:

$$\begin{tikzcd}
 & \mathbb{Q}^-_{\max} & \\
 & q_0 \arrow[u, phantom, sloped, "\in"] \arrow[r, "\sigma_{0,\omega_1}"] & q_{\omega_1} = (N_{\omega_1}, I_{\omega_1}, f) \\
p_0 \arrow[d, phantom, sloped, "\in"] \arrow[r, "\mu_{0,\omega_1^{q_0}}"] & p_{\omega_1^{q_0}} \arrow[u, phantom, sloped, "\in"] \arrow[r, "\mu_{\omega_1^{q_0},\omega_1}"] & p_{\omega_1} \arrow[u, phantom, sloped, "\in"] \arrow[d, equal] \\
\mathbb{Q}^-_{\max} & & ((H_{\omega_2})^{V[g]}, \mathrm{NS}_{\omega_1}^{V[g]}, f)
\end{tikzcd}$$

So that

($\mathbb{P}^\diamondsuit.i$) $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ are generic iterations of p_0, q_0 respectively,

($\mathbb{P}^\diamondsuit.ii$) $\mu_{0,\omega_1^{q_0}}$ witnesses $q_0 <_{\mathbb{Q}^-_{\max}} p_0$,

($\mathbb{P}^\diamondsuit.iii$) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^{q_0}})$ and

($\mathbb{P}^\diamondsuit.iv$) the generic iteration $\sigma_{0,\omega_1} \colon q_0 \to q_{\omega_1}$ is a $\diamondsuit$-iteration.

Claim 7.42. *f witnesses $\diamondsuit(\omega_1^{<\omega})$ in $V[g][h]$.*

Proof. By Lemma 4.19 and ($\mathbb{P}^\diamondsuit.iv$), $I_{\omega_1} = \mathrm{NS}_f^{V[g][h]} \cap N_{\omega_1}$, in particular f witnesses $\diamondsuit(\omega_1^{<\omega})$ in $V[g][h]$. □

It remains to show that the extension $V \subseteq V[g][h]$ has “frozen $\mathrm{NS}_{\omega_1}^V$ along f”. Let $S \in \mathcal{P}(\omega_1)^V$. It follows from ($\mathbb{P}^\diamondsuit.ii$), ($\mathbb{P}^\diamondsuit.iii$) and the definition of $<_{\mathbb{Q}^-_{\max}}$ (especially ($<_{\mathbb{Q}^-_{\max}}.ii$)) that one of the following holds:

- Either $S \in I_{\omega_1}$,
- or for some $p \in \mathrm{Col}(\omega, \omega_1)$ we have $S^f_p \subseteq S \mod I_{\omega_1}$.

As any $\diamondsuit$-iteration is correct, $I_{\omega_1} = \mathrm{NS}_{\omega_1}^{V[g][h]} \cap N_{\omega_1}$. It follows that

- either $S \in \mathrm{NS}_{\omega_1}^{V[g][h]}$,
- or for some $p \in \mathrm{Col}(\omega, \omega_1)$ we have $S^f_p \subseteq S \mod \mathrm{NS}_{\omega_1}^{V[g][h]}$,

which is what we had to show. □

Remark 7.43. Instead of closure of V under $X \mapsto M_1^\sharp$ we could just as well have assumed that there is a second Woodin cardinal with a measurable above.

Theorem 7.44. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$ and there is a supercompact limit of supercompact cardinals. Then there is a f-preserving forcing extension in which f witnesses* QM.

Proof. Let κ be a supercompact limit of supercompact cardinals and

$$L\colon V_\kappa \to V_\kappa$$

an associated Laver function. We describe a Q-iteration w.r.t. f

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$$

that forces QM. For any $\alpha < \kappa$, $\dot{\mathbb{Q}}_\alpha$ is a two step-iteration of the form

$$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_\alpha^0 * \ddot{\mathbb{Q}}_\alpha^1$$

with $|\dot{\mathbb{Q}}_\alpha| < \kappa$. If α is a successor (or 0) then

(*i*) $\dot{\mathbb{Q}}_\alpha^0$ is forced to be a f-preserving forcing that freezes NS_{ω_1} along f and

(*ii*) $\ddot{\mathbb{Q}}_\alpha^1$ is a name for a f-preserving partial order forcing SRP.

Note that $\dot{\mathbb{Q}}_\alpha^0$ exists by Lemma 7.41 and $\ddot{\mathbb{Q}}_\alpha^1$ exists by Corollary 3.76.

If α is a limit ordinal, then

(*i*) $\dot{\mathbb{Q}}_\alpha^0$ is $L(\alpha)$ if that is a $\mathbb{P}_\alpha$-name for a f-preserving forcing and the trivial forcing else,

(*ii*) $\ddot{\mathbb{Q}}_\alpha^1$ is as in the successor case.

It is clear that this constitutes a Q-iteration and hence $\mathbb{P}$ preserves f and in particular ω_1 is not collapsed. $\mathbb{P}$ is κ-c.c. by Fact 3.47. As we use f-preserving forcings guessed by L at limit steps, QM holds in the extension as witnessed by f by the usual argument. □

If one is only interested in forcing "NS_{ω_1} is ω_1-dense", a slightly weaker large cardinal assumption is sufficient.

Theorem 7.45. *Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$ and κ is an inaccessible limit of $<\kappa$-supercompact cardinals. Then there is a f-preserving forcing extension in which NS_{ω_1} is ω_1-dense.*

Proof. Indeed any nice iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \kappa, \beta < \kappa \rangle$$

so that for all $\gamma < \kappa$

$$V_\kappa \models \text{“}\mathbb{P}_\gamma \text{ is a } Q\text{-iteration w.r.t. } f\text{”}$$

preserves f and forces "NS_{ω_1} is ω_1-dense". To see this, first of all note that $\mathbb{P}$ is κ-c.c. by Fact 3.47. Now any $\mathbb{P}_\gamma$ for $\gamma < \kappa$ preserves f by Theorem 7.24 applied in V_κ and it follows immediately that $\mathbb{P}$ preserves f. Suppose now that G is $\mathbb{P}$-generic and

$$V[G] \models S \in \mathrm{NS}^+_{\omega_1}.$$

There must be some nonlimit $\gamma < \kappa$ with $S \in V[G_\gamma]$. As $\dot{\mathbb{Q}}_\gamma^{G_\gamma}$ freezes NS_{ω_1} along f in $V[G_\gamma]$, there must be some $b \in \mathrm{Col}(\omega, \omega_1)$ with $S^f_b \subseteq S$ mod NS_{ω_1} in $V[G_{\gamma+1}]$, hence in $V[G]$. □

Neither of these results answers Woodin's question, as Woodin asks specifically for a semiproper forcing, but Q-iterations are not stationary set preserving if NS_{ω_1} is not ω_1-dense to begin with. However, we have one more trick up our sleeves: For once we will pick f more carefully.

Lemma 7.46. *Suppose $\vec{S} = \langle S_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of pairwise disjoint stationary sets in ω_1 and $\diamondsuit(S_\alpha)$ holds for all $\alpha < \omega_1$. Then there is f witnessing $\diamondsuit(\omega_1^{<\omega})$ so that for all $\alpha < \omega_1$, there is $p \in \mathrm{Col}(\omega, \omega_1)$ with $S^f_p \subseteq S_\alpha$.*

Proof. From $\diamondsuit(S_\alpha)$, we get a witness f_α of $\diamondsuit(\omega_1^{<\omega})$ so that $f_\alpha(\beta)$ is the trivial filter if $\beta \notin S$, see (the proof of) Proposition 2.15. Let $\langle b_\alpha \mid \alpha < \omega_1\rangle$ be an enumeration of some maximal antichain in $\mathrm{Col}(\omega, \omega_1)$ of size $\aleph_1$. Now define $f\colon \omega_1 \to H_{\omega_1}$ as follows: For $\beta \in S_\alpha$ we let

$$f(\beta) = \{p \in \mathrm{Col}(\omega, \beta) \mid \exists p' \leqslant p \exists q \in f_\alpha(\beta)\ p' \leqslant b_\alpha{}^\frown q\}.$$

Note that there is at most one α with $\beta \in S_\alpha$. If β is not in any S_α, let $f(\beta)$ be the trivial filter. It is now clear that $S^f_{b_\alpha} \subseteq S_\alpha$, but we still need to verify that f indeed witnesses $\diamondsuit(\omega_1^{<\omega})$. So let $p \in \mathrm{Col}(\omega, \omega_1)$ and

$$\vec{D} = \langle D_\alpha \mid \alpha < \omega_1\rangle$$

be a sequence of dense subsets of $\mathrm{Col}(\omega, \omega_1)$. We have that show that

$$\{\beta < \omega_1 \mid p \in f(\beta) \wedge \forall \gamma < \beta\ f(\beta) \cap D_\gamma \neq \varnothing\}$$

is stationary. So let C be a club in ω_1. Find α so that b_α is compatible with p and note that we may assume further that $p \leqslant b_\alpha$. Hence we can write p as $p = b_\alpha{}^\frown q$. For $\gamma < \omega_1$, let

$$D'_\gamma = \{r \in \mathrm{Col}(\omega, \omega_1) \mid b_\alpha{}^\frown r \in D_\gamma\}$$

and note that D'_γ is dense. As f_α witnesses $\diamondsuit(\omega_1^{<\omega})$, we may find $\beta \in C$ large enough so that

$(\beta.i)$ $p \in \mathrm{Col}(\omega, \beta)$,

($\beta.ii$) $q \in f_\alpha(\beta)$ and

($\beta.iii$) $\forall\gamma < \beta\ f_\alpha(\beta) \cap D'_\gamma \neq \varnothing$.

It follows that $p \in f(\beta)$ and that

$$\forall\gamma < \beta\ f(\beta) \cap D_\gamma \neq \varnothing.$$

□

Corollary 7.47. *Assume there is a supercompact limit of supercompact cardinals. Then there is a semiproper forcing* $\mathbb{P}$ *with* $V^{\mathbb{P}} \models \mathrm{QM}$.

Proof. By otherwise taking advantage of the least supercompact, we may assume all stationary-set preserving forcings are semiproper. Next, we force with

$$\mathbb{P}_0 = \mathrm{Col}(\omega_1, 2^{\omega_1}).$$

Let G be $\mathbb{P}_0$-generic over V. There is then a partition $\langle T_\alpha \mid \alpha < \omega_1\rangle$ of ω_1 into stationary sets so that whenever $S \in V$ is stationary in ω_1, then $T_\alpha \cap S$ is stationary for all $\alpha < \omega_1$. Also, there is an enumeration

$$\langle S_\alpha \mid \alpha < \omega_1\rangle$$

of all stationary sets in V. Now in $V[G]$,

$$\langle S_\alpha \cap T_\alpha \mid \alpha < \omega_1\rangle$$

is a sequence of pairwise disjoint stationary sets. Moreover, $\diamondsuit_T$ holds for any stationary $T \subseteq \omega_1$. By Lemma 7.46, there is a witness f of $\diamondsuit(\omega_1^{<\omega})$ so that for any $\alpha < \omega_1$ there is $p \in \mathrm{Col}(\omega, \omega_1)$ with $S^f_p \subseteq (S_\alpha \cap T_\alpha)$. Thus for any stationary $S \in V$, S contains some S^f_p. Note that any further f-preserving forcing preserves the stationarity of any S^f_p and hence does not kill any stationary $S \in V$. By Theorem 7.44, there is an f-preserving $\dot{\mathbb{P}}_1$ that forces QM. It follows that back in V, the two-step forcing $\mathbb{P}_0 * \dot{\mathbb{P}}_1$ preserves stationary sets, hence is semiproper, and forces QM. □

Similarly, can prove the following from Theorem 7.45.

Corollary 7.48. *Assume there is an inaccessible* κ *that is a limit of* $<\kappa$-*supercompact cardinals. Then there is a stationary set preserving forcing* $\mathbb{P}$ *with*

$$V^{\mathbb{P}} \models \text{“}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

Assuming one more (sufficiently past κ-) supercompact cardinal below κ, one can replace stationary set preserving forcing by semiproper forcing.

So the answer to Woodin's question is yes assuming sufficiently large cardinals.

7.5 QM implies $\mathbb{Q}_{\max}$-(∗)

We apply the Blueprint Theorems to show that the relation between QM and $\mathbb{Q}_{\max}$-(∗) is analogous to the one of MM^{++} and (∗).
Typicality of $\mathbb{Q}_{\max}$ is witnessed by $\Psi^{\mathbb{Q}_{\max}}$ consisting of the formulae

- $\psi_0^{\mathbb{Q}_{\max}}(x) =$ "$x \in \dot{I}$",
- $\psi_1^{\mathbb{Q}_{\max}}(x) =$ "$x = \dot{f}$" and
- $\psi_2^{\mathbb{Q}_{\max}}(x) =$ "$x = \dot{f} \wedge x$ witnesses $\diamondsuit(\omega_1^{<\omega})$".

Note that $\psi_2^{\mathbb{Q}_{\max}}(x)$ is (in context equivalent to) a Π_1-formula.

Theorem 7.49. QM *implies* $\mathbb{Q}_{\max}$-(∗).

Proof. Suppose f witnesses QM. Then $\mathrm{MM}(f)$ holds and this entails SRP by Lemma 3.69. $\mathcal{H}_f$ is almost a $\mathbb{Q}_{\max}$-condition by Lemma 7.2. $\mathbb{Q}_{\max}$ accepts $\diamondsuit$-iterations by Lemma 4.19. $\mathbb{Q}_{\max}$-(∗) now follows from the First Blueprint Theorem 4.44. □

Definition 7.50. For $\Delta \subseteq \mathcal{P}(\mathbb{R})$, Δ-BQM states that there is f witnessing $\diamondsuit(\omega_1^{<\omega})$ so that
$$\Delta\text{-BFA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves } f\})$$
holds.

We mention that already BQM $= \varnothing$-BQM is enough to prove "NS_{ω_1} is ω_1-dense". In fact on can prove the following in a similar fashion as Lemma 7.5.

Lemma 7.51. *The following are equivalent for any* $\Delta \subseteq \mathcal{P}(\mathbb{R})$*:*

(*i*) Δ-BQM.

(*ii*) NS_{ω_1} *is* ω_1*-dense and there is* f *a witness of* $\diamondsuit(\omega_1^{<\omega})$ *with* Δ-$\mathrm{BMM}(f)$.

In contrast, BMM does not imply "NS_{ω_1} is saturated" (while MM does of course). See Lemma 10.103 in [Woo10][47].

Theorem 7.52. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(*i*) $\mathbb{Q}_{\max}$-(∗).

(*ii*) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-BQM.

[47] It is an open problem whether BMM implies "NS_{ω_1} is precipitous".

Proof. Let $\mathbb{Q}^{\dagger}_{\max}$ be defined in the same way as $\mathbb{Q}^{-}_{\max}$, except that $(<_{\mathbb{Q}^{-}_{\max}}.ii)$ is dropped in the definition of the order. Note that AD holds in $L(\mathbb{R})$. Lemma 7.40 shows that the inclusion $\mathbb{Q}_{\max} \hookrightarrow \mathbb{Q}^{\dagger}_{\max}$ is a dense embedding. Let $\mathbb{Q}^{\ddagger}_{\max}$ be the suborder of $\mathbb{Q}^{\dagger}_{\max}$ consisting only of conditions $(M, I, f) \in \mathbb{Q}^{\dagger}_{\max}$ with $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$. Results of Woodin, see Theorem 6.30 and Theorem 6.80 in [Woo10], imply that $\mathbb{Q}_{\max}$ is self-assembling and that there are densely many conditions $(M, I, f) \in \mathbb{Q}_{\max}$ with $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$ (assuming $\mathrm{AD}^{L(\mathbb{R})}$). It follows that $\mathbb{Q}^{\ddagger}_{\max}$ is self-assembling and dense in $\mathbb{Q}^{\dagger}_{\max}$. The point is that $\mathbb{Q}^{\ddagger}_{\max}$ has unique iterations by Fact 5.11 (while $\mathbb{Q}^{\dagger}_{\max}$ does not).
Consequently, it suffices to show that $\mathbb{Q}^{\ddagger}_{\max}$-$(*)$ is equivalent to (ii). $\mathbb{Q}^{\ddagger}_{\max}$ is a typical $\mathbb{P}_{\max}$-variation and this is witnessed by $\Psi^{\mathbb{Q}^{\ddagger}_{\max}} := \Psi^{\mathbb{Q}_{\max}}$. Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. Then

$$\Gamma_f^{\Psi^{\mathbb{Q}^{\ddagger}_{\max}}} = \Gamma_f^{\mathbb{Q}^{\ddagger}_{\max}}(\Psi^{\mathbb{Q}^{\ddagger}_{\max}})$$

by Theorem 3.60 as there is a proper class of Woodin cardinals. This equality is the reason we do not work with $\mathbb{Q}_{\max}$ directly. The equivalence now follows from the Second Blueprint Theorem 4.58. □

Finally, we remark that one can show that fragments of QM hold in $\mathbb{Q}_{\max}$-extensions of canonical models of determinacy. For example $\mathrm{QM}(\mathfrak{c})$, i.e. QM for forcings of size at most continuum, holds in the $\mathbb{Q}_{\max}$-extension of models of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" + $V = L(\mathcal{P}(\mathbb{R}))$ and BQM holds in the $\mathbb{Q}_{\max}$-extension of suitable $\mathbb{R}$-mice.
Finally we want to mention that Woodin has formulated a forcing axiom $\mathrm{FA}(\diamondsuit(\omega_1^{<\omega}))[\mathfrak{c}]$ somewhat similar to $\mathrm{QM}(\mathfrak{c})$ and has proven that it holds in the $\mathbb{Q}_{\max}$-extension of a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" + $V = L(\mathcal{P}(\mathbb{R}))$, see Theorem 9.54 in [Woo10][48] The global version $\mathrm{FA}(\diamondsuit(\omega_1^{<\omega}))$ of Woodin's axiom does *not* imply "NS_{ω_1} is ω_1-dense". The reason is that if f witnesses $\diamondsuit(\omega_1^{<\omega})$ and $\mathrm{MM}^{++}(f)$ holds then $\mathrm{FA}(\diamondsuit(\omega_1^{<\omega}))$ is true, however NS_{ω_1} is not ω_1-dense.

8 The $\mathbb{P}_{\max}$-Variation $\mathbb{F}_{\max}$

We pay our dues and show that $\mathbb{C}_{\max}$ is self-assembling, however we will not work with $\mathbb{C}_{\max}$ directly. The $\diamondsuit$-forcing approach offers a uniform treatment of

- semiproper forcing and hence MM^{++},

[48] We remark once again that Woodin has defined $\diamondsuit(\omega_1^{<\omega})$ slightly different than we have here.

- f-semiproper forcing and hence $\mathrm{MM}^{++}(f)$ for f a witness of $\diamondsuit(\mathsf{C})$ or $\diamondsuit(\omega_1^{<\omega})$ as well as
- semiproper forcing preserving a Suslin tree T and hence $\mathrm{MM}^{++}(T)$.

As we have shown in Section 5, these forcing axioms are related to $\mathbb{P}_{\max}$, $\mathbb{C}_{\max}$ and $\mathbb{S}^T_{\max}$ (if T is strongly homogeneous) respectively. This suggest that there should be a unified approach treating to $\mathbb{P}_{\max}$, $\mathbb{C}_{\max}$ and $\mathbb{S}^T_{\max}$. We present such an approach in this section.

Definition 8.1. A condition $p \in \mathbb{F}_{\max}$ is a generically iterable structure of the form[49]

$$p = (M, I, \mathbb{B}, \mathbf{f}, a)$$

with

($\mathbb{F}_{\max}.i$) $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$,

($\mathbb{F}_{\max}.ii$) $M \models$ "$\mathbf{f}$ uniformly witnesses $\diamondsuit^+_I(\mathbb{B})$" and

($\mathbb{F}_{\max}.iii$) $M \models$ "a is a subset of ω_1 with $\omega_1^{L[a]} = \omega_1$".

For $q, p \in \mathbb{F}_{\max}$, we have $q <_{\mathbb{F}_{\max}} p$ if and only if there is a generic iteration

$$\mu \colon p \to p^*$$

of p in q of length $\omega_1^N + 1$ with

($<_{\mathbb{F}_{\max}}.i$) $I^q \cap p^* = I^{p^*}$,

($<_{\mathbb{F}_{\max}}.ii$) $\mathbb{B}^{p^*} = \mathbb{B}^q$, $\mathbf{f}^{p^*} = \mathbf{f}^q$ and $a^{p^*} = a^q$.

Note that $\mathbb{F}_{\max}$ has unique iterations by ($\mathbb{F}_{\max}.i$) and Fact 5.11.
As we said already in the beginning of this section, $\mathbb{F}_{\max}$ is a sort of amalgamation of different nicer $\mathbb{P}_{\max}$-variations.

Definition 8.2. If $\mathbb{P}$ is a forcing then we call a suborder $\mathbb{Q} \subseteq \mathbb{P}$ a component of $\mathbb{P}$ if no $p \in \mathbb{P} - \mathbb{Q}$ is compatible with any $q \in \mathbb{Q}$.

In fact we will have that $\mathbb{P}_{\max}, \mathbb{C}_{\max}$ and $\mathbb{S}^T_{\max}$ are (assuming $\mathrm{AD}^{L(\mathbb{R})}$ forcing equivalent to) components of $\mathbb{F}_{\max}$. Thus $\mathbb{F}_{\max}$ is not homogeneous and thus the $\mathbb{F}_{\max}$-extension of $L(\mathbb{R})$ is not exactly canonical, even though the components mentioned above produce canonical extensions of $L(\mathbb{R})$, assuming $\mathrm{AD}^{L(\mathbb{R})}$ of course. We view $\mathbb{F}_{\max}$ as a creature brought to life by different components that do not want to fit together, similar to Frankenstein's monster. This explains the letter $\mathbb{F}$.

[49] We will take liberties to write M^p for M, I^p for I, etc. and also to confuse p with M.

8.1 The $\mathbb{F}_{\max}$-extension of $L(\mathbb{R})$

For the rest of this section we will pretend that if $(M, I, \mathbb{B}, \mathbf{f}, a)$ is a $\mathbb{F}_{\max}$-condition then $\mathbf{f}$ is a usual witness of $\diamondsuit(\mathbb{B})$ in M and not a sequence of uniform witnesses. In essence we pretend that $\mathbf{f}$ is constant. As before, all arguments will generalize.

Definition 8.3. Suppose $g \subseteq \mathbb{F}_{\max}$ is a filter.

(i) $\mathcal{P}(\omega_1)_g$ is the set of all $X \subseteq \omega_1$ so that there is $p \in g$ and $\bar{X} \in p$ as well as an iteration $\mu\colon p \to p^*$ guided by g with $X \in p^*$.

(ii) $\mathbb{B}_g = \bigcup_{p \in g} \mathbb{B}^p$, $f_g = \bigcup_{p \in g} f^p$ and $a_g = \bigcup_{p \in g} a^p$.

Lemma 8.4. *Suppose (M, I) is a generically iterable structure and $\diamondsuit$ holds. Then there is a $\diamondsuit$-iteration of (M, I).*

Proof. A standard trick shows that under $\diamondsuit$, a seemingly stronger principle holds: There is a sequence

$$\vec{a} = \langle a_\beta \mid \beta < \omega_1 \rangle$$

and a partition $\mathcal{S} = \{S_i \mid i < \omega_1\}$ of ω_1 into ω_1-many stationary sets so that $\vec{a}$ witnesses $\diamondsuit_S$ for all $S \in \mathcal{S}$ simultaneously. Let us enumerate $\mathcal{S}$ as

$$\langle S_{\alpha,n} \mid \alpha < \omega_1, n < \omega \rangle.$$

We fix a uniform way of coding sequences of the form $\mathbf{Z} = \langle \vec{Z}_i \mid i < \omega_1 \rangle$ where $\vec{Z}_i \in \mathcal{P}(\omega_1)^{\omega_1}$ for all $i < \omega_1$ into a subset of ω_1. For $\vec{Z} \in \mathcal{P}(\omega_1)^{\omega_1}$ and $\alpha < \omega_1$, $\vec{Z} \restriction\restriction \alpha$ denotes the map

$$\alpha \to \mathcal{P}(\alpha), \ \beta \mapsto \vec{Z}(\beta) \cap \alpha$$

And for $\mathbf{Z} = \langle \vec{Z}_i \mid i < \omega_1 \rangle$ as above, $\alpha < \omega_1$, we define

$$\mathbf{Z} \restriction\restriction\restriction \alpha = \langle \vec{Z}_i \restriction\restriction \alpha \mid i < \alpha \rangle.$$

We require from the coding that if A codes $\mathbf{Z}$ then on a club, $A \cap \alpha$ codes $\mathbf{Z} \restriction\restriction\restriction \alpha$. We leave it to the reader to find such a coding.
We define a generic iteration

$$\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$$

of $(M_0, I_0) = (M, I)$ by induction. We only have to define the generic filter g_α for any stage α. At each stage, we will choose some enumeration $\langle X_n^\alpha \mid n < \omega \rangle$ of $I_\alpha^+ \cap M_\alpha$.
So suppose (M_α, I_α) is defined.
<u>Case 1:</u> a_α codes a sequence $\langle \vec{Z}_i \mid i < \alpha \rangle$ so that for all $i < \alpha$

($\vec{Z}.i$) $\vec{Z}_i \in \mathcal{P}(\alpha)^\alpha$ and

($\vec{Z}.ii$) D_i is dense in $I_\alpha^+ \cap M_\alpha$, where $D_i := \operatorname{ran}(\vec{Z}_i)$.

Let β, n be unique with $\alpha \in S_{\beta,n}$. Then let g_α be an $I_\alpha^+ \cap M_\alpha$-generic filter over M_α so that

($g_\alpha.i$) g_α meets any D_i for all $i < \alpha$ and

($g_\alpha.ii$) if $\beta \leqslant \alpha$ then $\mu_{\beta,\alpha}(X_n^\beta) \in g_\alpha$.

Case 2: Case 1 fails. Again let β, n be unique with $\alpha \in S_{\beta,n}$. Then let g_α be an arbitrary $I_\alpha^+ \cap M_\alpha$-filter generic over M_α with ($g_\alpha.ii$).

It remains to show that $\mu = \mu_{0,\omega_1}\colon M_0 \to M_{\omega_1}$ is a $\diamondsuit$-iteration. Let $S \in I_{\omega_1}^+ \cap M_{\omega_1}$ and $\langle D_i \mid i < \omega_1\rangle$ a sequence of dense subsets of $I_{\omega_1}^+ \cap M_{\omega_1}$ and let $\vec{Z}_i$ enumerate D_i in ordertype ω_1. Let $A \subseteq \omega_1$ be a code for

$$\mathbf{Z} := \langle \vec{Z}_i \mid i < \omega_1\rangle.$$

Find $\beta < \omega_1$ with $S \in \operatorname{ran}(\mu_{\beta,\omega_1})$, say $S = \mu_{\beta,\omega_1}(\bar{S})$. There is now $n < \omega$ with $\bar{S} = X_n^\beta$. Next, we may find a club $C \subseteq \omega_1$ so that for all $\alpha \in C$

($C.i$) $A \cap \alpha$ codes $\mathbf{Z} \restriction\restriction\restriction \alpha$,

($C.ii$) $\omega_1^{M_\alpha} = \alpha$,

($C.iii$) $\operatorname{ran}(\mu_{\alpha,\omega_1}) \cap D_i = \vec{Z}_i[\alpha]$ for all $i < \alpha$ and

($C.iv$) $\mu_{\alpha,\omega_1}^{-1}[D_i]$ is dense in $I_\alpha^+ \cap M_\alpha$ for all $i < \alpha$.

Let E be the club of iteration points of μ, i.e. $E = \{\omega_1^{M_\alpha} \mid \alpha < \omega_1\}$. It follows from the construction of the generic iteration that S contains a tail of $E \cap S_{\beta,n}$. As $\vec{a}$ witnesses $\diamondsuit_{S_{\beta,n}}$, there is a stationary set $T \subseteq S$ such that $a_\alpha = A \cap \alpha$ for all $\alpha \in T$.

Claim 8.5. $T \cap C \subseteq \{\alpha \in S \mid \forall i < \alpha\ g_\alpha \cap \mu_{\alpha,\omega_1}^{-1}[D_i] \neq \varnothing\}$.

Proof. Let $\alpha \in T \cap C$. As $\alpha \in T$, $a_\alpha = A \cap \alpha$ and by ($C.i$), a_α codes $\mathbf{Z} \restriction\restriction\restriction \alpha$. Let

$$\bar{D}_i := \operatorname{ran}(Z_i \restriction\restriction \alpha)$$

for $i < \alpha$. It follows from ($C.ii$) and ($C.iii$) that

$$\bar{D}_i = \mu_{\alpha,\omega_1}^{-1}[D_i]$$

for all $i < \alpha$. Next, ($C.iv$) implies that $\bar{D}_i$ is dense in $I_\alpha^+ \cap M_\alpha$ for all $i < \alpha$ and hence g_α has been defined according to Case 1. By ($g_\alpha.i$), we can conclude

$$g_\alpha \cap \mu_{\alpha,\omega_1}^{-1}[D_i] = g_\alpha \cap \bar{D}_i \neq \varnothing$$

for all $i < \omega_1$. □

As T is stationary, $\{\alpha \in S \mid \forall i < \alpha\ g_\alpha \cap \mu^{-1}_{\alpha,\omega_1}[D_i] \neq \varnothing\}$ is stationary, which is what we had to show. $\square$

Corollary 8.6. *Assume $\diamondsuit$. Let*

$$p = (M, I, \mathbb{B}, f, a)$$

be a $\mathbb{F}_{\max}$-condition. Then there is a generic iteration

$$j\colon p \to p^* = (M^*, I^*, \mathbb{B}^*, f^*, a^*)$$

so that

(*i*) *f^* witnesses $\diamondsuit(\mathbb{B}^*)$ and*

(*ii*) *$I^* = \mathrm{NS}_{f^*} \cap p^*$.*

Proof. This follows immediately from Lemma 8.4 and Lemma 4.19. $\square$

Fact 8.7 (Woodin)**.** *Suppose M is a countable transitive model of* ZFC*, $I \in M$ and*

$$M \models \text{“}I \text{ is a normal uniform precipitous ideal”}.$$

If $\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma\rangle$ is any generic iteration of $(M_0, I_0) = (M, I)$ and $\gamma \in \mathrm{Ord} \cap M$ then M_γ is wellfounded.

Corollary 8.8. *Suppose $\mathbb{R}$ is closed under $x \mapsto M_1^\sharp(x)$. Then for any real x and $\mathbb{F}_{\max}$-condition p, there is a $\mathbb{F}_{\max}$-condition q with*

(*q.i*) *$q <_{\mathbb{F}_{\max}} p$,*

(*q.ii*) *$x \in M^q$.*

(*q.iii*) *$M^q \models$ ZFC and*

(*q.iv*) *$M^q \models$ “$I^q = \mathrm{NS}_{\omega_1}$ is saturated”.*

Proof. Let $x \in \mathbb{R}$ and y be a real coding p. Let $z = x \oplus y$ and set $M = M_1^\sharp(z) \| \kappa$ where κ is the critical point of the active extender of $M_1^\sharp(z)$. Note that $\diamondsuit$ holds in M. By Corollary 8.6, in M, there is a generic iteration

$$\mu\colon p \to p^* = (M^*, I^*, \mathbb{B}^*, f^*, a^*)$$

so that f^* witnesses $\diamondsuit(\mathbb{B}^*)$ and $I^* = \mathrm{NS}_{f^*} \cap p^*$. By Theorem 3.60, there is a f^*-stationary set preserving forcing extension $M[g]$ of M with

(*i*) $M[g] \models$ “NS_{ω_1} is saturated”,

(*ii*) $M[g] \models$ “f^* witnesses $\diamondsuit^+(\mathbb{B}^*)$” and *(iii)* $M[g] \models \psi_{\mathrm{AC}}$.

Let
$$q := (M[g], \mathrm{NS}_{\omega_1}^{M[g]}, \mathbb{B}^*, f^*, a^*).$$

Note that q is generically iterable by (i), Fact 8.7 and the choice of M. Thus $q \in \mathbb{F}_{\max}$. The iteration $\mu \colon p \to p^*$ now witnesses $q <_{\mathbb{F}_{\max}} p$. Wehave

$$I^* = \mathrm{NS}_{\omega_1}^{M[g]} \cap p^*$$

since $I^* = \mathrm{NS}_{f^*}^{M[g]} \cap p^*$, by (ii) and as the extension $M \subseteq M[g]$ preserved f^*-stationary sets. □

It follows that $\mathbb{F}_{\max}$ is a $\mathbb{P}_{\max}$-variation.

Applying the methods we developed so far, the following can be proven analogously as the corresponding results for $\mathbb{P}_{\max}$.

Lemma 8.9. *If $\mathbb{R}$ is closed under $x \mapsto M_1^\sharp(x)$ then $\mathbb{F}_{\max}$ is σ-closed.*

Lemma 8.10. *Assume* AD *in $L(\mathbb{R})$. Then for any $X \subseteq \mathbb{R}$, $X \in L(\mathbb{R})$, for any $p \in \mathbb{F}_{\max}$ there is $q = (M, I, \mathbb{B}, f, a) <_{\mathbb{F}_{\max}} p$ so that*

(i) *(M, I) is X-iterable and*

(ii) *$(H_{\omega_1}^M, X \cap M) \prec (H_{\omega_1}, A)$.*

Theorem 8.11. *Suppose* AD *holds in $L(\mathbb{R})$. Let g be $\mathbb{F}_{\max}$-generic over $L(\mathbb{R})$. Then in $L(\mathbb{R})[g]$ we have*

(i) *$\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_g$,*

(ii) *f_g witnesses $\diamondsuit^+(\mathbb{B}_g)$,*

(iii) *NS_{ω_1} is saturated,*

(iv) *ψ_{AC} is true and*

(v) *the axiom of choice holds.*

Proof. (i), $(iii)-(v)$ can be shown just as for $\mathbb{P}_{\max}$, from Lemma 8.10. Also, (ii) follows from Lemma 4.49. □

Corollary 8.12. *If $\mathrm{AD}^{L(\mathbb{R})}$ holds then $\mathbb{F}_{\max}$ is self-assembling.*

It follows that $\mathbb{C}_{\max}$ is self-assembling under $\mathrm{AD}^{L(\mathbb{R})}$.

We can now apply the Blueprint Theorems directly to $\mathbb{F}_{\max}$, we leave the details to the reader.

Theorem 8.13. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. Then* $\mathrm{MM}^{++}(f)$ *implies* $\mathbb{F}_{\max}$-$(*)$ *and moreover, there is a filter witnessing* $\mathbb{F}_{\max}$-$(*)$ *that produces $\mathbb{B}, f, A$ for some $A \subseteq \omega_1$.*

Theorem 8.14. *Suppose there is a proper class of Woodin cardinals and f witnesses $\diamondsuit(\mathbb{B})$. The following are equivalent:*

(*i*) *There is a filter witnessing* $\mathbb{F}_{\max}$-$(*)$ *that produces $\mathbb{B}, f, A$ for some A.*

(*ii*) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(f)$.

9 Maximal Models of $\mathfrak{d} = \aleph_1$

We continue here what we begun in Subsection 5.4, that is we provide Σ_2-sentences ϕ that Shelah-Zapletal proved to be Π_2-compact with instances of $\mathrm{MM}^{++} \Rightarrow (*)$. We will do so here for

$$\phi^{\mathfrak{d}} = \text{“}\mathfrak{d} = \aleph_1\text{”}$$

and consider the bounding number in Section 10. On the forcing side, Miyamoto has proven an iteration theorem for semiproper forcing not adding unbounded reals. This yields the consistency of MM^{++} conditioned on the existence of a witness of $\mathfrak{d} = \aleph_1$.
We will show in this section that these two approaches cohere in the sense that this forcing axiom implies $P_{\phi^{\mathfrak{d}}}$-$(*)$ where $P_{\phi^{\mathfrak{d}}}$ is the forcing Shelah-Zapletal constructed to prove the Π_2-compactness of $\phi^{\mathfrak{d}}$. The dominating number is defined as follows.

Definition 9.1. (*i*) $\leqslant^*$ denotes the partial order of eventual domination on ${}^{\omega}\omega$, i.e. if $f, g\colon \omega \to \omega$ are functions then $f \leqslant^* g$ iff there is some $k < \omega$ so that $\forall n \geqslant k\ f(n) \leqslant g(n)$.

(*ii*) A family $\mathcal{D} \subseteq {}^{\omega}\omega$ is *dominating* if for any $f \in {}^{\omega}\omega$ there is $g \in \mathcal{D}$ that eventually dominates f, i.e. $f \leqslant^* g$.

(*iii*) The dominating number, denoted by $\mathfrak{d}$, is the least cardinality of which there is a dominating family, that is

$$\mathfrak{d} := \min\{|\mathcal{D}| \mid \mathcal{D} \subseteq {}^{\omega}\omega \text{ is dominating}\}.$$

Definition 9.2 (Shelah-Zapletal,[SZ99])**.** A sequence $\vec{d} = \langle d_\alpha \mid \alpha < \omega_1\rangle$ of reals is called a *good dominating sequence* if

($\vec{d}.i$) $\{d_\alpha \mid \alpha < \omega_1\}$ is a dominating family,

($\vec{d}.ii$) if $\alpha \leqslant \beta < \omega_1$ then $d_\alpha \leqslant^* d_\beta$ and

($\vec{d}.iii$) for any function $h \in {}^{\omega}\omega$ the set

$$S_h := \{\alpha < \omega_1 \mid f \leqslant d_\alpha \text{ pointwise}\}$$

is stationary.

If I is a normal uniform ideal on ω_1, then $\vec{d}$ is a *I-good dominating sequence* if ($\vec{d}.i$)-($\vec{d}.ii$) hold and $S_h \in I^+$ for any $h \in {}^{\omega}\omega$.

Fact 9.3 (Shelah-Zapletal, [SZ99])**.** *The following are equivalent.*

(*i*) $\mathfrak{d} = \aleph_1$.

(*ii*) *There is a good dominating sequence.*

We translate the forcing $P_{\mathfrak{d}=\aleph_1}$ of Shelah-Zapletal to a $\mathbb{P}_{\max}$-variant suitable for our context. Essentially, we replace iterations coming from the stationary tower forcing at a Woodin cardinal with iterations given by forcing with precipitous ideals on ω_1.

Definition 9.4. Conditions in $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ are generically iterable structures of the form

$$p = (M, I, \vec{d}, a)$$

so that

($\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}.i$) $(M; \in, I) \models \psi_{\mathrm{AC}}(I)$,

($\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}.ii$) $(M; \in, I) \models$ "$\vec{d}$ is a I-good dominating sequence" and

($\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}.iii$) $M \models$ "$a \subseteq \omega_1$ and $\omega_1^{L[a]} = \omega_1$".

The order is given by $q = (N, J, \vec{e}, b) <_{\mathfrak{d}=\aleph_1} p$ iff there is a generic iteration

$$j \colon p \to p^* = (M^*, I^*, \vec{d}^*, a^*)$$

of length $\omega_1^q + 1$ in q with

($<_{\mathfrak{d}=\aleph_1}.i$) $I^* = J \cap M^*$,

($<_{\mathfrak{d}=\aleph_1}.ii$) $\vec{d}^* = \vec{e}$ and

($<_{\mathfrak{d}=\aleph_1}.iii$) $a^* = b$.

It follows with methods of Woodin and results of Shelah-Zapletal [SZ99] that $P_{\mathfrak{d}=\aleph_1}$ and $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ achieve the same thing.

Corollary 9.5. *Assume* $V = L(\mathbb{R}) \models \mathrm{AD}$*. Then* $P_{\mathfrak{d}=\aleph_1}$ *and* $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ *are forcing equivalent.*

9.1 The forcing axiom $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$

Definition 9.6. A forcing $\mathbb{P}$ is *${}^\omega\omega$-bounding* if whenever G is $\mathbb{P}$-generic then $({}^\omega\omega)^V$ is a dominating family in $V[G]$.

Definition 9.7. The axiom $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ holds if

(*i*) $\mathfrak{d} = \aleph_1$ and

(*ii*) FA^{++}(stationary set preserving and ${}^\omega\omega$-bounding) is true.

As usual, proving the consistency of $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ goes through the principle $\mathrm{SPFA}^{++}(\mathfrak{d} = \aleph_1)$ which is $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ restricted to ${}^\omega\omega$-bounding semiproper forcings. The relevant iteration theorem is:

Fact 9.8 (Miyamoto, [Miy01]). *Assume* CH. *If*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$$

is a nice iteration of limit length γ so that

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is semiproper and } {}^\omega\omega\textit{-bounding“}$$

for all $\alpha < \gamma$ then $\mathbb{P}$ is semiproper and ${}^\omega\omega$-bounding.

To be precise, Miyamoto has proven the above result for so called simple iterations, the version above then follows from results of Miyamoto on the relation between nice iterations and simple iterations[50].
The consistency of $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ is essentially proven as Theorem 3.3 in [Miy01], even though it is not spelled out explicitly. We give only a few details.

Theorem 9.9. *If* ZFC + *“there is a supercompact cardinal” is consistent, then so is* $\mathrm{ZFC} + \mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$.

Proof. (Sketch) We modify the construction of a model of MM^{++} from a supercompact. Start in a model $V \models$ ZFC + CH + “κ is supercompact”. Let $\mathbb{P}$ be the nice iteration of ${}^\omega\omega$-bounding semiproper forcings of length κ where each iterand is chosen by a Laver function on κ, cf. the proof of Lemma 6.7. By Fact 9.8, $\mathbb{P}$ is semiproper and ${}^\omega\omega$-bounding. It follows that $\mathcal{D} = ({}^\omega\omega)^V$ is a dominating family of size ω_1 in $V^{\mathbb{P}}$ and thus $\mathfrak{d} = \aleph_1$ holds in $V^{\mathbb{P}}$. It follows that $\mathrm{SPFA}^{++}(\mathfrak{d} = \aleph_1)$ holds in $V^{\mathbb{P}}$. In particular,

$$V^{\mathbb{P}} \models \text{“Every (pruned) tree of height } \omega_1 \text{ which is semiproper has a branch”}$$

as forcing with such trees does not add reals. Miyamoto has shown that this implies SRP, see Lemma 3.5 in [Miy01]. Thus all stationary set preserving forcings are semiproper, so that $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ holds true. □

[50] We also note that nothing in this section changes if we would use simple iterations instead of nice iterations.

9.2 $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$-forcing

We want to apply the theory of Section 4 to the $\mathbb{P}_{\max}$-variation $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$. However, $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ does not seem to accept $\diamondsuit$-iterations and because of this we need to come up with a replacement for Theorem 4.20.

Theorem 9.10. *Suppose that*

(*i*) *generic projective absoluteness holds for generic extensions by forcings of size* ω_2*,*

(*ii*) NS_{ω_1} *is saturated and* $\mathcal{P}(\omega_1)^\sharp$ *exists,*

(*iii*) $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \vec{d}, A)$ *is almost a* $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$*-condition and*

(*iv*) $D \subseteq \mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ *is* 2^{ω_2}*-universally Baire and dense in* $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ *in any generic extension by a forcing of size* 2^{ω_2}*, as witnessed by trees* T, S *with* $p[T] = D$*.*

Then there is a ${}^\omega\omega$*-bounding forcing* $\mathbb{P}^{\mathfrak{d}}$ *so that in* $V^{\mathbb{P}^{\mathfrak{d}}}$ *the following picture exists*

$$\begin{tikzcd}
 & p[T] & \\
 & q_0 \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}" description] \arrow[r, "\sigma_{0,\omega_1}"] & q_{\omega_1} \\
p_0 \arrow[r, "\mu_{0,\omega_1^N}"] \arrow[d, phantom, "\rotatebox[origin=c]{270}{$\in$}" description] & p_{\omega_1^N} \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}" description] \arrow[r, "\mu_{\omega_1^N,\omega_1}"] & p_{\omega_1} \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}" description] \arrow[d, equal] \\
\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1} & & ((H_{\omega_2})^V, \mathrm{NS}_{\omega_1}^V, \vec{d}, A)
\end{tikzcd}$$

so that

($\mathbb{P}^{\mathfrak{d}}.i$) $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ *are generic iterations of* p_0*,* q_0 *respectively,*

($\mathbb{P}^{\mathfrak{d}}.ii$) μ_{0,ω_1^N} *witnesses* $q_0 <_{\mathfrak{d}=\aleph_1} p_0$*,*

($\mathbb{P}^{\mathfrak{d}}.iii$) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$ *and*

($\mathbb{P}^{\mathfrak{d}}.iv$) $\sigma_{0,\omega_1} \colon q_0 \to q_{\omega_1}$ *is a correct iteration*[51]*.*

We go on and give a sketch of the proof, it is quite similar to the proof of Theorem 4.20. Let κ denote ω_3 and assume $\diamondsuit_\kappa$. It is along the lines of Section 4, i.e. we modify the forcing of Asperó-Schindler to the given context and add a condition that potential certificates have to satisfy in order to establish the appropriate preservation. We borrow all notation from Section 4 (in the special case that $\mathbb{V}_{\max} = \mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$). The changes to the language etc. are straightforward. However we store an additional bit

[51]We could also arrange σ_{0,ω_1} to be a $\diamondsuit$-iteration, but we have no use of this.

of information in potential certificates, namely for every $\xi \in K$ a natural number k_ξ. Thus potential certificates are of the form

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, k_\xi, X_\xi \mid \xi \in K\rangle\rangle.$$

We do not put any demand on k_ξ for potential certificates except that k_ξ is an integer. To the language we add formulas of the form

$$\ulcorner\dot{k}_\xi = \underline{k}\urcorner$$

for $\xi < \kappa$, $k < \omega$ that capture this in the obvious way. That is, for Σ to (λ-)precertify $\mathfrak{C}$, we demand that if $\xi \in K$ then

$$\ulcorner\dot{k}_\xi = \underline{k}\urcorner \in \Sigma \Leftrightarrow k_\xi = k$$

for all $k < \omega$.

We now start to describe an additional condition that we will exploit in the end to show $\mathbb{P}^{\mathfrak{d}}$ to be ${}^\omega\omega$-bounding. The idea is as follows: In the end, $\mathbb{P}^{\mathfrak{d}}$ adds a semantic certificate

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle(k_n, \alpha_n) \mid n < \omega\rangle, \langle\lambda_\xi, k_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

We want to show that every function $f\colon \omega \to \omega$ in $V^{\mathbb{P}^{\mathfrak{d}}}$ is dominated by a point d_ξ of the good dominating sequence $\vec{d}$. The strategy to achieve this is: Find a stage $\lambda \in C$ so that the reduct to Q_λ of (a code for) a name $\dot{f}$ is exactly A_λ. The sentence $\ulcorner\underline{\xi} \mapsto \underline{\lambda}\urcorner$ should then capture that the evaluation of $\dot{f}$ is eventually dominated by d_ξ. We make this precise now. The following definition is part of the induction on $\lambda \in C \cup \{\kappa\}$ that eventually defines $\mathbb{P}^{\mathfrak{d}} = \mathbb{P}^{\mathfrak{d}}_\kappa$.

Definition 9.11. For $\bar\lambda \in C \cap \lambda$, a $\bar\lambda$-code for a real is a set

$$Z \subseteq \mathbb{P}^{\mathfrak{d}}_{\bar\lambda} \times \omega \times \omega$$

so that

($Z.i$) for any $p \in \mathbb{P}^{\mathfrak{d}}_{\bar\lambda}$ and $n < \omega$ there is $q \leqslant p$ and $k < \omega$ with $(q, n, k) \in Z$,

($Z.ii$) if $(p, n, k), (p, n, l)$ are both in Z then $k = l$ and

($Z.iii$) if $(p, n, k) \in Z$ and $q \leqslant p$ then $(q, n, k) \in Z$.

If Σ is a precertificate and $Z_0 \subseteq Z$ then the evaluation of Z_0 by Σ is defined as

$$Z_0^\Sigma = \{(n, k) \in \omega \times \omega \mid \exists p \in [\Sigma]^{<\omega}\ (p, n, k) \in Z_0\}$$

Note that this is a (potentially partial) function $Z^\Sigma\colon \omega \to \omega$.

Let

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k_n, \alpha_n) \mid n < \omega\rangle, \langle \lambda_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

be a λ-precertificate certified by Σ. $\mathfrak{C}$ is a semantic λ-certificate and Σ a syntactic λ-certificate if additionally

$(\Sigma.8)^{\mathfrak{d}=\aleph_1}$ if $\xi \in K$ and $E \subseteq \mathbb{P}^{\mathfrak{d}}_{\lambda_\xi}$ is dense and definable over

$$(Q_\lambda; \in, \mathbb{P}^{\mathfrak{d}}_{\lambda_\xi}, A_{\lambda_\xi})$$

from parameters in X_ξ then $X_\xi \cap E \cap [\Sigma]^{<\omega} \neq \emptyset$ and

$(\Sigma.9)^{\mathfrak{d}=\aleph_1}$ whenever $\xi \in K$ and $Z = A_{\lambda_\xi}$ is a λ_ξ-code for a real then for any $k_\xi \leqslant n < \omega$

$$Z^\Sigma(n) \leqslant d_\xi(n).$$

$(\Sigma.8)^{\mathfrak{d}=\aleph_1}$ is exactly the genericity condition of Asperó-Schindler adapted to the context here. $(\Sigma.9)^{\mathfrak{d}=\aleph_1}$ is a new "preservation condition". Note that If $\mathfrak{C}$ is a semantic certificate certified by Σ, then $(\Sigma.8)^{\mathfrak{d}=\aleph_1}$ guarantees that Z^Σ is a total function on ω if $\xi \in K$ and $Z = A_{\lambda_\xi}$ is a λ_ξ-code for a real.

Definition 9.12. Conditions $p \in \mathbb{P}^{\mathfrak{d}}_\lambda$ are finite sets of $\mathcal{L}^\lambda$ formulae so that

$$V^{\mathrm{Col}(\omega,\omega_2)} \models \text{“}\exists \Sigma \subseteq \mathcal{L}^\lambda\ \Sigma \text{ certifies } p\text{”}.$$

Lemma 9.13. *Suppose $\lambda \in C \cup \{\kappa\}$ and $g \subseteq \mathbb{P}^{\mathfrak{d}}_\lambda$ is a filter that meets any dense subset of $\mathbb{P}^{\mathfrak{d}}_\lambda$ which is definable over*

$$(Q_\lambda; \in, \mathbb{P}^{\mathfrak{d}}_\lambda, A_\lambda)$$

Then $\bigcup g$ is a syntactic λ-certificate.

Proof. We will only show that $\bigcup g$ satisfies $(\Sigma.9)^{\mathfrak{d}=\aleph_1}$. Let

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k_n, \alpha_n) \mid n < \omega\rangle, \langle \lambda_\xi, k_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

be the semantic precertificate that comes from g. Now suppose $\xi \in K$ and $Z = A_{\lambda_\xi}$ is a λ_ξ-code for a real. Suppose $k_\xi \leqslant n < \omega$. Let

$$p = \{\ulcorner \underline{\xi} \mapsto \underline{\lambda_\xi}\urcorner, \ulcorner \dot{k}_\xi = \underline{k_\xi}\urcorner\} \in g$$

and let $q \leqslant p$. Find Σ that certifies q and find

$$\mathfrak{C}' = \langle\langle M'_i, \mu'_{i,j}, N'_i, \sigma'_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k'_n, \alpha'_n) \mid n < \omega\rangle, \langle \lambda'_\rho, k'_\rho, X'_\rho \mid \rho \in K'\rangle\rangle$$

the corresponding semantic certificate. Then $\xi \in K'$ and $k'_\xi = k_\xi$ as $p \in [\Sigma]^{<\omega}$. Thus using $(\Sigma.8)^{\mathfrak{d}=\aleph_1}$, we find $r \in [\Sigma]^{<\omega}$, $r \leqslant q$ and $l < \omega$ so that

$$(r, n, l) \in Z$$

and by $(\Sigma.9)^{\mathfrak{d}=\aleph_1}$, $l \leqslant d_\xi(n)$.
This is a density argument which shows that there is $s \leqslant p$, $s \in g$ and $j \leqslant d_\xi(n)$ with $(s, n, j) \in Z$. This shows

$$Z^{\bigcup g}(n) = j \leqslant d_\xi(n).$$

□

The above argument is the reason we had added the bit k_ξ for $\xi \in K$ to the information a certificate carries. If we had not done this, we would still get $d_\xi \nleqslant^* Z^{\bigcup g}$ in the argument above, but it is no longer clear whether d_ξ eventually dominates $Z^{\bigcup g}$.

The next two lemmas can be proven exactly as in [AS21]. For the first one, recall that SRP is a consequence of $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$, so that the universe is closed under $X \mapsto M_\omega^\sharp(X)$.

Lemma 9.14. $\mathbb{P}^{\mathfrak{d}}_{\min C} \neq \varnothing$.

Lemma 9.15. $\mathbb{P}^{\mathfrak{d}}$ *preserves stationary sets.*

In order to prove $\mathbb{P}^{\mathfrak{d}}$ to be ${}^\omega\omega$-bounding, the following fact is crucial.

Fact 9.16 (Shelah-Zapletal, [SZ99, Corollary 2.5])**.** *Let M be a countable transitive model of* ZFC *and let $\mathbb{Q} \in M$ be a forcing, $\dot{f} \in M^{\mathbb{Q}}$ with*

(i) $M \models \mathbb{1}_{\mathbb{Q}} \Vdash$ "*$\dot{f}$ is a dominating real*" *and*

(ii) *for any $g \in ({}^\omega\omega)^M$, there is $p \in \mathbb{Q}$ with*

$$M \models p \Vdash \text{“}\check{g} \leqslant \dot{f} \text{ pointwise”}.$$

Let $h \in {}^\omega\omega$, not necessarily in M. Then there is a filter $G \subseteq \mathbb{Q}$ generic over M so that $h \leqslant^* \dot{f}^G$.

Lemma 9.17. $\mathbb{P}^{\mathfrak{d}}$ *is ${}^\omega\omega$-bounding.*

Proof. ω_1 will always denote ω_1^V in the following argument.
Assume $\dot{f} \in V^{\mathbb{P}^{\mathfrak{d}}}$ and $q \in \mathbb{P}^{\mathfrak{d}}$ forces $\dot{f}$ to be an element of ${}^\omega\omega$. We define a κ-code $\hat{Z} \subseteq \mathbb{P}^{\mathfrak{d}} \times \omega \times \omega$ for a real: We let $(q, n, k) \in \hat{Z}$ iff

$$q \Vdash \dot{f}(\check{n}) = \check{k}.$$

Next, find $\lambda \in C$ with $p \in \mathbb{P}^{\mathfrak{d}}_\lambda$ so that

$$Z := A_\lambda = \hat{Z} \cap Q_\lambda.$$

Then by elementarity, Z is a λ-code for a real. We will find $\xi < \omega_1$ with

$$p \cup \{\ulcorner \underline{\xi} \to \underline{\lambda} \urcorner\} \in \mathbb{P}^{\mathfrak{d}}.$$

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V.

Claim 9.18. *In $V[h]$, there are filters $g \subseteq \mathbb{P}^{\mathfrak{d}}_\lambda$ and G which satisfy conditions (i) and (ii) below.*

(*i*) *g meets all dense $E \subseteq \mathbb{P}^{\mathfrak{d}}_\lambda$ that are definable over*

$$(Q_\lambda; \in, \mathbb{P}^{\mathfrak{d}}_\lambda, A_\lambda).$$

Let

$$\mathfrak{C} = \langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k_n, \alpha_n) \mid n < \omega\rangle, \langle \lambda_\xi, k_\xi, X_\xi \mid \xi \in K\rangle\rangle$$

be the certificate corresponding to g.

(*ii*) *G is generic over N_{ω_1} for $(I^+)^{N_{\omega_1}}$ and if*

$$j : N_{\omega_1} \to \mathrm{Ult}(N_{\omega_1}, G) =: N_{\omega_1+1}$$

is the resulting ultrapower then

$$Z^{\bigcup g} \leqslant^* d_{\omega_1}$$

where $j(\vec{d}) = \langle d_\alpha \mid \alpha < \omega_1^{N_{\omega_1+1}}\rangle$.

Proof. Work in a further extension W of $V[h]$ in which a sufficiently large cardinal has been collapsed to ω. In W, there will be a filter $g' \subseteq \mathbb{P}^{\mathfrak{d}}_\lambda$ that is V-generic. Let

$$\mathfrak{C}' = \langle\langle M'_i, \mu'_{i,j}, N'_i, \sigma'_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle, \langle (k'_n, \alpha'_n) \mid n < \omega\rangle, \langle \lambda'_\rho, k'_\rho, X'_\rho \mid \rho \in K'\rangle\rangle$$

be the corresponding semantic certificate. By applying Fact 9.16 in W, we find a filter G' that is $(I^+)^{N'_{\omega_1}}$-generic over N'_{ω_1} so that if

$$j : N'_{\omega_1} \to \mathrm{Ult}(N'_{\omega_1}, G) =: N'$$

is the resulting ultrapower and

$$j(\vec{d}) = \langle d_\alpha \mid \alpha < \omega_1^{N'}\rangle$$

then

$$Z^{\bigcup g'} \leqslant^* d_{\omega_1}.$$

Here we use that $\vec{d}$ is a good dominating sequence to meet all assumptions of Fact 9.16. The existence of filters with these properties is Σ^1_1 in a real code for $(Q_\lambda; \in, \mathbb{P}^{\mathfrak{d}}_\lambda, A_\lambda)$, so such filters exist in $V[h]$ by absoluteness. □

Let $g, G \in V[h]$ be filters as above. The remaining argument is close to the proof of Lemma 4.32, we carry it out briefly. Note that N_0 is still generically iterable in $V[h]$. Extend the iteration

$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \omega_1\rangle$$

to one of length $\kappa = \omega_1^{V[h]} + 1$, say to

$$\langle N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle$$

so that the ultrapower $\sigma_{\omega_1,\omega_1+1}\colon N_{\omega_1} \to N_{\omega_1+1}$ is given by G. This also extends the iteration of M_0 to one of length $\kappa + 1$ and we can lift its tail which is an iteration of $(H^V_{\omega_2}, \mathrm{NS}^V_{\omega_1})$ to an iteration

$$\langle M^+_i, \mu^+_{i,j} \mid \omega_1 \leqslant i \leqslant j \leqslant \kappa\rangle$$

of $M^+_0 = V$. Let $\mu^+ := \mu^+_{\omega_1,\kappa}$ and $M^+ := M^+_\kappa$. Find $k < \omega$ so that for all $k \leqslant n < \omega$, $Z^{\bigcup g}(n) \leqslant d_{\omega_1}(n)$ and set

$$q = \mu^+(p) \cup \{\ulcorner \underline{\omega_1} \mapsto \underline{\mu^+(\lambda)}\urcorner\}.$$

Claim 9.19. *$q \in \mu^+(\mathbb{P}^{\mathfrak{d}})$.*

Proof. Let $\mathfrak{C}^*$ denote

$$\langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle, \langle(k_n, \mu^+(\alpha_n)) \mid n < \omega\rangle, \langle\lambda^*_\xi, k^*_\xi, X^*_\xi \mid \xi \in K^*\rangle\rangle$$

where

$$\langle\langle M_i, \mu_{i,j}, N_i, \sigma_{i,j} \mid i \leqslant j \leqslant \kappa\rangle, \langle(k_n, \mu^+(\alpha_n)) \mid n < \omega\rangle, \langle\lambda^*_\xi, X^*_\xi \mid \xi \in K^*\rangle\rangle$$

is defined as in Claim 4.34, $k^*_\xi = k_\xi$ for $\xi \in K$ and $k^*_{\omega_1} = k$. Let Σ^+ precertify $\mathfrak{C}^*$ with $\mu^+[\bigcup g] \subseteq \Sigma^+$, we will show that Σ^+ indeed certifies $\mathfrak{C}^*$. We will only argue that $(\Sigma.9)^{\mathfrak{d}=\aleph_1}$ is satisfied at $\xi = \omega_1$. Note that $\mu^+(Z) = \mu^+(A_\lambda)$ is a $\mu^+(\lambda)$-code for a real in M^+ w.r.t. $\mu^+(\mathbb{P}^{\mathfrak{d}})$. It is easy to see that

$$\mu^+(Z)^{\Sigma^+} = Z^{\bigcup g}$$

and we have

$$Z^{\bigcup g}(n) \leqslant d_{\omega_1}(n)$$

for all $k \leqslant n < \omega$ by our choice of g, G. $\square$

Thus by elementarity, in V there must be $\xi < \omega_1$ so that

$$q := p \cup \{\ulcorner \underline{\xi} \mapsto \underline{\lambda}\urcorner\} \in \mathbb{P}^{\mathfrak{d}}.$$

It is now straightforward to show that $q \Vdash \dot{f} \leqslant^* \check{d}_\xi$. $\square$

9.3 $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ implies $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$

$\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ is a typical $\mathbb{P}_{\max}$-variation and this is witnessed by $\Psi^{\mathfrak{d}=\aleph_1}$ consisting of

- $\psi_0^{\mathfrak{d}=\aleph_1}(x) =$ "$x \in \dot{I}$",
- $\psi_1^{\mathfrak{d}=\aleph_1}(x) =$ "$x = \dot{\vec{d}}$",
- $\psi_2^{\mathfrak{d}=\aleph_1}(x) =$ "$x = \dot{a}$" and
- $\psi_3^{\mathfrak{d}=\aleph_1}(x) =$ "$x = \dot{\vec{d}} \wedge x$ is a dominating sequence".

Theorem 9.20. $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ *implies* $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$.

Proof. Assume $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$. By Fact 9.3 there is a good dominating sequence $\vec{d}$. Let $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$. We have already seen that SRP is a consequence of $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$ and thus NS_{ω_1} is saturated and ψ_{AC} holds. It follows that

$$(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \vec{d}, A)$$

is almost a $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-condition. Arguing as in the proof of the First Blueprint Theorem 4.44 with Theorem 9.10 in place of Theorem 4.20 shows that $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$ holds. □

Next up we define the relevant bounded forcing axioms in an effort to find an equivalence of $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$ in terms of a forcing axiom assuming large cardinals exist.

Definition 9.21. For $\Delta \subseteq \mathcal{P}(\mathbb{R})$, Δ-$\mathrm{BMM}^{++}(\mathfrak{d} = \aleph_1)$ holds if

(*i*) $\mathfrak{d} = \aleph_1$ and

(*ii*) Δ-$\mathrm{BFA}(\{\mathbb{P} \mid \mathbb{P}$ is ${}^\omega\omega$-bounding and stationary set preserving$\})$ is true.

$\mathrm{BMM}^{++}(\mathfrak{d} = \aleph_1)$ is $\varnothing$-$\mathrm{BMM}^{++}(\mathfrak{d} = \aleph_1)$.

Theorem 9.22. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(*i*) $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-$(*)$.

(*ii*) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(\mathfrak{d} = \aleph_1)$.

Proof. Results of Shelah-Zapletal in [SZ99] together with methods of Woodin imply that $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ is self-assembling.

Claim 9.23. *We have*

$$\Gamma^{\Psi^{\mathfrak{d}=\aleph_1}}_{(\vec{d},A)} = \Gamma^{\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}}_{(\vec{d},A)}(\Psi^{\mathfrak{d}=\aleph_1})$$

whenever $\vec{d}$ is a good dominating sequence and $A \subseteq \omega_1$ with $\omega_1^{L[A]} = \omega_1$.

Proof. Let $V[G]$ be any ${}^{\omega}\omega$-bounding and stationary set preserving extension of V. We have to show that there is a further ${}^{\omega}\omega$-bounding and stationary-set preserving extension of $V[G]$ in which $\mathcal{H}_{(\vec{d},A)}$ is almost a $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$-condition. We may assume CH in $V[G]$, otherwise force with $\mathrm{Add}(\omega_1,1)$. Observe that there is a proper class of Woodin cardinals in $V[G]$. Let $V[G][H]$ be an extension by the forcing $\mathbb{P}$ constructed in the proof of Theorem 3.60 in the case that $\mathbb{B}=\{\mathbb{1}\}$ is the trivial forcing. Thus in $V[G][H]$

$(H.i)$ NS_{ω_1} is saturated and

$(H.ii)$ ψ_{AC} holds.

$\mathbb{P}$ is a nice iteration of semiproper antichain-sealing forcings, forcings of the form $\mathbb{P}(S,T)$ from Lemma 3.57 and σ-closed forcings. All these forcings are semiproper and σ-distributive, the latter follows by arguments similar to the proofs of f-semiproperness 3.57 and the proof that antichain sealing forcings are stationary set preserving (if the antichain to be sealed is maximal). It follows from Fact 9.8 that $\mathbb{P}$ is semiproper and ${}^{\omega}\omega$-bounding. It follows that $\vec{d}$ is still a good dominating sequence in $V[G][H]$. □

The proof of the Second Blueprint Theorem 4.58 with Theorem 4.20 replaced by Theorem 9.10 gives the desired equivalence. □

10 Maximal Models of $\mathfrak{b}=\aleph_1$

We show that the results of section 9 also holds for the cardinal invariant dual to $\mathfrak{d}$, the bounding number invariant $\mathfrak{b}$.

Definition 10.1. (i) For $f\colon\omega\to\omega$ and $\mathcal{F}\subseteq{}^{\omega}\omega$, we say that f *dominates* $\mathcal{F}$ if f eventually dominates every member of $\mathcal{F}$.

(ii) A family $\mathcal{F}\subseteq\mathbb{R}$ is called *unbounded* if it is not dominated by any $f\in{}^{\omega}\omega$.

(iii) $\mathfrak{b}$ is the least size of an unbounded family, i.e.

$$\mathfrak{b}:=\min\{|\mathcal{F}|\mid\mathcal{F}\subseteq{}^{\omega}\omega\text{ is unbounded}\}.$$

In case that $\mathfrak{b}=\aleph_1$, there are particularly nice sequences witnessing this.

Definition 10.2. A sequence $\vec{u}=\langle u_\alpha\mid\alpha<\omega_1\rangle$ is called an *unbounded sequence* if

(i) $u_\alpha\leqslant^* u_\beta$ for all $\alpha<\beta<\omega_1$,

(ii) any u_α is increasing and

(iii) $\{u_\alpha\mid\alpha<\omega_1\}$ is an unbounded family.

$\vec{u}$ is a *good unbounded sequence* if additionally

(*iv*) for any $s \in {}^{<\omega}\omega$, the set

$$\{\alpha < \omega_1 \mid s \leqslant u_\alpha \restriction \mathrm{dom}(s) \text{ pointwise}\}$$

is stationary.

Fact 10.3 (Shelah-Zapletal, [SZ99])**.** *The following are equivalent:*

(*i*) $\mathfrak{b} = \aleph_1$.

(*ii*) *There is a good unbounded sequence.*

If $\vec{u}$ is an unbounded sequence, we say that a forcing $\mathbb{P}$ *preserves* $\vec{u}$ if $\vec{u}$ is still unbounded in $V^{\mathbb{P}}$.

10.1 The forcing axiom $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$

We want to identify the maximal MM^{++}-style forcing axiom conditioned on $\mathfrak{b} = \aleph_1$. In order to be able to prove the consistency of it, we will need a suitable iteration theorem.

Theorem 10.4. *Suppose $\vec{u}$ is an unbounded sequence. If*

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$$

is a nice iteration so that for all $\alpha < \gamma$

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathbb{Q}}_\alpha \textit{ is semiproper and preserves } \check{\vec{u}}\text{”}$$

then $\mathbb{P}$ preserves $\vec{u}$.

This can be recast to resemble Fact 9.8.

Corollary 10.5. *Assume* CH. *If $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ is a nice iteration of semiproper forcings of limit length and $\mathbb{P}_\alpha$ does not add dominating reals for all $\alpha < \gamma$, then $\mathbb{P}$ does not add dominating reals.*

We will prove Theorem 10.4 by applying yet another result of Miyamoto.

Fact 10.6 (Miyamoto, [Miy02])**.** *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a nice iteration of semiproper forcings, θ sufficiently large and regular, $X \prec H_\theta$ countable with $\mathbb{P} \in X$. For any limit ordinal $\beta \leqslant \gamma$, $p \in \mathbb{P}_\beta \cap X$ and $\langle E_n \mid n < \omega \rangle$, if for all $n < \omega$*

(*i*) $E_n \subseteq \mathbb{P}_\beta$ *and* (*ii*) *for all* $r \leqslant p$*, all* $\alpha < \beta$ *and any* $\mathbb{P}_\alpha$*-name* $\dot{Y}$ *we have*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\textit{if } \check{X} \sqsubseteq \dot{Y},\ \dot{G}_\alpha \in \dot{Y} \prec H_{\check{\theta}}^{V[\dot{G}_\alpha]} \textit{ is countable, } \check{r} \in \dot{Y},\ \check{r} \restriction \check{\alpha} \in \dot{G}_\alpha \\ \textit{then there is } s \in \check{E}_n \cap \dot{Y} \textit{ with } s \leqslant \check{r} \textit{ and } s \restriction \check{\alpha} \in \dot{G}_\alpha\text{”},$$

then there is $q \in \mathbb{P}_\beta$*,* $q \leqslant p$ *so that for all* $n < \omega$

$$q \Vdash \check{E}_n \cap \dot{G} \neq \varnothing.$$

Proof of Theorem 10.4. The proof is by induction on γ, so we may assume that $\mathbb{P}_\beta$ preserves $\vec{u}$ for all $\beta < \gamma$. Assume now that $p \in \mathbb{P}$ and p forces that $\dot{f}$ is an element of ${}^\omega\omega$. Let θ be sufficiently large and regular and $X \prec H_\theta$ countable with $\mathbb{P}, p, \vec{u}, \dot{f} \in X$. For $n < \omega$ define

$$E_n := \{q \in \mathbb{P} \mid \exists A \in [\omega]^n\ \forall m \in A\ q \Vdash \dot{f}(m) < \check{u}_{\delta^X}(m)\}.$$

We will show that the assumptions of Fact 10.6 are satisfied. Let $n < \omega$. Let $r \leqslant p$, $\alpha < \gamma$ and assume that G_α is $\mathbb{P}_\alpha$-generic and countable $Y \in V[G_\alpha]$ with

($Y.i$) $X \sqsubseteq Y \prec H_\theta^{V[G_\alpha]}$ and

($Y.ii$) $r, G_\alpha \in Y$ and $r \restriction \alpha \in G_\alpha$.

In Y, we may define a descending sequence $\langle r_m \mid m < \omega\rangle$ in $\dot{\mathbb{P}}_{\alpha,\gamma}^{G_\alpha}$ so that for all $m < \omega$

($\vec{r}.i$) $r_m \leqslant r$ and

($\vec{r}.ii$) $r_m \Vdash \dot{f}(\check{m}) = \check{i}_m$ for some $i_m < \omega$.

Define the function $h \in {}^\omega\omega$ by $h(m) = i_m$. Then $h \in Y$. As $\mathbb{P}_\alpha$ preserves $\vec{u}$, there is some $\xi < \delta^Y = \delta^X$ so that

$$u_\xi \nleqslant^* h$$

and as $u_\xi \leqslant^* u_{\delta^X}$ we have

$$u_{\delta^X} \nleqslant^* h.$$

Thus $r_m \in Y \cap E_n$ and $r_m \restriction \alpha \in G_\alpha$ for all large enough $m < \omega$.
It follows that there is $q \leqslant p$ so that

$$q \Vdash \forall n < \omega\ \check{E}_n \cap \dot{G} \neq \varnothing$$

and thus $q \Vdash \check{u}_{\delta^X} \nleqslant^* \dot{f}$. □

Definition 10.7. $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$ holds if

(*i*) $\mathfrak{b} = \aleph_1$ and

(*ii*) $\mathrm{FA}^{++}(\Gamma)$ holds where

$$\Gamma := \{\mathbb{P} \mid \mathbb{P} \text{ preserves stationary sets and all unbounded sequences}\}.$$

Note that this is once again a maximal forcing axiom. If $\vec{u}$ is an unbounded sequence and $\mathbb{P}$ is a forcing so that

$$\Vdash_{\mathbb{P}} \text{“}\vec{u} \text{ is bounded”}$$

then $\mathrm{FA}(\{\mathbb{P}\})$ fails.

As always, we first formulate a SPFA^{++}-style axiom which we show equivalent to $\mathrm{MM}^{++}(\mathfrak{d} = \aleph_1)$. It turns out that in this case, it does not matter whether we iterate all semiproper forcings which preserve a distinguished unbounded sequence or all semiproper forcings which preserve all unbounded sequences. Both approaches end up with “the same model”. This was trivially true before in Section 9, as one cannot destroy one dominating sequence without destroying all dominating sequences, but here the situation is more subtle. We reflect this in our presentation.

Definition 10.8. Let $\vec{u}$ be an ω_1-sequence of reals. $\mathrm{SPFA}^{++}(\vec{u})$ holds if $\vec{u}$ is an unbounded sequence and

$$\mathrm{FA}^{++}(\text{semiproper forcings preserving } \vec{u})$$

holds.

We will show the following.

Lemma 10.9. $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$ *is equivalent to* $\exists\vec{u}\ \mathrm{SPFA}^{++}(\vec{u})$.

Definition 10.10 (Shelah-Zapletal, [SZ99])**.** Two unbounded sequences

$$\langle u_\alpha \mid \alpha < \omega_1\rangle,\ \langle v_\alpha \mid \alpha < \omega_1\rangle$$

are *locked* if there is $x \in [\omega]^\omega$ so that for all $\alpha < \omega_1$

(*i*) there is $\beta < \omega_1$ with $u_\alpha \upharpoonright x \leqslant^* v_\beta \upharpoonright x$ and

(*ii*) there is $\beta < \omega_1$ with $v_\alpha \upharpoonright x \leqslant^* u_\beta \upharpoonright x$.

Fact 10.11 (Shelah-Zapletal, [SZ99])**.** *There is a forcing*[52] $\mathbb{P}$ *so that*

($\mathbb{P}$.*i*) $\mathbb{P}$ *is proper,*

($\mathbb{P}$.*ii*) $\mathbb{P}$ *preserves all unbounded sequences and*

($\mathbb{P}$.*iii*) *any two unbounded sequences in* V *are locked in* $V^{\mathbb{P}}$.

[52] In fact, the forcing is simply Mathias forcing.

The following is an immediate consequence.

Proposition 10.12. *If $\vec{u}$ is an unbounded sequence and* $\mathrm{SPFA}^{++}(\vec{u})$ *holds then any forcing that preserves $\vec{u}$ in fact preserves all unbounded sequences.*

The following can be proven exactly as in Section 9.

Proposition 10.13. $\exists\vec{u}$ $\mathrm{SPFA}^{++}(\vec{u})$ *implies* SRP *and hence that all stationary set preserving forcings are semiproper.*

Lemma 10.9 is a consequence of Proposition 10.12 and 10.13.

Theorem 10.14. *If* ZFC + "*there is a supercompact cardinal*" *is consistent, then so is* $\mathrm{ZFC} + \mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$.

Proof. (Sketch) Start with a model $V \models \mathrm{ZFC} + \mathrm{CH} +$ "κ is supercompact". Thus there is an unbounded sequence $\vec{u}$. Let $\mathbb{P}$ be the nice iteration of semiproper forcings preserving $\vec{u}$ with the iterands chosen by some Laver function on κ. Then $\mathbb{P}$ is semiproper and preserves $\vec{u}$ by Theorem 10.4, hence $\mathfrak{b} = \aleph_1$ in $V^{\mathbb{P}}$. The usual argument shows that

$$V^{\mathbb{P}} \models \mathrm{FA}^{++}(\text{semiproper forcings preserving } \vec{u})$$

thus $\mathrm{SPFA}^{++}(\vec{u})$ holds in $V^{\mathbb{P}}$. By Lemma 10.9, $\mathbb{P}$ forces $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$. $\square$

$P_{\mathfrak{b}=\aleph_1}$ and $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$ are defined in the same way as $P_{\mathfrak{d}=\aleph_1}$ and $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ with "good dominating sequence" replaced by "good unbounded sequence". We denote the order on $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$ by $<_{\mathfrak{b}=\aleph_1}$. Shelah-Zapletal have proven analogous results for $P_{\mathfrak{b}=\aleph_1}$ as they did for $P_{\mathfrak{d}=\aleph_1}$.

10.2 $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$-$(*)$-forcing

Theorem 10.15. *Suppose that*

(*i*) *generic projective absoluteness holds for generic extensions by forcings of size* ω_2,

(*ii*) NS_{ω_1} *is saturated and* $\mathcal{P}(\omega_1)^\sharp$ *exists,*

(*iii*) $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \vec{u}, A)$ *is almost a* $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$*-condition and*

(*iv*) $D \subseteq \mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ *is* 2^{ω_2}*-universally Baire and dense in* $\mathbb{P}_{\max}^{\mathfrak{d}=\aleph_1}$ *in any generic extension by a forcing of size* 2^{ω_2}*, as witnessed by trees* T, S *with* $p[T] = D$.

Then there is a $\vec{u}$-preserving forcing $\mathbb{P}^{\mathfrak{b}}$ so that in $V^{\mathbb{P}^{\mathfrak{b}}}$ the following picture exists

$$\begin{array}{ccccc}
 & & p[T] & & \\
 & & \Cup & & \\
 & & q_0 & \xrightarrow{\quad\sigma_{0,\omega_1}\quad} & q_{\omega_1} \\
 & & \Cup & & \Cup \\
p_0 & \xrightarrow{\quad\mu_{0,\omega_1^N}\quad} & p_{\omega_1^N} & \xrightarrow{\quad\mu_{\omega_1^N,\omega_1}\quad} & p_{\omega_1} \\
\Cap & & & & \| \\
\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1} & & & & ((H_{\omega_2})^V, \mathrm{NS}_{\omega_1}^V, \vec{u}, A)
\end{array}$$

so that

($\mathbb{P}^{\mathfrak{b}}$.*i*) $\mu_{0,\omega_1}, \sigma_{0,\omega_1}$ *are generic iterations of* p_0, q_0 *respectively,*

($\mathbb{P}^{\mathfrak{b}}$.*ii*) μ_{0,ω_1^N} *witnesses* $q_0 <_{\mathfrak{b}=\aleph_1} p_0$,

($\mathbb{P}^{\mathfrak{b}}$.*iii*) $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$ *and*

($\mathbb{P}^{\mathfrak{b}}$.*iv*) $\sigma_{0,\omega_1} : q_0 \to q_{\omega_1}$ *is a correct iteration.*

The argument is similar to the proof of Theorem 9.10. Assume $\diamondsuit_\kappa$ and define the club C and forcings $(\mathbb{P}^{\mathfrak{b}}_\lambda)_{\lambda \in C \cup \{\kappa\}}$ by induction as usual. In the definition of potential certificates, the good dominating sequences $\vec{d}$ are now replaced by good unbounded sequences $\vec{u}$. In this case, we do not have to add additional information to certificates. The correct "genericity conditions" for a syntactic λ-precertificate Σ to be a full λ-certificate are now:

$(\Sigma.8)^{\mathfrak{b}=\aleph_1}$ If $\xi \in K$ and $E \subseteq \mathbb{P}^{\mathfrak{b}}_{\lambda_\xi}$ is dense and definable over

$$(Q_\lambda; \in, \mathbb{P}^{\mathfrak{b}}_{\lambda_\xi}, A_{\lambda_\xi})$$

from parameters in X_{λ_ξ} then

$$E \cap X_\xi \cap [\Sigma]^{<\omega} \neq \varnothing$$

and

$(\Sigma.9)^{\mathfrak{b}=\aleph_1}$ whenever $\xi \in K$ and $Z = A_{\lambda_\xi}$ is a λ_ξ-code for a real then $u_\xi \nleq^* Z^\Sigma$.

In this case it might be more natural to amalgamate both conditions as in Section 4, but we try to keep as close to Section 9 as possible.

Lemma 10.16. $\mathbb{P}^{\mathfrak{b}}_{\min(C)} \neq \varnothing$.

The following can be proved in a similar fashion as Lemma 9.13.

Lemma 10.17. *If* $\lambda \in C \cup \{\kappa\}$ *and* $g \subseteq \mathbb{P}^{\mathfrak{b}}_\lambda$ *is a filter that meets and dense* $E \subseteq \mathbb{P}^{\mathfrak{b}}_\lambda$ *definable with parameters over*

$$(Q_\lambda; \in, \mathbb{P}^{\mathfrak{b}}_\lambda, A_\lambda)$$

then $\bigcup g$ *is a syntactic* λ*-certificate.*

Lemma 10.18. $\mathbb{P}^{\mathfrak{b}}$ *preserves stationary sets.*

Relevant to this specific situation is now:

Lemma 10.19. $\vec{u}$ *is an unbounded sequence in* $V^{\mathbb{P}^{\mathfrak{b}}}$*, in particular* $\mathbb{P}^{\mathfrak{b}}$ *does not add dominating reals.*

Fact 10.20 (Shelah-Zapletal, [SZ99])**.** *Let M be a countable transitive model of* ZFC *and let $\mathbb{Q} \in M$ be a forcing, $\dot{f} \in M^{\mathbb{Q}}$ with*

(*i*) $M \models \Vdash_{\mathbb{Q}}$ "*$\dot{f}$ is an unbounded real*" *and*

(*ii*) *for any $s \in {}^{<\omega}\omega$, there is $p \in \mathbb{Q}$ with*

$$M \models p \Vdash \text{“}\check{s} \leqslant \dot{f} \restriction \mathrm{dom}(\check{s}) \textit{ pointwise.”}$$

Let $h \in {}^{\omega}\omega$, not necessarily in M. Then there is a filter $G \subseteq \mathbb{Q}$ generic over M so that $\dot{f}^G \nleqslant^* h$.

This is not explicitly stated in [SZ99] but follows immediately from Claim 5.9 of that paper.

Proof of Lemma 10.19. Let $p \in \mathbb{P}^{\mathfrak{b}}$ and $\dot{f}$ a name for an element of ${}^{\omega}\omega$ and let Z be the natural κ-code for a real for $\dot{f}$. Repeat the argument of Lemma 10.19. The version of Claim 9.18 here now demands of g, G that

$$u_{\omega_1} \nleqslant^* Z^{\bigcup g}$$

which can be achieved using Fact 10.20. To do this, use that $\vec{u}$ is a *good* unbounded sequence. The rest of the argument works as before and shows that we can find $\xi < \omega_1$ so that

$$q = p \cup \{\ulcorner \underline{\xi} \mapsto \underline{\lambda} \urcorner\} \in \mathbb{P}^{\mathfrak{b}}$$

where $\lambda \in C$ is large enough with $A_\lambda = Z \cap Q_\lambda$. Clearly $q \Vdash \check{u}_\xi \nleqslant^* \dot{f}$. □

10.3 $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$ implies $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$-$(*)$

$\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$ is a typical $\mathbb{P}_{\max}$-variation and this is witnessed by $\Psi^{\mathfrak{b}=\aleph_1}$ consisting of

- $\psi_0^{\mathfrak{b}=\aleph_1}(x) = \text{“}x \in \dot{I}\text{”}$,
- $\psi_1^{\mathfrak{b}=\aleph_1}(x) = \text{“}x = \dot{\vec{u}}\text{”}$,
- $\psi_2^{\mathfrak{b}=\aleph_1}(x) = \text{“}x = \dot{a}\text{”}$ and
- $\psi_3^{\mathfrak{b}=\aleph_1}(x) = \text{“}x = \dot{\vec{u}} \wedge x \text{ is an unbounded sequence”}$.

Theorem 10.21. $\mathrm{MM}^{++}(\mathfrak{b} = \aleph_1)$ *implies* $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$-$(*)$.

Proof. One can argue analogous to the proof of Theorem 9.20. Use Theorem 10.15 instead of Theorem 9.10. □

Theorem 10.22. *Suppose there is a proper class of Woodin cardinals. The following are equivalent:*

(*i*) $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$-$(*)$.

(*ii*) $(\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$-$\mathrm{BMM}^{++}(\mathfrak{b} = \aleph_1)$.

Proof. Results in [SZ99] imply that $\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}$ is self-assembling. A similar proof as for Theorem 9.22 works. Use Theorem 10.4 instead of Fact 9.8 to show

$$\Gamma_{(\vec{u},A)}^{\Psi^{\mathfrak{b}=\aleph_1}} = \Gamma_{(\vec{u},A)}^{\mathbb{P}_{\max}^{\mathfrak{b}=\aleph_1}}(\Psi^{\mathfrak{b}=\aleph_1})$$

whenever $\vec{u}$ is a good unbounded sequence and $A \subseteq \omega_1$ satisfies $\omega_1^{L[A]} = \omega_1$. □

11 Appendix

We prove a few more results mostly related to $\diamondsuit$-forcing that did not fit in earlier.

11.1 A version of Martin's axiom conditioned on $\diamondsuit(\mathbb{B})$

The following is almost certainly due to Shelah.

Fact 11.1. *The following are equivalent for any forcing* $\mathbb{P}$*:*

(*i*) $\mathbb{P}$ *satisfies the countable chain condition.*

(*ii*) *For any sufficiently large regular* θ *and any countable* $X \prec H_\theta$ *with* $\mathbb{P} \in X$*, any* $p \in \mathbb{P}$ *is* $(X, \mathbb{P})$*-generic.*

See for example [Mek84]. This suggest a natural version of c.c.c. forcings conditioned on a witness f of $\diamondsuit(\mathbb{B})$.

Definition 11.2. Suppose f witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is f*-c.c.c.* if for any sufficiently large regular θ and any f-slim $X \prec H_\theta$, any condition in $\mathbb{P}$ is $(X, \mathbb{P}, f)$-generic.

We have f-c.c.c. $\Rightarrow$ f-proper$\wedge$ c.c.c.. Lemma 5.65 gives an example of a f-proper c.c.c. forcing which is not f-c.c.c..

We get a natural analog of Fact 11.1.

Proposition 11.3. *Suppose* f *witnesses* $\diamondsuit(\mathbb{B})$*. The following are equivalent for any forcing* $\mathbb{P}$*:*

(*i*) *Whenever* θ *is sufficiently large regular,* $X \prec H_\theta$ *is* f*-slim with* $\mathbb{P} \in X$ *and* $M_X[f(\delta^X)] \models$ "$\mathcal{A} \subseteq \bar{\mathbb{P}}$ *is a maximal antichain*" *then* $\pi_X[\mathcal{A}]$ *is a maximal antichain in* $\mathbb{P}$.

(*ii*) $\mathbb{P}$ *is* f*-c.c.c..*

It is straightforward to see that if $\mathbb{P}$ is f-c.c.c. and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is f-c.c.c." then $\mathbb{P} * \dot{\mathbb{Q}}$ is f-c.c.c..

Lemma 11.4. *Suppose* f *witnesses* $\diamondsuit(\mathbb{B})$ *and* $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ *is a finite support iteration of* f*-c.c.c. forcings. Then* $\mathbb{P}$ *is* f*-c.c.c..*

Proof. We proceed by induction and only prove the limit step. So assume that $\gamma \in \mathrm{Lim}$ and let θ be sufficiently large and regular, $X \prec H_\theta$ be f-slim with $\mathbb{P} \in X$. Suppose that $D \in \mathbb{M}_X[f(\delta^X)]$ is dense in $\bar{\mathbb{P}}$, we will show that $\pi_X[D]$ is predense in $\mathbb{P}$. Suppose $p \in \mathbb{P}$. As γ is a limit, there is $\alpha \in X \cap \gamma$ so that $\mathrm{supp}(p) \cap X \subseteq \alpha$. Let

$$E = \{q \in \bar{\mathbb{P}}_{\bar{\alpha}} \mid \exists r \in D\ r \upharpoonright \bar{\alpha} = q \upharpoonright \bar{\alpha}\}$$

and note that $E \in M_X[f(\delta^X)]$ is dense in $\bar{\mathbb{P}}_{\bar{\alpha}}$. By induction, $\mathbb{P}_\alpha$ is f-c.c.c. and hence $\pi_X[E]$ is predense in $\mathbb{P}_\alpha$. Find some $q \in \mathbb{P}_\alpha$ with $q \leqslant p \upharpoonright \alpha$ so that q is below some condition $\pi_X(r) \in \pi_X[E]$. By definition of E, there is $s \in \bar{\mathbb{P}}$ so that $s \upharpoonright \bar{\alpha} = r$ and $s \in D$. By choice of α,

$$\mathrm{supp}(p) \cap (\mathrm{supp}(\pi_X(s)) - \mathrm{supp}(q)) = \varnothing$$

and hence $t = q \cup \pi_X(s)$ is a condition, $t \leqslant p$ and t is clearly below a condition in $\pi_X[D]$. $\square$

This allows us to force a fragment of Martin's Axiom that is consistent with $\diamondsuit(\mathbb{B})$ with f-c.c.c. forcing.

Definition 11.5. Suppose f witnesses $\diamondsuit(\mathbb{B})$ and κ is an uncountable cardinal.

(*i*) $\mathrm{MA}_\kappa(f)$ holds if for any f-c.c.c. forcing $\mathbb{P}$ and any collection $\mathcal{D}$ of at most κ-many dense subsets of $\mathbb{P}$ there is a filter $g \subseteq \mathbb{P}$ with $g \cap D \neq \varnothing$ for all $D \in \mathcal{D}$.

(*ii*) $\mathrm{MA}(f)$ states that $\mathrm{MA}_\kappa(f)$ holds for all uncountable cardinals $\kappa < 2^\omega$.

Suppose κ is weakly compact, that is for every $A \subseteq \kappa$ there is a κ-model[53] and an elementary embedding $j\colon M \to N$ with $\mathrm{crit}(j) = \kappa$ and N transitive. We say that $L\colon \kappa \to V_\kappa$ is a *Laver function for* κ *weakly compact* if for any $A, B \subseteq \kappa$ there is $j\colon M \to N$ as before with $A \in M$ and $j(L)(\kappa) = B$.

[53] A κ-model is a model transitive M of sufficiently much of ZFC of size κ with $\kappa+1 \subseteq M$ and $M^{<\kappa} \subseteq M$

Fact 11.6 (Hamkins, [Ham00]). *If κ is weakly compact then there is a Laver function for κ weakly compact in a forcing extension which does not add subsets of ω_1.*

Theorem 11.7. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and there is a Laver function for κ weakly compact. Then there is a f-c.c.c. forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$*

(i) $2^\omega = \kappa$ *and*

(ii) $\mathrm{MA}(f)$ *holds.*

Proof (Sketch). Let L be a Laver function for κ weakly compact. Let $\mathbb{P}$ be the finite support iteration of f-c.c.c. forcings as guessed by L. Then $\mathbb{P}$ is f-c.c.c. and as Cohen forcing is f-c.c.c., $2^\omega = \kappa$ in $V^{\mathbb{P}}$. The typical lifting argument shows that in $V^{\mathbb{P}}$, $\mathrm{MA}(f)$ holds restricted to all forcings of size κ. This suffices for full $\mathrm{MA}(f)$: Suppose $\mathbb{Q}$ is any f-c.c.c. forcing in $V^{\mathbb{P}}$ and $\mathcal{D}$ is a set of dense subsets of $\mathbb{Q}$ of size $<\kappa$. We have that $\mathbb{Q}$ is c.c.c. and $\kappa = 2^\omega$, so $\mathbb{Q}$ has a complete subforcing $\mathbb{Q}_0 \subseteq \mathbb{Q}$ with $D \cap \mathbb{Q}_0$ dense in $\mathbb{Q}_0$ for all $D \in \mathcal{D}$. Clearly, $\mathbb{Q}_0$ if f-c.c.c. and hence there is a filter $g \subseteq \mathbb{Q}_0$ meeting all $D \cap \mathbb{Q}_0$ for $D \in \mathcal{D}$ which in turn generates a filter $\hat{g} \subseteq \mathbb{Q}$ which meets all $D \in \mathcal{D}$.

□

It seems likely that the large cardinal can be removed, though we have not tried to do so. If additionally f witnesses $\diamondsuit^+(\mathbb{B})$ in V, then f will still witness $\diamondsuit^+(\mathbb{B})$ in $V^{\mathbb{P}}$, so we can have $\mathrm{MA}(f)$ + "f witnesses $\diamondsuit^+(\mathbb{B})$" and the continuum large.

Lemma 11.8. *Suppose f witnesses $\diamondsuit^+(\mathbb{B})$. Then f witnesses $\diamondsuit^+(\mathbb{B})$ in any generic extension by f-c.c.c. forcing.*

Proof. Suppose $\mathbb{P}$ is f-c.c.c., G is $\mathbb{P}$-generic over V and θ is sufficiently large, regular. Let $S \in V[G]$ be stationary in ω_1, we have to show that S is f-stationary. Let $\mathcal{C} \subseteq \left[H_\theta^{V[G]}\right]^\omega$ be a club. As $\mathbb{P}$ is c.c.c., we can find some countable $Y \prec H_\theta^{V[G]}$ with

$(X.i)$ $f, \kappa \in Y \in \mathcal{C}$,

$(X.ii)$ $\delta^Y \in S$,

$(X.iii)$ $X := Y \cap H_\theta^V \in V$ and

$(X.iv)$ $Y = X[G]$.

As f witnesses $\diamondsuit^+(\mathbb{B})$ in V, X is f-slim and as $\mathbb{P}$ is f-c.c.c., G trivially contains a $(X, \mathbb{P}, f)$-generic condition, hence $Y = X[G]$ is f-slim. □

11.2 More consequences of PFA(f)

Many interesting structural consequences of PFA are in fact already implied by Todorčević's P-ideal dichotomy or by Moore's Mapping Reflection Principle (MRP). Among them are e.g. $2^\omega = \omega_2$, global failure of square and the Singular Cardinal Hypothesis. Unfortunately, if $\diamondsuit(\omega_1^{<\omega})$ holds then both the P-ideal dichotomy and MRP fail. For The P-ideal dichotomy, this immediately follows from a result of Todorčević.

Fact 11.9 (Todorčević, [Tod00])**.** *If the P-ideal dichotomy holds then there are no Suslin trees.*

By Fact 5.6, $\diamondsuit(\omega_1^{<\omega})$ entails the existence of a Suslin tree and hence the failure of the P-ideal dichotomy. We now introduce MRP.

Definition 11.10. Suppose A is an uncountable set.

(i) For $a \in [A]^{<\omega}$ and $N \in [A]^\omega$, we define
$$[a, N] = \{B \in [A]^\omega \mid a \subseteq B \subseteq N\}.$$

(ii) The *Ellentuck topology* on $[A]^\omega$ is generated by basic open sets $[a, N]$ for all $a \in [A]^{<\omega}$ and $N \in [A]^\omega$.

(iii) We say that a map Σ with range in $\mathcal{P}([A]^\omega)$ is open if $\Sigma(X)$ is open in $[A]^\omega$ equipped with the Ellentuck topology for all $X \in \mathrm{dom}(\Sigma)$.

(iv) If θ is regular and $X \prec H_\theta$ is countable with $A \in X$ then a set $\mathcal{S} \subseteq [A]^\omega$ is X-stationary if $\mathcal{C} \cap \mathcal{S} \cap X$ is nonempty for all clubs $\mathcal{C} \subseteq [A]^\omega$, $\mathcal{C} \in X$.

Definition 11.11 (Moore, [Moo05])**.** Suppose A is uncountable and
$$\Sigma\colon \mathcal{C} \to \mathcal{P}([A]^\omega)$$
so that

($\Sigma.i$) for some regular $\theta = \theta^\Sigma \geqslant \omega_2$, $\mathcal{C}$ is a club of countable elementary substructures of H_θ and

($\Sigma.ii$) Σ is open and for all $X \in \mathcal{C}$, $\Sigma(X)$ is X-stationary in $[A]^\omega$.

Then we call Σ an *open stationary set mapping*.

Definition 11.12 (Moore, [Moo05])**.** The *Mapping Reflection Principle* MRP holds if for any open stationary set mapping
$$\Sigma\colon \mathcal{C} \to \mathcal{P}([A]^\omega)$$
there is a continuous increasing chain $\vec{X} := \langle X_\alpha \mid \alpha < \omega_1\rangle$ of elements of $\mathcal{C}$ so that if $\alpha \in \omega_1 \cap \mathrm{Lim}$ then there is some $\beta < \alpha$ with
$$\forall \xi \in (\beta, \alpha)\ X_\xi \cap A \in \Sigma(X_\alpha).$$

Proposition 11.13. MRP *implies* $\neg\diamondsuit(\omega_1^{<\omega})$.

Proof. Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$. We employ the same trick as in the proof of Proposition 5.15, namely we consider $f(\alpha)$ as a maximal filter in $\mathrm{Col}_{\mathrm{inc}}(\omega,\alpha)$ for all $\alpha < \omega_1$. Let $r_\alpha = \mathrm{ran}(\bigcup f(\alpha))$ for $\alpha < \omega_1$ and we will assume that r_α is cofinal in α for all limit $\alpha < \omega_1$. Let $\mathcal{C}$ be the set of all countable $X \prec H_{\omega_2}$. We define $\Sigma\colon \mathcal{C} \to [\omega_1]^\omega$ by

$$\Sigma(X) = \{Y \in [\delta^X]^\omega \mid \sup Y \notin r_{\delta^X}\}.$$

If $Y \in \Sigma(X)$ is nonempty then there is a maximal point $\alpha \in r_{\delta^X} \cup \{0\}$ below $\sup Y$. If $\beta \in Y, \beta > \alpha$ then $[\{\beta\}, Y] \subseteq \Sigma(X)$ and hence Σ is open. Now we show that $\Sigma(X)$ is X-stationary. If $C \in X$, $C \subseteq \omega_1$ is club then, as $\mathrm{otp}(C \cap \delta^X) = \delta^X$ and $\mathrm{otp}(r_{\delta^X}) = \omega$, it follows easily that there is $\alpha \in C \cap \mathrm{Lim} \cap \delta^X$ with $\alpha \notin r_{\delta^X}$. Hence $\alpha \in C \cap X \cap \Sigma(X)$.
Let $\vec{X} := \langle X_\alpha \mid \alpha < \omega_1\rangle$ witness the instance of MRP given by Σ. Let $C = \{\alpha < \omega_1 \mid \alpha = \delta^{X_\alpha}\}$. Note that this is club and for all $\alpha \in C$, $r_\alpha \cap C$ is bounded in α. Let $\theta > \omega_2$ be regular. As f witnesses $\diamondsuit(\omega_1^{<\omega})$, we can find some f-slim $Y \prec H_\theta$ with $C \in Y$. As $f(\delta^Y)$ is generic over Y, $r_{\delta^Y} \cap C$ is unbounded in δ^Y, but also $\delta^Y \in C$, contradiction. $\square$

Remark 11.14. The relevant instance of MRP used above is Example 2.5 in [Moo05].

As SRP and MRP are both reflection principles that follow from MM and PFA respectively and already imply many of their celebrated consequences, it is natural to ask about the relation between these principles. As MM is stronger than PFA, one should not expect MRP $\Rightarrow$ SRP and indeed there are well known models in which MRP holds and SRP fails. For example Beaudoin [Bea91] constructs a model of PFA in which there is a stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cof}(\alpha) = \omega\}$ which does not reflect[54]. All such sets reflect under SRP, so SRP fails. Also Shelah [She98, XVII Theorem 3.3] shows that PFA does not imply "NS_{ω_1} is saturated".
What about the reverse implication? As of yet, no model of SRP $\wedge$ $\neg$MRP appears in the literature.

Corollary 11.15. *Suppose* ZFC + "*There is a supercompact cardinal*" *is consistent. Then so is* ZFC + SRP + $\neg$MRP.

Any model of MM(f) where f witnesses $\diamondsuit(\omega_1^{<\omega})$ works. Justin Moore asked the author whether there are strong club guessing sequences in these models of SRP + $\neg$MRP, as they would constitute strong counterexamples to MRP. For sets a, b, $a \subseteq^* b$ holds if $a - b$ is finite.

[54] $S \subseteq \omega_2$ reflects if there is some $\alpha < \omega_2$ of cofinality ω_1 so that $S \cap \alpha$ is stationary in α.

Definition 11.16. A *strong club guessing sequence* is a sequence

$$\langle C_\alpha \mid \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$$

so that

($\vec{C}.i$) $C_\alpha \subseteq \alpha$ is cofinal and of ordertype ω for all $\alpha \in \mathrm{Lim} \cap \omega_1$

($\vec{C}.ii$) For any club $C \subseteq \omega_1$ we have $C_\alpha \subseteq^* C$ for all but possibly nonstationary many $\alpha \in C \cap \mathrm{Lim}$.

We will show that under QM there are no such sequences. The technique we use to do so shows that even though QM does not have a $+$, some arguments that seem to require a $+$-style forcing axiom can be carried out for QM.

Lemma 11.17. *If* BQM *holds then there is no strong club guessing sequence.*

Proof. Suppose $\vec{C} := \langle C_\alpha \mid \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ satisfies ($\vec{C}.i$) from the definition of a strong club guessing sequence. We will show that ($\vec{C}.ii$) fails. Suppose that f witnesses BQM.

Claim 11.18. *In $V^{\mathrm{Add}(\omega_1,1)}$, there is a club C so that*

$$\{\alpha \in C \mid C_\alpha \nsubseteq^* C\} \cap T \in \mathrm{NS}_f^+$$

for all f-stationary $T \in V$.

Proof. Let θ be large enough regular, $X \prec H_\theta$ be f-slim and $p \in \mathrm{Add}(\omega_1, 1) \cap X$. Then it is straightforward to find some $q \leqslant p$ which is $(X, \mathrm{Add}(\omega_1, 1), f)$-generic so that $C_{\delta^X} \nsubseteq^* \mathrm{Lim}(q^{-1}[\{1\}])$. This shows that if G is $\mathrm{Add}(\omega_1, 1)$-generic and $A \subseteq \omega_1$ is the generic subset added by G then $C = \mathrm{Lim}(A)$ works. □

Let G be $\mathrm{Add}(\omega_1, 1)$-generic and $C \in V[G]$ as provided by Claim 11.18. Let

$$S := \{\alpha \in C \mid C_\alpha \nsubseteq^* C\}$$

and note that $\omega_1 - S$ does not contain any of the S_p^f mod NS_f. If $\mathbb{P}$ is the forcing to shoot a club through S then the argument from Lemma 7.2 shows that $\mathbb{P}$ is f-preserving in $V[G]$. Thus in $V[G]^{\mathbb{P}}$, S contains a club and hence the statement "There are clubs $C, D \subseteq \omega_1$ with $C_\alpha \nsubseteq C$ for all $\alpha \in C \cap D$" reflects down to V by BQM. □

Fact 11.9 and Proposition 11.13 leave open how to derive for example global failure of square from $\mathrm{PFA}(f)$. It also does not seem like Todorčević's original proof of this in [Tod84] from PFA generalizes to $\mathrm{PFA}(f)$. Nonetheless, we aim to give a proof of the following result.

Theorem 11.19. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and* PFA(f) *holds. Then*

(i) $\neg\Box(\kappa)$ *for all cardinals* $\kappa \geqslant \omega_2$ *and*

(ii) SCH *holds.*

We will do so by suitably modifying MRP. More concretely, we will adapt the notion of X-stationarity to the $\diamondsuit$-context as we should take into account clubs that "come from $M_X[f(\delta^X)]$". The proof of Proposition 11.13 will then no longer go through as it relied on $\mathrm{otp}(C \cap \delta^X) = \delta^X$ for a club $C \in X$. However, arguments on larger structures should not be affected as much as collapsing ω_1 is a small forcing.

Definition 11.20. Suppose A is an uncountable set,f witnesses $\diamondsuit(\mathbb{B})$, $\theta \geqslant \omega_2$ is regular and $X \prec H_\theta$ is f-slim with $A \in X$. A set $\mathcal{S} \subseteq [A]^\omega$ is *X-f-stationary* in $[A]^\omega$ if for all $C \in M_X[f(\delta^X)]$ so that

$$M_X[f(\delta^X)] \models \text{“}C \text{ is club in } [\bar{A}]^\omega\text{”}$$

there is some $Y \in C$ with $\pi_X[Y] \in \mathcal{S}$.

Definition 11.21. Suppose A is uncountable, f witnesses $\diamondsuit(\mathbb{B})$ and

$$\Sigma\colon \mathcal{C} \to \mathcal{P}([A]^\omega)$$

with

($\Sigma.i$) for some regular $\theta = \theta^\Sigma \geqslant \omega_2$, $\mathcal{C}$ consists of f-slim substructures $X \prec H_\theta$ with $A \in X$,

($\Sigma.ii$) almost all f-slim $X \prec H_\theta$ are in $\mathcal{C}$, i.e. $[H_\theta]^\omega - \mathcal{C}$ is f-nonstationary and

($\Sigma.iii$) Σ is open and for all $X \in \mathcal{C}$, $\Sigma(X)$ is X-f-stationary in $[A]^\omega$.

Then we call Σ an *open f-stationary set mapping*.

Definition 11.22. If f witnesses $\diamondsuit(\mathbb{B})$ then the *f-Mapping Reflection Principle* f-MRP holds if for any open f-stationary set mapping $\Sigma\colon \mathcal{C} \to \mathcal{P}([A]^\omega)$ there is a continuous $\in$-chain $\langle X_\alpha \mid \alpha < \omega_1\rangle$ of countable elementary substructures of H_{θ^Σ} so that for

$$S = \{\alpha < \omega_1 \mid X_\alpha \text{ is } f\text{-slim}\}$$

we have

($\vec{X}.i$) if $\alpha \in S$ then $X_\alpha \in \mathcal{C}$ and

($\vec{X}.ii$) if $\alpha \in S \cap \mathrm{Lim}$ then there is some $\beta < \alpha$ so that for all $\beta < \xi < \alpha$, $X_\xi \cap A \in \Sigma(X_\alpha)$.

Lemma 11.23. *Suppose f witnesses $\Diamond(\mathbb{B})$. Then* $\mathrm{PFA}(f)$ *implies* f-MRP.

Proof. Suppose $\Sigma\colon \mathcal{C} \to \mathcal{P}([A]^\omega)$ is an open f-stationary set mapping and let $\theta = \theta^\Sigma$. We may assume $A \in H_\theta$. Let $\mathbb{P}$ be the forcing consisting of approximations of successor length to a sequence that f-MRP dictates to exist in the instance of Σ. More precisely, a condition $p \in \mathbb{P}$ is a continuous $\in$-chain

$$p = \langle X_\alpha \mid \alpha \leqslant \beta\rangle$$

of length some $\beta + 1 < \omega_1$ so that $(\vec{X}.i)$ and $(\vec{X}.ii)$ hold for

$$S = S^p := \{\alpha \leqslant \beta \mid X_\alpha \text{ is } f\text{-slim}\}.$$

The order on $\mathbb{P}$ is given by end-extensions. We will show that $\mathbb{P}$ is f-proper. Let $\lambda > \theta^+$ be sufficiently large and regular, $X \prec H_\lambda$ be f-slim with $\mathbb{P} \in X$.

Claim 11.24. *For any $p \in \mathbb{P} \cap X$ and dense $D \subseteq \pi_X^{-1}(\mathbb{P})$, $D \in M_X[f(\delta^X)]$, there is $q \leqslant p$ so that $q \in \pi_X[D]$ and $\Sigma(Y_\alpha) \cap A \in \Sigma(X \cap H_\theta)$ for all $\alpha \in \mathrm{length}(q) - \mathrm{length}(p)$.*

Proof. As $\Sigma(X \cap A)$ is X-f-stationary, we can find some $Y \prec H_{\theta^+}$ so that

$(Y.i)$ $p, \Sigma \in Y \subseteq X$,

$(Y.ii)$ $\pi_X[D] \cap Y$ is dense in $\mathbb{P} \cap Y$ and

$(Y.iii)$ $Y \cap A \in \Sigma(X \cap H_\theta)$

since the set

$$\{\pi_X^{-1}[Y] \mid Y \prec H_{\theta^+} \text{ satisfies } (Y.i) \text{ and } (Y.ii)\} \cap M_X[f(\delta^X)]$$

is believed by $M_X[f(\delta^X)]$ to contain a club in $[\pi_X^{-1}(H_{\theta^+})]^\omega$. As Σ is open, there is some finite $a \subseteq Y \cap A$ so that $[a, Y \cap A] \subseteq \Sigma(X \cap H_\theta)$ and note that $a \in X$. Now we can easily find $Z \in Y$ so that $a \subseteq Z$ and

$$p' = p^\frown Z$$

is a condition in $\mathbb{P}$. By $(Y.ii)$, there is $q \in \pi_X[D]$, $q \in Y$ so that $q \leqslant p'$. If

$$q = \langle Z_\alpha \mid \alpha \leqslant \beta\rangle$$

and Z_α does not appear in p, then $a \subseteq Z_\alpha \in Y$ and hence $Z_\alpha \in [a, Y] \subseteq \Sigma(X \cap H_\theta)$ as desired. $\square$

Now suppose $p \in \mathbb{P} \cap X$. Using the claim above, it is straightforward to construct a sequence $\vec{p} = \langle p_n \mid n < \omega\rangle$ descending in $\mathbb{P}$ so that

$(\vec{p}.i)$ $p_0 = p$,

($\vec{p}.ii$) for any dense $D \subseteq \pi_X^{-1}(\mathbb{P})$, $D \in M_X[f(\delta^X)]$, there is $n < \omega$ so that $p_n \in \pi_X[D]$ and

($\vec{p}.iii$) for $n < \omega$, if Z appears in p_n but not in p then $Z \cap A \in \Sigma(X \cap H_\theta)$.

It follows that

$$q = \left(\bigcup_{n<\omega} p_n \right)^\frown (X \cap H_\theta)$$

is a condition in $\mathbb{P}$ and that q is $(X, \mathbb{P}, f)$-generic. Thus $\mathbb{P}$ is f-proper. Note that if G is $\mathbb{P}$-generic, then $\bigcup G$ is necessarily a sequence of length ω_1, as otherwise ω_1 would be collapsed. Hence the set

$$D_\alpha = \{p \in \mathbb{P} \mid \text{length}(p) \geqslant \alpha\}$$

is dense in $\mathbb{P}$ for all $\alpha < \omega_1$. By PFA(f), there is a filter $G \subseteq \mathbb{P}$ that meets all D_α, $\alpha < \omega_1$. It follows that $\bigcup G$ witnesses the instance of f-MRP for Σ. □

Proposition 11.25. *Suppose $\mathbb{B}$ is c.c.c. and f witnesses $\diamondsuit(\mathbb{B})$. If A is uncountable, θ is sufficiently large and regular and $X \prec H_\theta$ is f-slim with $A \in X$ then any X-stationary subset of $[A]^\omega$ is X-f-stationary.*

Proof. Assume that $\mathcal{S} \subseteq [A]^\omega$ is X-stationary. Suppose $C \in M_X[f(\delta^X)]$ so that

$$M_X[f(\delta^X)] \models \text{“}C \text{ is club in } [\bar{A}]^\omega\text{”}.$$

Note that

$$M_X \models \text{“}\bar{A} \text{ is uncountable and } \bar{\mathbb{B}} \text{ is c.c.c.”}$$

and hence there is $D \in M_X$ with

$$M_X \models \text{“}D \text{ is club in } [\bar{A}]^\omega\text{”}$$

and $D \subseteq C$. As $\mathcal{S}$ is X-stationary,

$$\varnothing \neq \pi_X(D) \cap X \cap \mathcal{S} \subseteq \pi_X[C] \cap \mathcal{S}.$$

□

Corollary 11.26. *If $\mathbb{B}$ is c.c.c. and f witnesses $\diamondsuit(\mathbb{B})$ then* PFA(f) *implies* MRP.

This also holds if f is replaced by a sequence $\mathbf{f}$ uniformly witnessing $\diamondsuit(\mathbb{B})$. Thus this generalizes a result of Miyamoto published in [MY13], who has shown that if S is a (coherent) Suslin tree then PFA(S) implies MRP.

In order to prove Theorem 11.19, we have no other choice but to go through the arguments of Moore and Viale that prove these consequences from MRP and check that f-MRP suffices.

Definition 11.27 (Todorčević)**.** Suppose κ is an uncountable regular cardinal. $\square(\kappa)$ holds if there is a sequence $\langle C_\alpha \mid \alpha < \kappa \cap \mathrm{Lim}\rangle$ so that

($\vec{C}.i$) $C_\alpha \subseteq \alpha$ is cofinal and closed below α for all $\alpha < \kappa$,

($\vec{C}.ii$) if $\beta < \alpha < \kappa$ and β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$ and

($\vec{C}.iii$) there is no club $C \subseteq \kappa$ so that $C_\alpha = C \cap \alpha$ for all limit points α of C.

We call $\vec{C}$ a $\square(\kappa)$-*sequence*.

Lemma 11.28. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and f-*MRP *holds. Then $\neg\square(\kappa)$ for all regular $\kappa \geqslant \omega_2$.*

We adapt the argument given in Section 6 of [Moo05].

Proof. Suppose $\kappa \geqslant \omega_2$ is regular and $\vec{C} := \langle C_\alpha \mid \alpha < \kappa\rangle$ is a $\square(\kappa)$-sequence. Let $\theta > 2^\kappa$ regular and let $\mathcal{C}$ consist of all f-slim $X \prec H_\theta$ with $\vec{C} \in X$. Define $\Sigma : \mathcal{C} \to [\kappa]^\omega$ via

$$\Sigma(X) = \{Y \in [X \cap \kappa]^\omega \mid \sup Y \notin C_{\sup(X\cap\kappa)}\}.$$

We will only show that $\Sigma(X)$ is X-f-stationary for all $X \in \mathcal{C}$, the rest works as in Section 6 of [Moo05]. So suppose that $D \in M_X[f(\delta^X)]$ so that

$$M_X[f(\delta^X)] \models \text{“}D \text{ is club in } [\bar{\kappa}]^\omega\text{”}.$$

Consider $E = \{\sup(Y) \mid Y \in D\}$. Then $E \in M_X[f(\delta^X)]$ and $E \subseteq \bar{\kappa}$ is unbounded.

Claim 11.29. $E \nsubseteq \pi_X^{-1}[C_{\sup(X\cap\kappa)}]$.

Proof. Otherwise $C = \bigcup\{\bar{C}_\alpha \mid \alpha \text{ is a limit point of } E\} \in M_X[f(\delta^X)]$ is coherent with $\langle \bar{C}_\alpha \mid \alpha < \bar{\kappa}\rangle := \pi_X^{-1}(\vec{C})$. But $\pi_X^{-1}(\vec{C})$ is a $\square(\bar{\kappa})$-sequence in M_X and small forcing cannot destroy a $\square(\bar{\kappa})$-sequence, contradiction. □

This shows that there is $Y \in D$ with $\sup(\pi_X[Y]) \notin C_{\sup(X\cap\kappa)}$, i.e. $\pi_X[Y] \in \Sigma(X)$. □

We now turn to proving SCH from f-MRP. Matteo Viale studied certain covering properties in his PhD thesis that allowed him to give elegant proofs of SCH or large fragments thereof from a range of different assumptions, including MRP.

Definition 11.30 (Viale, [Via08])**.** Suppose κ is an infinite cardinal. A *covering matrix* is a sequence $\mathcal{D} = \langle K(n,\alpha) \mid n < \omega, \alpha < \kappa\rangle$ so that for all $\alpha < \kappa$ and $n < \omega$

($\mathcal{D}.i$) $\alpha + 1 = \bigcup_{m<\omega} K(m,\alpha)$,

($\mathcal{D}$.*ii*) $|K(n,\alpha)| < \kappa$,

($\mathcal{D}$.*iii*) $K(n,\alpha) \subseteq K(m,\alpha)$ for all $n \leqslant m < \omega$,

($\mathcal{D}$.*iv*) for all $\beta \in (\alpha,\kappa)$, there is $m < \omega$ with $K(n,\alpha) \subseteq K(m,\beta)$ and

($\mathcal{D}$.*v*) for all $X \in [\kappa]^\omega$, there is $\gamma_X < \kappa$ so that for all $\beta < \kappa$ and $m < \omega$ there is $l < \omega$ so that $K(m,\beta) \cap X \subseteq K(l,\gamma_X)$.

Definition 11.31 (Viale, [Via08])**.** Suppose $\mathcal{D}$ is a covering matrix at an infinite cardinal κ. The *covering property* $\mathrm{CP}(\mathcal{D})$ holds for $\mathcal{D}$ if there is an unbounded $A \subseteq \kappa$ so that $[A]^\omega$ is *covered* by $\mathcal{D}$, that is

$$[A]^\omega \subseteq \bigcup\{[K(n,\beta)]^\omega \mid n < \omega, \beta < \kappa\}.$$

Recall that the Singular Cardinal Hypothesis (SCH) states that for any singular cardinal λ, if $2^{\mathrm{cof}(\lambda)} < \lambda$ then $\lambda^{\mathrm{cof}(\lambda)} = \lambda^+$.

Fact 11.32 (Viale, [Via08])**.** *Suppose whenever $\kappa = \lambda^+$ and λ is a singular cardinal of countable cofinality, and $\mathcal{D}$ is a covering matrix at κ with all entries closed sets of ordinals, then* $\mathrm{CP}(\mathcal{D})$*. Then* SCH *holds.*

Lemma 11.33. *Suppose f witnesses $\diamondsuit(\mathbb{B})$ and f-*MRP *holds. Then* SCH *holds true.*

We will follow Section 7 of [Via08].

Proof. Suppose $\kappa = \lambda^+$, λ is a singular cardinal, $\mathrm{cof}(\lambda) = \omega$ and $\mathcal{D} = \langle K(n,\beta) \mid n < \omega, \beta < \kappa\rangle$ is a covering matrix at κ with all $K(n,\beta)$ closed sets of ordinals. We will show that $\mathrm{CP}(\mathcal{D})$ holds, so that SCH then follows from Fact 11.32. So suppose $\mathrm{CP}(\mathcal{D})$ fails. Let $\langle C_\alpha \mid \alpha < \kappa, \mathrm{cof}(\alpha) = \omega\rangle$ be a ladder system, that is $C_\alpha \subseteq \alpha$ is cofinal of ordertype ω for $\alpha < \kappa$, $\mathrm{cof}(\alpha) = \omega$. Let θ be sufficiently large, regular. For $X \prec H_\theta$ countable, let γ_X denote the ordinal $\gamma_{X\cap\kappa}$ from ($\mathcal{D}$.*v*) and let $\xi_X := \sup(X \cap \kappa)$. Let $\mathcal{C}$ be the set of all f-slim $X \prec H_\theta$ with $\mathcal{D} \in X$. Define $\Sigma\colon \mathcal{C} \to \mathcal{P}([\kappa]^\omega)$ by

$$\Sigma(X) = \{Y \in [X \cap \kappa]^\omega \mid \sup(Y) \notin K(|C_{\xi_X} \cap \sup(Y)|, \gamma_X)\}.$$

We will only show that $\Sigma(X)$ is f-X-stationary. Let us work in $M_X[f(\delta^X)]$ for the moment and note that $\bar\kappa$ is a regular cardinal there. Let $C \subseteq [\bar\kappa]^\omega$ be a club. We can find $h\colon \bar\kappa^{<\omega} \to \bar\kappa^\omega$ so that

$$\{Z \in [\bar\kappa]^\omega \mid h[Z^{<\omega}] \subseteq [Z]^\omega\} \subseteq C.$$

Let $D = \{\alpha < \bar\kappa \mid h[\alpha^{<\omega}] \subseteq [\alpha]^\omega\}$, a club in $\bar\kappa$. We may find some countable $N \prec (H_{\bar\kappa^+}; \in, H^{M_X}_{\bar\kappa^+})$ with $\mathcal{D}, D, h \in N$. Let

$$n := |\sup(\pi_X[N \cap \bar\kappa]) \cap C_{\xi_X}| < \omega$$

and find some $\alpha \in N \cap \bar\kappa$ so that

$$n = |\pi_X(\alpha) \cap C_{\xi_X}|.$$

For $p \in f(\delta^X)$, let

$$A_p := \{\beta < \bar\kappa \mid p \Vdash \check\beta \in \dot A\}$$

where $\dot A \in M_X$ is a $\bar{\mathbb{B}}$-name with $\dot A^{f(\delta^X)} = D - \alpha$. For some $p \in f(\delta^X)$, A_p must be unbounded in $\bar\kappa$. Note that $\mathrm{Lim}(A_p) \subseteq D$ as D is a club, so that $B := \mathrm{Lim}(A_p) \cap (E^{\bar\kappa}_\omega)^{M_X} \in M_X$ is unbounded in $\bar\kappa$, too. By assumption, B is not covered by $\bar{\mathcal{D}}$ so that there is some $Z \in \left([B]^\omega\right)^{M_X}$ with $Z \nsubseteq \bar K(n, \beta)$ for all $\beta < \bar\kappa$. By elementarity,

$$\pi_X[Z] = \pi_X(Z) \nsubseteq K(n, \gamma_X)$$

so we may pick some $\zeta \in Z - \pi_X^{-1}[K(n, \gamma_X)]$. By elementarity of N in

$$\left(H^{M_X[f(\delta^X)]}_{\bar\kappa^+}; \in, H^{M_X}_{\kappa^+}\right)$$

we may assume without loss of generality that $Z \in N$ so that $\zeta \in N$ as well. Let $Y_0 \in N$ be a cofinal subset of ζ of ordertype ω and let Y be the closure of Y_0 under h so that $Y \in C$. As $\zeta \in D$, $\sup(Y) = \zeta$.

Claim 11.34. $\pi_X[Y] \in \Sigma(X)$.

Proof. First observe that $\pi_X(\zeta) = \sup(\pi_X[Y])$ as $\mathrm{cof}(\zeta)^{M_X} = \omega$ and hence π_X is continuous at ζ. We have $\sup(\pi_X[Y]) = \pi_X(\zeta) \notin K(n, \gamma_X)$ and $\alpha < \zeta$, $Y \subseteq N$ implies $|\sup(\pi_X[Y]) \cap C_{\xi_X}| = |\sup(\pi_X[N \cap \bar\kappa]) \cap C_{\xi_X}| = n$. □

It follows that $\Sigma(X)$ is indeed f-X-stationary. Now let $\langle X_i \mid i < \omega_1\rangle$ witness the instance of f-MRP at Σ. Let

$$C = \{\xi_{X_\alpha} \mid \alpha < \omega_1\}$$

and $\xi_* = \sup_{\alpha<\omega_1} \xi_{X_\alpha}$. It is a consequence of $(\mathcal{D}.i)$ that there is some $n < \omega$ so that

$$C \cap K(n, \xi_*)$$

is unbounded in ξ_*. As $K(n, \xi_*)$ is closed, $C \subseteq K(n, \xi_*) \mod \mathrm{NS}_{\xi_*}$. Without loss of generality we may assume $C \subseteq K(n, \xi_*)$ as we could otherwise throw away nonstationary many elements of $\langle X_i \mid i < \omega_1\rangle$. Let $\alpha < \omega_1$ be a limit so that X_α is f-slim. There is now some $\beta < \alpha$ so that

$$X_\nu \cap \kappa \in \Sigma(X_\alpha)$$

for all $\nu \in (\beta, \alpha)$. We can also find some $m < \omega$ so that

$$K(n, \xi_*) \cap (X_\alpha \cap \kappa) \subseteq K(m, \gamma_{X_\alpha})$$

by our choice of γ_{X_α}. As $\xi_{X_\alpha} = \sup_{\nu<\alpha} \xi_{X_\nu}$, we can find some $\nu \in (\beta, \alpha)$ large enough with

$$l := |C_{\xi_{X_\alpha}} \cap \xi_{X_\nu}| \geqslant m.$$

We have that

$$\xi_{X_\nu} \notin K(l, \gamma_{X_\alpha})$$

but $\xi_{X_\nu} \in K(n, \xi_*) \cap (C \cap \xi_{X_\alpha}) \subseteq K(n, \xi_*) \cap (X \cap \kappa)$, contradiction. □

This completes the proof of Theorem 11.19.

Moore's proof of $2^{\omega_1} = \omega_2$ from MRP does not seem to go through for f-MRP in case f witnesses $\diamondsuit(\omega_1^{<\omega})$. However this is essentially the only problematic case.

Lemma 11.35. *Suppose there is $b \in \mathbb{B}$ so that $\mathbb{B} \restriction b$ preserves ω_1 and f witnesses $\diamondsuit(\mathbb{B})$. Then f-MRP implies $2^{\omega_1} = \omega_2$. Thus if PFA(f) holds then $2^\omega = \omega_2$.*

We leave this one for the reader to check. This still leaves open the size of the continuum under PFA(f) if f witnesses $\diamondsuit(\omega_1^{<\omega})$. We note that Todorčević's original proof of $2^\omega = \omega_2$ from PFA does not apply either, as it goes through the open coloring axiom. The open coloring axiom implies $\mathfrak{b} = \omega_2$, while $\mathfrak{b} = \omega_1$ follows from $\diamondsuit(\omega_1^{<\omega})$.

Question 11.36. Suppose f witnesses $\diamondsuit(\omega_1^{<\omega})$ and PFA(f) holds. Must $2^\omega = \omega_2$?

A negative answer would be seriously surprising.

11.3 When is Namba forcing f-semiproper?

In this section we will investigate Namba forcing in the context of f-semiproper forcings where f witnesses $\diamondsuit(\mathbb{B})$. Let us first fix some notation.

Definition 11.37. Assume $T \subseteq \omega_2^{<\omega}$ is a tree.

(*i*) For $t \in T$, the set of ordinals immediately succeeding T is defined as

$$\mathrm{succ}_T(t) = \{\alpha < \omega_2 \mid t^\frown\alpha \in T\}.$$

(*ii*) We say $t \in T$ splits iff $|\mathrm{succ}_T(t)| = \omega_2$.

(*iii*) For $n < \omega$ and $t \in T$ we let $t \in \mathrm{split}_n(T)$ iff t splits and there are exactly n proper initial segments of t that split.

(*iv*) If $t \in \mathrm{split}_n(T)$ and $m \leqslant n$ then $t \restriction^{\mathrm{sp}} m$ is the unique initial segment $s \subseteq t$ which is in $\mathrm{split}_m(T)$.

(v) The stem of T, denoted by stem(T), is defined (if it exists) as the largest $t \in T$ with
$$\forall s \in T \ s \leqslant t \vee t \leqslant s.$$

(vi) If $t \in T$ then $T \restriction t = \{s \in t \mid s \leqslant t \vee t \leqslant s\}$.

Trees grow upwards in our notation, as they should.

Definition 11.38. *Namba forcing*, denoted Nm, consists of trees $p \subseteq \omega_2^{<\omega}$ so that for any $t \in p$, there is $t \leqslant_T s$ that splits. The order on Nm is inclusion.

Shelah was interested in the question when Nm is semiproper and proved the following theorem.

Fact 11.39. *(Shelah, [She98, XII Theorem 2.2]) The following are equivalent:*

(i) Nm *is semiproper.*

(ii) *There is a semiproper forcing* $\mathbb{P}$ *with* $\mathrm{cof}(\omega_2^V)^{V^{\mathbb{P}}} = \omega$.

(iii) $\mathrm{SCC}_{\mathrm{cof}}$ *holds.*

We generalize this result here and prove Nm to be f-semiproper if and only if a variant of $\mathrm{SCC}_{\mathrm{cof}}$.

Definition 11.40. Suppose f witnesses $\diamondsuit(\mathbb{B})$. f-$\mathrm{SCC}_{\mathrm{cof}}$ holds iff for any sufficiently large regular θ and f-slim $X \prec H_\theta$ the following is true: There are cofinally many $\alpha < \omega_2$ so that there is an f-slim $X \sqsubseteq Y \prec H_\theta$ with $\alpha \in Y$.

Theorem 11.41. *Suppose f witnesses $\diamondsuit(\mathbb{B})$. The following are equivalent:*

(i) Nm *is f-semiproper.*

(ii) *There is an f-semiproper forcing* $\mathbb{P}$ *with* $\mathrm{cof}(\omega_2^V)^{V^{\mathbb{P}}} = \omega$.

(iii) f-$\mathrm{SCC}_{\mathrm{cof}}$ *holds.*

We remark that the proof of the interesting direction (iii)$\Rightarrow$(i) we present here differs significantly from Shelah's proof, who used games to show Nm to be semiproper from $\mathrm{SCC}_{\mathrm{cof}}$. Instead, we produce (X, Nm, f)-semigeneric conditions directly.

Proof. (i)$\Rightarrow$(ii): This holds as the generic branch given by Namba forcing witnesses $\mathrm{cof}(\omega_2^V) = \omega$ in V^{Nm}.

(ii)$\Rightarrow$(iii): Let $\mathbb{P}$ be a f-semiproper forcing so that $\mathrm{cof}(\omega_2^V)^{V^{\mathbb{P}}} = \omega$. Let θ be sufficiently large and regular and
$$X \prec (H_\theta; \in, \lhd) =: \mathcal{H}$$

be f-slim, where $\lhd$ is some wellorder on H_θ. Let $\alpha < \omega_2$. By assumption, there is $q \in \mathbb{P}$ that is $(X, \mathbb{P}, f)$-semigeneric, so that if G is $\mathbb{P}$-generic with $q \in G$ then both

$$\delta^{X[G]} = \delta^X$$

and $X[G]$ is f-slim. Also, $X[G] \cap \omega_2^V$ is cofinal in ω_2^V. Let $\alpha < \beta \in X[G]$ and let

$$Y := \mathrm{Hull}^{\mathcal{H}}(X \cup \{\beta\}).$$

We have $Y \in V$, $Y \subseteq X[G] \cap V$ hence $X \sqsubseteq Y$ and $\beta \in Y$. Moreover, $X[G]$ is f-slim and as $Y \sqsubseteq X[G]$, Y is f-slim as well.

$(iii) \Rightarrow (i)$: Let θ be sufficiently large and

$$X \prec (H_\theta; \in, \lhd) =: \mathcal{H}$$

countable and f-slim, where again $\lhd$ is some wellorder of H_θ. For a set $t \in H_\theta$, we let

$$X_t := \mathrm{Hull}^{\mathcal{H}}(X \cup \{t\}).$$

Furthermore, we choose some

$$h \colon \omega \to \omega \times \omega$$

surjective with $i \leqslant n$ whenever $h(n) = (i, j)$. h will be used for bookkeeping purposes.

Now let $p \in \mathsf{Nm} \cap X$ and let us assume $\mathrm{stem}(p) = \emptyset$ for convenience. We will define a descending sequence $\langle p^n \mid n < \omega \rangle$ through Nm and sets

$$\langle D_j^t \mid t \in \mathrm{split}_{\mathrm{lh}(t)}(p^{\mathrm{lh}(t)}),\ j < \omega \rangle$$

satisfying

$(\vec{p}.i)$ $p^0 = p$,

$(\vec{p}.ii)$ $\forall n \leqslant m < \omega\ \mathrm{split}_n(p^m) = \mathrm{split}_n(p^n)$,

$(\vec{p}.iii)$ for $n < \omega$ and $t \in \mathrm{split}_n(p^n)$, $X \sqsubseteq X_t$ and X_t is f-slim,

$(\vec{p}.iv)$ for $n < \omega$ and $t \in \mathrm{split}_n(p^n)$, $\langle D_j^t \mid j < \omega \rangle$ is an enumeration of all dense subsets of $\mathsf{Nm}^{M_{X_t}}$ that are in $M_{X_t}[f(\delta^X)]$,

$(\vec{p}.v)$ for $n < \omega$ and $t \in \mathrm{split}_n(p^n)$, $p^n \upharpoonright t \in X_t$ and

$(\vec{p}.vi)$ if $h(n) = (i, j)$ and $t \in \mathrm{split}_{n+1}(p^{n+1})$ then

$$p^{n+1} \upharpoonright t \in \pi_{X_t}[\mu^+_{t \upharpoonright^{\mathrm{sp}} i, t}(D_j^{t \upharpoonright^{\mathrm{sp}} i})].$$

Here, $\mu^+_{s,t}$ denotes

$$\mu^+_{X_s,X_t}\colon M_{X_s}[f(\delta^X)] \to M_{X_t}[f(\delta^X)].$$

We also let M_t denote M_{X_t} and $\pi_t := \pi_{X_t}$ if X_t is defined.
Suppose p^n is defined already. We will prune p^n to get to p^{n+1}, but we are not allowed to change $\mathrm{split}_n(p^n)$. For any $t \in \mathrm{split}_n(p^n)$, we will find a set

$$S_t \in [\mathrm{succ}_{p^n}(t)]^{\omega_2}$$

and find strengthenings $q^{t,\alpha} \leqslant p^n \upharpoonright t^\frown\alpha$ for any $\alpha \in S_t$. p^{n+1} will then be all the $q^{t,\alpha}$ glued together. We let

$$S_t := \{\alpha \in \mathrm{succ}_{p^n}(t) \mid X \sqsubseteq X_{t^\frown\alpha} \text{ is } f\text{-slim}\}.$$

Claim 11.42. *S_t is cofinal in ω_2.*

Proof. Let $\gamma < \omega_2$. By $(\vec{p}.iii)$, $X \sqsubseteq X_t$ and X_t is f-slim. By f-$\mathrm{SCC}_{\mathrm{cof}}$ there is then $\gamma \leqslant \xi < \omega_2$ and $Y \prec H_\theta$ with $X_t \sqsubseteq Y$, Y is f-slim and $\gamma \in Y$. By (v), $p^n \upharpoonright t \in X_t \subseteq Y$. As

$$Y \models |\mathrm{succ}_{p^n \upharpoonright t}(t)| = \omega_2$$

there must be some $\alpha \in Y$, $\alpha \geqslant \xi$ with $\alpha \in \mathrm{succ}_{p^n \upharpoonright t}(t)$. Clearly, $\alpha \in S_t$. $\square$

For any $\alpha \in S_t$, we choose $q^{t,\alpha}$ to be some condition in $\mathsf{Nm} \cap X_{t^\frown\alpha}$ below $p^n \upharpoonright t^\frown\alpha$ that is in

$$\pi_{X_{t^\frown\alpha}}[\mu^+_{t\upharpoonright^{\mathrm{sp}} i, t^\frown\alpha}(D_j^{t\upharpoonright^{\mathrm{sp}} i})]$$

where $h(n) = (i,j)$. This is possible as

$$M_{X_{t^\frown\alpha}}[f(\delta^X)] \models \text{“}\mu^+_{t\upharpoonright^{\mathrm{sp}} i, t^\frown\alpha}(D_j^{t\upharpoonright^{\mathrm{sp}} i}) \text{ is dense in } \mathsf{Nm}\text{”}.$$

This defines p^{n+1}. To see that $(\vec{p}.iii)$ holds for $n+1$, note that if $s \in \mathrm{split}_{n+1}(p^{n+1})$, $t = s \upharpoonright^{\mathrm{sp}} n$ and α unique with

$$t \subseteq t^\frown\alpha \subseteq s$$

Then $X_s = X_{t^\frown\alpha}$, as s is definable from $q^{t,\alpha}$, and

$$p^{n+1} \upharpoonright s = p^{n+1} \upharpoonright t^\frown\alpha \in X_{t^\frown\alpha}$$

by construction.

By $(\vec{p}.ii)$, there exists a fusion $q \in \mathsf{Nm}$ below all p^n, $n < \omega$. Simply define q as the downwards closure of $\bigcup_{n<\omega} \mathrm{split}_n(p^n)$. We will show that q is (X, Nm, f)-semigeneric. Let G be Nm-generic with $q \in G$ and let $b = \bigcap G$ be the generic branch. Let $X_b := X[G] \cap V$. By Proposition 3.33, it is enough to show that

($X_b.i$) $X \sqsubseteq X_b$,

($X_b.ii$) X_b is f-slim and

($X_b.iii$) $\pi_{X_b}^{-1}[G]$ is generic over $M_{X_b}[f(\delta)]$.

Claim 11.43. $X_b = \bigcup_{t \in b} X_t$.

Proof. The inclusion $\supseteq$ is true as $X \subseteq X[G]$, $b \in X[G]$ and thus $X \cup b \subseteq X_b$. To see $\subseteq$, let $\dot{x} \in X$ be a Nm-name with

$$\Vdash_{\mathsf{Nm}} \dot{x} \in V.$$

Then the set of conditions deciding the value of $\dot{x}$ must be

$$D = \pi_X(D_j^{\varnothing})$$

for some $j < \omega$. Find $n < \omega$ with $h(n) = (0, j)$ and let t be the unique node in $b \cap \mathrm{split}_{n+1}(q)$. Then $q \upharpoonright t \in G$ and $q \upharpoonright t \leqslant p^{n+1} \upharpoonright t$ so that

$$p^{n+1} \upharpoonright t \in G.$$

By ($\vec{p}.vi$), we have

$$p^{n+1} \upharpoonright t \in \pi_t[\mu^+_{\varnothing,t}(D_j^{\varnothing})] \subseteq D$$

so that $p^{n+1} \upharpoonright t$ decides $\dot{x}$ to be y for some y. By ($\vec{p}.v$), $p^{n+1} \upharpoonright t \in X_t$ and hence $\dot{x}^G = y \in X_t$. □

($X_b.i$) and ($X_b.ii$) follow from the above as X_t for $t \in b$ is f-slim and satisfies $X \sqsubseteq X_t$ by ($\vec{p}.iii$). It remains to show ($X_b.iii$). Set

$$M_b := M_{X_b} \text{ and } \pi_b := \pi_{X_b}.$$

It follows from Claim 11.43 that we can let

$$\langle M_b[f(\delta^X)], \mu^+_{t,b} \mid t \in b\rangle = \varinjlim \langle M_s[f(\delta^X)], \mu^+_{s,t} \mid s \leqslant t \in b\rangle.$$

Hence if $D \in M_b[f(\delta^X)]$ is dense in Nm^{M_b}, then for some $s \in b$ and $j < \omega$,

$$D = \mu^+_{s,b}(D_j^s).$$

We may assume that s is a splitting node of q, say $s \in \mathrm{split}_i(q)$. Find n with $h(n) = (i, j)$. Let t be the unique node in $b \cap \mathrm{split}_{n+1}(q)$. Then as before we get

$$p^{n+1} \upharpoonright t \in G$$

and also by ($\vec{p}.vi$)

$$p^{n+1} \upharpoonright t \in \pi_t[\mu^+_{s,t}(D_j^s)].$$

Note that $\pi_t \circ \mu_{s,t} = \pi_b \circ \mu_{s,b}$. It follows that

$$p^{n+1} \upharpoonright t \in G \cap \pi_b[D]$$

so that $\pi_b^{-1}[G] \cap D \neq \varnothing$. This establishes ($X_b.iii$). □

Remark 11.44. Observe that if we plug in $\mathbb{B} = \{0\}$ the trivial forcing in the theorem above, we recover exactly Shelah's result Fact 11.39.

11.4 The $\mathrm{MM}^{++}(f)$-$\mathbb{F}_{\max}$-$(*)$ diagram

This subsection is joint work with Ralf Schindler.

Definition 11.45. $\mathbb{V}_{\max}$ is *rich* if for all $p \in \mathbb{V}_{\max}$ and $X \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, there is $q <_{\mathbb{V}_{\max}} p$ so that

(*i*) q is X-iterable and

(*ii*) $(H^q_{\omega_1}, \in) \prec (H_{\omega_1}, \in)$.

In practice, establishing richness of a $\mathbb{P}_{\max}$-variation $\mathbb{V}_{\max}$ makes use of (the full power of) AD and is a crucial ingredient in showing that $\mathbb{V}_{\max}$ is self-assembling.

We take a closer look at the different axiom related and investigate their relationship. For the rest of the section, we will assume that $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variations with all the good and convenient features and the V has sufficiently many large cardinals. Assume that

- there is a proper class of Woodin cardinals,
- $\mathbb{V}_{\max}$ has unique iterations and accepts $\diamondsuit$-iterations,
- $\mathbb{V}_{\max}$ is typical and this is witnessed by a set $\Psi^{\mathbb{V}_{\max}}$ of $(\Sigma_1 \cup \Pi_i)$-formulae,
- $\Gamma^{\Psi}_{A} = \Gamma^{\mathbb{V}_{\max}}_{A}(\Psi)$ and this class contains all σ-closed forcings,
- $\mathbb{V}_{\max}$ is rich and self-assembling and
- $n^{\mathbb{V}}_{\max} = 0$ (this is purely for convenience).

We also fix some set A, think of A as some set produced by forcing with $\mathbb{V}_{\max}$.

Under these assumptions, we find the following diagram:

$$\begin{tikzcd}
\mathbb{V}_{\max}\text{-}(*)^{++,\mathrm{uB}}_A \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathbb{V}_{\max}\text{-}(*)^{++}_A \arrow[d, Rightarrow] & & \mathrm{MM}^{++}(A) \arrow[dll, Rightarrow, "/" marking] \arrow[dd, Rightarrow] & \\
\mathbb{V}_{\max}\text{-}(*)^{+,\mathrm{uB}}_A \arrow[r, Rightarrow] \arrow[drrr, Rightarrow] & \mathbb{V}_{\max}\text{-}(*)^{+}_A & & & \\
& & & \mathrm{MM}^{++}_{\mathfrak{c}}(A) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
& & \mathrm{MM}^{++}(A,\mathfrak{c}) \arrow[ddr, Rightarrow, "/" marking] & \text{uB-BMM}^{++}(A) \arrow[d, Leftrightarrow] & \mathrm{MM}_{\mathfrak{c}}(A) \arrow[ddd, Rightarrow, "/" marking] \\
& & & \mathbb{V}_{\max}\text{-}(*)^{\mathrm{uB}}_A \arrow[d, Rightarrow] & \\
& & & \mathbb{V}_{\max}\text{-}(*)_A \arrow[dl, Leftrightarrow] & \\
& & (\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))\text{-BMM}^{++}(A) \arrow[rr, Rightarrow] & & \mathrm{BMM}^{++}(A)
\end{tikzcd}$$

We use a number of shorthand notation for principles we already defined as well as some principles we have not defined earlier.

Definition 11.46. $\mathbb{V}_{\max}$-$(*)^+_A$ is the statement: For any $X \subseteq \mathbb{R}$ there is $B \subseteq \mathbb{R}$ such that

$(a)^+$ $L(B, \mathbb{R}) \models \mathrm{AD}^+$ and

$(b)^+$ there is a $\mathbb{V}_{\max}$-filter g generic over $L(B, \mathbb{R})$ so that $X \in L(B, \mathbb{R})[g]$ and g produces A.

$\mathbb{V}_{\max}$-$(*)^{+,\mathrm{uB}}_A$ is the statement: For any $X \subseteq \mathbb{R}$ there is $B \subseteq \mathbb{R}$ such that $(a)^+$ and $(b)^+$ from above hold and additionally

$(c)^+$ all sets of reals in $L(B, \mathbb{R})$ are universally Baire.

$\mathbb{V}_{\max}$-$(*)^{++}_A$ holds if there is a pointclass Σ so that

$(a)^{++}$ $L(\Sigma, \mathbb{R}) \models \mathrm{AD}^+$ and

$(b)^{++}$ there is a $\mathbb{V}_{\max}$-filter g generic over $L(\Sigma, \mathbb{R})$ so that $\mathcal{P}(\mathbb{R}) \in L(\Sigma, \mathbb{R})[g]$ and g produces A.

$\mathbb{V}_{\max}$-$(*)^{++,\mathrm{uB}}_A$ holds if $(a)^{++}$ and $(b)^{++}$ from above hold for $\Sigma = \mathrm{uB}$ and additionally

$(c)^{++}$ all sets of reals in $L(\mathrm{uB}, \mathbb{R})$ are universally Baire, i.e.

$$\mathcal{P}(\mathbb{R}) \cap L(\mathrm{uB}, \mathbb{R}) = \mathrm{uB}.$$

Definition 11.47. (*i*) $\mathrm{MM}^{++}(A)$ is short for

$$\mathrm{FA}^{\Psi}_{A}(\Gamma^{\Psi}_{A})$$

and $\mathrm{MM}(A)$ denotes

$$\mathrm{FA}(\Gamma^{\Psi}_{A}).$$

(*ii*) $\mathrm{MM}^{++}_{\mathfrak{c}}(A)$ is $\mathrm{MM}^{++}(A)$ restricted to instances where the dense sets are generated[55] by maximal antichains of size $\leqslant\mathfrak{c}$. $\mathrm{MM}_{\mathfrak{c}}(A)$ is the analogous fragment of $\mathrm{MM}(A)$.

(*iii*) $\mathrm{MM}^{++}(A,\mathfrak{c})$ is $\mathrm{MM}^{++}(A)$ restricted to all forcings of size $\leqslant\mathfrak{c}$.

(*iv*) For a pointclass Δ, Δ-$\mathrm{BMM}^{++}(A)$ is short for

$$\Delta\text{-}\mathrm{BFA}^{\Psi}_{A}(\Gamma^{\Psi}_{A})$$

where we omit Δ if $\Delta=\varnothing$.

(*v*) For a pointclass Γ, $\mathbb{V}_{\max}$-$(*)^{\Gamma}_{A}$ holds if

(a) all sets in Γ are determined and

(b) there is a filter $g\subseteq\mathbb{V}_{\max}$ so that

(*g.i*) if $D\in\Gamma$ is (or rather codes) a dense set in $\mathbb{V}_{\max}$ then $g\cap D\neq\varnothing$,

(*g.ii*) $\mathcal{P}(\omega_1)=\mathcal{P}(\omega_1)_g$ and

(*g.iii*) g produces A.

We omit Γ if $\Gamma=\mathcal{P}(\mathbb{R})\cap L(\mathbb{R})$.

All implications in the upper left rectangle of the diagram are trivial, we note that the vertical implications holds as suitably nice $\mathbb{V}_{\max}$-names for a subset of ω_1 can be coded as sets of reals. We want to mention that Woodin has proved the following remarkable theorem.

Fact 11.48 (Woodin,[Woo])**.** *$(*)^{+}$ and $(*)^{++}$ are equivalent.*

Here, $(*)^{+}$ and $(*)^{++}$ denote $\exists A\ \mathbb{P}_{\max}$-$(*)^{+}_{A}$, $\exists A\ \mathbb{P}_{\max}$-$(*)^{++}_{A}$ respectively. The exact relation between $(*)^{+}$ and MM^{++} is still a mystery. Woodin demonstrated that $(*)^{+}$ is false in the standard model of MM^{++} (i.e. in the ones that result after iterating semiproper forcings guided by a Laver function on a supercompact cardinal). However, whether or not $(*)^{+}$ is consistent with MM^{++} (or MM for that matter) remains open.

The only new implication in the diagram is $\mathbb{V}_{\max}$-$(*)^{+,\mathrm{uB}}_{A}\Rightarrow\mathrm{MM}^{++}_{\mathfrak{c}}(A)$, which we aim to prove now.

[55] A dense set $D\subseteq\mathbb{P}$ is generated by A if $D=\{p\in\mathbb{P}\mid\exists a\in A\ p\leqslant a\}$.

Theorem 11.49. *Suppose there is a proper class of Woodin cardinals. Then* $\mathbb{V}_{\max}$-$(*)^{+,\mathrm{uB}}_A$ *implies* $\mathrm{MM}^{++}_{\mathfrak{c}}(A)$.

Definition 11.50. Let $\mathcal{M} = \langle M; \in, C_0, \dots, C_n\rangle$ be a transitive structure and ζ a Σ_1-formula with one free variable in the language $\{\in, \dot{R}^\psi \mid \psi \in \Psi\}$. Then $\Omega(\mathcal{M}, \zeta)$ is the statement that there is a transitive structure $\bar{\mathcal{M}}$ of size ω_1 as well as an embedding $\mu\colon \bar{\mathcal{M}} \to \mathcal{M}$ so that $(H_\theta; \in, R^\psi_A \mid \psi \in \Psi) \models \zeta(\bar{\mathcal{M}})$ for any/all large enough regular θ.

Claverie-Schindler [CS12] have proven the following fact in the MM-context. Their argument generalizes to:

Fact 11.51. *Let κ be a regular uncountable cardinal. Then the following are equivalent:*

(*i*) $\mathrm{MM}^{++}_{\mathfrak{c}}(A)$.

(*ii*) *Whenever $\mathbb{P}$ is a (Ψ, A)-preserving forcing and*

$$\mathcal{M} = \langle M; \in, A_0, \dots, A_n\rangle$$

is a transitive structure of size at most $\mathfrak{c}$ and ζ a Σ_1-formula in the language $\{\in, \dot{R}^\psi \mid \psi \in \Psi\}$ with one free variable then

$$(V; \in, R^\psi_A \mid \psi \in \Psi)^V \models \Omega(\mathcal{M}, \zeta) \Leftrightarrow (V^{\mathbb{P}}; \in, R^\psi_A \mid \psi \in \Psi)^{V^{\mathbb{P}}} \models \Omega(\mathcal{M}, \zeta).$$

In the remainder of this section we will sometimes confuse sets H with the structure $(H; \in, R^\psi_A \mid \psi \in \Psi)^H$ for readability purposes. We will take a look at how to establish $\mathrm{MM}^{++}_{\mathfrak{c}}(A)$ via *lovely* $\mathbb{V}_{\max}$ *conditions*. Suppose that B is a set of reals and G is a $\mathbb{V}_{\max}$-generic filter so that

- $L(B, \mathbb{R}) \models \mathrm{AD}^+$,
- G is generic over $L(B, \mathbb{R})$ and produces A and
- $\mathcal{P}(\omega_1) \subseteq L(B, \mathbb{R})$.

We will try to show that (*ii*) of Fact 11.51 holds for instances "in $L(B, \mathbb{R})[G]$". Let $\mathcal{M}$ be a transitive structure of size $\mathfrak{c}$, we may assume that $\mathcal{M}$ is of the form $\langle M; \in, C\rangle$. In V, there is a surjection of $\mathbb{R}$ onto $\mathfrak{c}$. This yields a model $\mathcal{R}$ on $\mathbb{R}$ of the form

$$\mathcal{R} = \langle \mathbb{R}; I, C\rangle$$

as well as an equivalence relation $\sim \in V$ on $\mathbb{R}$ so that

$$\mathcal{R}/\sim\ \cong\ \mathcal{M}.$$

Note that $\sim$ really is definable from $\mathcal{R}$ by $x \sim y$ iff $\forall z \in \mathbb{R}\ zI^{\mathcal{R}}x \Leftrightarrow zI^{\mathcal{R}}y$. Let us additionally assume that

$$\langle \mathbb{R}; I, C\rangle \in L(B, \mathbb{R})[G].$$

If $\mathcal{N}$ is any structure on the reals in the same signature as $\mathcal{R}$ then by $\underline{\mathcal{N}}$ we will denote the transitive isomorph (if it exists) of $\mathcal{N}/\sim$ where $\sim$ is defined as above, so for example $\underline{\mathcal{R}} = \mathcal{M}$. Let $\dot{I}$ and $\dot{C}$ be canonical $\mathbb{V}_{\max}$-names for I and C in $L(B, \mathbb{R})$ in the sense that

$$\dot{I} = \left\{\left(\widetilde{(x,y)}, p\right) \mid p \Vdash_{\mathbb{V}_{\max}} (\check{x}, \check{y}) \in \dot{I}\right\}^{L(B,\mathbb{R})}$$

and

$$\dot{C} = \{(\check{x}, p) \mid p \Vdash \check{x} \in \dot{C}\}^{L(B,\mathbb{R})}.$$

Definition 11.52. Let ζ be a Σ_1-formula with one free parameter in the signature $\{\in, \dot{R}^\psi \mid \psi \in \Psi\}$. Let us call a condition $p = (N, J, a) \in \mathbb{V}_{\max}$ *lovely* w.r.t. $\mathcal{R}, \zeta$ if there are $g, d \in N$ so that for all $y_0, \ldots, y_n \in d$ and all $\in$-formulas θ we have

($\heartsuit.i$) $g \subseteq \mathbb{V}_{\max}$ is a filter with $p <_{\mathbb{V}_{\max}} q$ for all $q \in g$ and $d \subseteq \mathbb{R}$,

($\heartsuit.ii$) p is $T \oplus \dot{I} \oplus \dot{C}$-iterable[56] where

$$T = \{(q, \theta', z_0, \ldots z_m) \mid m < \omega \wedge q \Vdash (\mathbb{R}, \dot{I}, \dot{C}) \models \theta'(\check{z}_0, \ldots, \check{z}_n)\},$$

($\heartsuit.iii$) g decides[57] $(\mathbb{R}; \dot{I}, \dot{C}) \models \theta(\check{y}_0, \ldots, \check{y}_n)$,

($\heartsuit.iv$) if $\exists q \in g\ q \Vdash (\mathbb{R}; \dot{I}, \dot{C}) \models \exists x\ \theta(x, \check{y}_0, \ldots, \check{y}_n)$ then there is $x \in d$ and $r \in g$ with

$$r \Vdash (\mathbb{R}; \dot{I}, \dot{C}) \models \theta(\check{x}, \check{y}_0, \ldots, \check{y}_n)$$

and

($\heartsuit.v$) $(N; \in, R^\psi_A \mid \psi \in \Psi)^N \models \zeta(\underline{\mathcal{R}'})$ where $\mathcal{R}' = (d; (\dot{I} \cap N)^g, (\dot{C} \cap N)^g)$ (in particular $(\dot{C} \cap N)^g \subseteq d$ and similarly for $\dot{I}$).

Proposition 11.53. *If p is lovely (w.r.t. $\mathcal{R}, \zeta$) as witnessed by (g, d) and*

$$j : p \to q$$

is a countable iteration of p then q is lovely (w.r.t. $\mathcal{R}, \zeta$) as witnessed by $(j(g), j(d))$.

Proof. It is clear that conditions ($\heartsuit.i$) and ($\heartsuit.ii$) remain true. Conditions ($\heartsuit.iii$)-($\heartsuit.v$) can be phrased as first order statements about $T \cap p$, $\dot{I} \cap p$ and $\dot{C} \cap p$ as well as d and g. By ($\heartsuit.ii$), $j(T \cap p) = T \cap q$ and similar for $\dot{I}, \dot{C}$ and hence by elementarity of j these statements remain true for $(j(g), j(d))$ in q. □

[56] $A \oplus B$ denotes a canonical way to code two sets of reals into one set of reals.
[57] g decides φ if there is $p \in g$ with $p \Vdash \varphi$ or $p \Vdash \neq \varphi$.

Lemma 11.54. *Suppose that there is $p \in G$ that is lovely w.r.t $\mathcal{R}, \zeta$ and let*

$$j\colon p \to p^* = (N^*, J^*, A)$$

be the G-iteration of p. Let

$$\mathcal{R}_0 := (j(d), (\dot{I} \cap N^*)^{j(g)}, (\dot{C} \cap N^*)^{j(g)}).$$

Then $V \models \zeta(\underline{\mathcal{R}_0})$ and $\mathcal{R}_0 \prec \mathcal{R}$.

Proof. We will focus on the latter first. We start by showing that $\mathcal{R}_0$ is indeed a substructure of $\mathcal{R}$. We will prove

$$(\dot{C} \cap N^*)^{j(g)} = C \cap j(d)$$

and an analogous argument yields the corresponding statement about I. It follows from $(\heartsuit.v)$ and the elementarity of j that $(\dot{C} \cap N^*)^{j(g)} \subseteq j(d)$. Now suppose $y \in j(d)$ and since y appears in a countable iterate of p along p's G-iteration, we may assume $y \in d$ using Proposition 11.53. Then g decides whether or not $\check{y} \in \dot{C}$ by $(\heartsuit.iii)$ and since $g \subseteq G$ by $(\heartsuit.i)$, g forces the same decision as G, that is

$$(\exists q \in g\ q \Vdash \check{y} \in \dot{C}) \text{ iff } y \in C.$$

Since the name $\dot{C}$ is chosen canonically, this means

$$y \in (\dot{C} \cap N^*)^{j(g)} \text{ iff } y \in C.$$

Next, we will apply the Tarski criterion to check for elementarity of $\mathcal{R}_0$ in $\mathcal{R}$. So assume $y_0, \dots, y_n \in j(d)$ and

$$\mathcal{R} \models \exists x\ \theta(x, y_0, \dots, y_n).$$

As above, we may assume that in fact $y_0, \dots, y_n \in d$. By $(\heartsuit.iii)$, there is $q \in g$ that decides whether or not

$$(\mathbb{R}; \dot{I}, \dot{C}) \models \exists x\ \theta(x, \check{y}_0, \dots, \check{y}_n)$$

and by $(\heartsuit.i)$ and as $p \in G$, we have $q \in G$. Thus we can conclude

$$q \Vdash (\mathbb{R}; \dot{I}, \dot{C}) \models \exists x\ \theta(x, \check{y}_0, \dots, \check{y}_n).$$

By $(\heartsuit.iv)$, there is $x \in d$ and $r \in g$ so that

$$r \Vdash (\mathbb{R}; \dot{I}, \dot{C}) \models \theta(\check{x}, \check{y}_0, \dots, \check{y}_n).$$

Thus $x \in j(d)$ and $\mathcal{R} \models \theta(x, y_0, \dots, y_n)$ as desired.

Finally, $V \models \zeta(\underline{\mathcal{R}_0})$ holds true as

$$(N^*; \in, R_A^\psi \mid \psi \in \Psi)^{N^*} \models \zeta(\underline{\mathcal{R}_0})$$

by $(\heartsuit.v)$, the elementarity of j and, since

$$(N^*; \in, R_A^\psi \mid \psi \in \Psi)^{N^*} \prec_{\Sigma_1} (H_{\omega_2}; \in, R_A^\psi \mid \psi \in \Psi)^{L(B,\mathbb{R})[G]} = (H_{\omega_2}; \in, R_A^\psi \mid \psi \in \Psi)^V$$

and ζ is Σ_1, the truth of ζ about $\underline{\mathcal{R}_0}$ in $(N^*; \in, R_A^\psi \mid \psi \in \Psi)^{N^*}$ is upwards absolute to $(H_{\omega_2}; \in, R_A^\psi \mid \psi \in \Psi)$. Note that $\mathcal{R}_0$ is of size at most ω_1 as N^* has size ω_1. Finally, the elementarity of $\mathcal{R}_0$ in $\mathcal{R}$ yields an elementary embedding $\mu\colon \underline{\mathcal{R}_0} \to \underline{\mathcal{R}}$. □

Thus if there are enough lovely conditions with correct iterations then $\mathrm{MM}^{++}_{\mathfrak{c}}(f)$ holds true.

Proof of Theorem 11.49. We make use of the lovely strategy. Let $\mathbb{P}$ be a (Ψ, A)-preserving forcing. Again it will be enough to check that

$$V^{\mathbb{P}} \models \Psi(\underline{\mathcal{R}}, \zeta) \Rightarrow V \models \Psi(\underline{\mathcal{R}}, \zeta)$$

for ζ an appropriate Σ_1-formula and a structure $\mathcal{R}$ of the form $(\mathbb{R}; I, C)$. By $\mathbb{V}_{\max}$-$(*)^{+,\mathrm{uB}}_A$, there is a set B of reals and a filter $G \subseteq \mathbb{V}_{\max}$ so that

(*i*) $L(B, \mathbb{R}) \models \mathrm{AD}^+$,

(*ii*) all sets of reals in $L(B, \mathbb{R})$ are universally Baire,

(*iii*) G is generic over $L(B, \mathbb{R})$ and produces A,

(*iv*) $\mathcal{R} \in L(B, \mathbb{R})[G]$ and

(*v*) $\mathcal{P}(\omega_1) \subseteq L(B, \mathbb{R})[G]$.

Again, we may take names $\dot{I}$ and $\dot{C}$ in $L(B, \mathbb{R})$ for I and C respectively as before. Let T be defined as before and let $S = T \oplus \dot{C} \oplus \dot{I}$. It suffices to show that there is a p in G which is lovely w.r.t. $\mathcal{R}, \zeta$. Let H be $\mathbb{P}$-generic so that in $V[H]$ there is a witness X for $\zeta(\underline{\mathcal{R}})$. As $\Gamma^\Psi_A = \Gamma^{\mathbb{V}_{\max}}_A(\Psi)$ and since we assumed that all σ-closed forcings are contained in this class, there is a further (Ψ, A)-preserving generic extension $V[H][h]$ of $V[H]$ in which

(*h.i*) NS_{ω_1} is saturated,

(*h.ii*) $p = (H_{\omega_2}, \mathrm{NS}_{\omega_1}, A)^{V[H][h]}$ is almost a $\mathbb{V}_{\max}$-condition and

(*h.iii*) $|\mathcal{R}| = \omega_1$.

Note that $\zeta(\mathcal{R})$ still holds true in

$$(H_{\omega_2}; \in, R_A^\psi \mid \psi \in \Psi)^{V[H][h]}.$$

Work in a further extension W in which p is countable, so that $p \in \mathbb{V}_{\max}$ then.

Claim 11.55. *In W, p is lovely w.r.t. $\mathcal{R}, \zeta$.*

Proof. Let $g := G$ and $d := \mathbb{R}^V$. We will prove that (g, d) witness p to be lovely. We will make frequent use of

$$(L(B, \mathbb{R}^V); \in, B, S, x \mid x \in \mathbb{R}^V) \overset{(\clubsuit)}{\equiv} (L(B^*, \mathbb{R}^{V[H][h]}); \in, B^*, S^*, x \mid x \in \mathbb{R}^V)$$

which is a consequence of the existence of a proper class of Woodin cardinals in V.

Clearly, g is a $\mathbb{V}_{\max}$-filter and d is a set of reals. All conditions in g are above p as the extension $V \subseteq V[H][h]$ is (Ψ, A)-preserving and by the proof of Claim 4.57. Hence $(\heartsuit.i)$ holds. By construction, p is S^*-iterable where S^* is the version of S in $V[H][h]$ and we can write S^* as $T^* \oplus \dot{I}^* \oplus \dot{C}^*$. By $(\clubsuit)$, T^* has the correct meaning, so that $(\heartsuit.ii)$ holds true. To establish $(\heartsuit.iii)$, consider any $y_0, \ldots, y_n \in d$ and let D be the dense set of $\mathbb{V}_{\max}$ conditions deciding

$$(\mathbb{R}; \dot{I}^*, \dot{C}^*) \models \theta(\check{y}_0, \ldots, \check{y}_n).$$

Again by $(\clubsuit)$, $D = E^*$ where E has the same definition in $L(B, \mathbb{R}^V)$ so that g meets E. Now $E \subseteq D$, which gives

$$g \text{ decides } (\mathbb{R}; \dot{I}^*, \dot{C}^*) \models \theta(\check{y}_0, \ldots, \check{y}_n)$$

A similar argument yields $(\heartsuit.iv)$. We already know that $(\heartsuit.v)$ holds, so p is indeed lovely. $\square$

A final use of $(\clubsuit)$ shows that there must be a p in $\mathbb{V}_{\max}^{L(B,\mathbb{R})}$ which is lovely w.r.t. $\mathcal{R}, \zeta$. In fact, the argument above shows that there is such a condition below every element of G. Thus there must be such a p in G and hence $\Psi(\underline{\mathcal{R}}, \zeta)$ holds in V by Lemma 11.54. $\square$

Remark 11.56. Some assumption beyond $\mathbb{V}_{\max}$-$(*)_A^+$ is necessary to prove $\mathrm{MM}_{\mathfrak{c}}^{++}(A)$. For example in the case $\mathbb{V}_{\max} = \mathbb{P}_{\max}$, $\mathrm{MM}_{\mathfrak{c}}^{++}$ does not hold in the $\mathbb{P}_{\max}$ extension of a model of the form $L(\Gamma, \mathbb{R})$ as BMM entails closure of the universe under sharps. This was proven by Ralf Schindler in [Sch04] and even more than that later in [Sch06].

Let us briefly discuss the remaining diagram. All implications we have not discussed yet are either trivial or have been proven before. For example, uB-$\mathrm{BMM}^{++}(A) \Rightarrow \mathbb{V}_{\max}$-$(*)_A^{\mathrm{uB}}$ follows from (the proof of) the Second Blueprint Theorem 4.58 making use of the following fact.

Fact 11.57 (Woodin)**.** *If there is a proper class of Woodin cardinals, then*

(*i*) uB *is closed under projections and*

(*ii*) *the* $*$ *and* $\exists^{\mathbb{R}}$ *operators commute on* uB*, i.e. in any forcing extension*

$$(\exists^{\mathbb{R}}A)^* = \exists^{\mathbb{R}}A^*$$

for ∞*-universally Baire sets* A*.*

The upshot of this is that if there are a proper class of Woodins and B is projective in a ∞-universally Baire set A then B is itself ∞-universally Baire and in any forcing extension B^* has the same definition from A^* as B has from A in V. The relevant consequence in the current context is that ∞-universally Baire dense subsets of $\mathbb{V}_{\max}$ have dense interpretations in any forcing extension.
Let us now consider the remaining two negative results. Regarding the first, Woodin [Woo10, Theorem 10.90] produced a model of $\mathrm{MM}^{++}(\mathfrak{c})$ in which $(*)$ fails. Regarding the other one, i.e. $\mathrm{MM}_{\mathfrak{c}}(f) \not\Rightarrow \mathrm{BMM}^{++}(f)$, Shelah has shown that MM does not imply PFA^+, see [She98, XVII §3]. He did this by producing a model of MM in which there is a proper forcing $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{S}$ for a stationary set so that for no $\mathbb{P}$-filter F, the interpretation $\dot{S}^F$ is stationary where

$$\dot{S}^F = \{\alpha \mid \exists p \in F\ p \Vdash \check{\alpha} \in \dot{S}\}.$$

A simple trick shows that then the bounded version of PFA^+ fails, but we do not bother defining this axiom, so officially we prove:

Proposition 11.58. *Assume that there is a stationary set preserving forcing* $\mathbb{P}$ *and a* $\mathbb{P}$*-name* $\dot{S}$ *for a stationary subset of* ω_1 *so that for no* $\mathbb{P}$*-filter* $F \in V$ *the interpretation* $\dot{S}^F$ *is stationary. Then* BMM^{++} *fails.*

Proof. We may assume that $\mathbb{P}$ is a complete Boolean algebra and that $\dot{S}$ is of the form

$$\dot{S} = \{(\check{\beta}, a_\beta) \mid \beta < \omega_1\}.$$

Find θ large enough with $\mathbb{P} \in H_\theta$ and let $X \prec H_\theta$ be of size ω_1 with $\omega_1 \subseteq X$, $\dot{S} \in X$. Let $\bar{\mathbb{P}}, \dot{T}, \bar{a}_\beta$ be the π_X-preimage of $\mathbb{P}, \dot{S}$ and a_β respectively for $\beta < \omega_1$.

Claim 11.59. $(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \models$ "*there is a filter* F *so that* $\dot{T}^F \notin \mathrm{NS}_{\omega_1}$"*.*

Proof. Let G be $\mathbb{P}$-generic. Then $F = \pi_X^{-1}[G]$ is a $\bar{\mathbb{P}}$-filter: It is clearly upwards closed and if $p, q \in F$ then $p \wedge q \in F$. Moreover by our assumption,

$$\dot{T}^F = \{\beta \mid \pi_X^{-1}(a_\beta) \in F\} = \{\beta \mid a_\beta \in G\} = \dot{S}^G$$

is stationary. $\square$

On the other hand this statement is not true in V: If F is a $\bar{\mathbb{P}}$-filter then the upwards closure $\hat{F}$ of $\pi_X[F]$ is a $\mathbb{P}$-filter with $\dot{T}^F = \dot{S}^{\hat{F}}$ non-stationary. $\square$

It follows that MM does not imply BMM^{++}.

11.5 Disrespectful forcing and (‡)

It seems plausible that a better understanding of respectful forcings could lead to a reduction of the large cardinal assumption necessary to force QM, perhaps to an argument that QM could be forced from one supercompact cardinal only. One would thus hope for some sort of iteration theorem involving respectful forcings. We hint at some possible difficulties implementing this strategy: We will show that in L, the σ-closed forcing $\mathrm{Add}(\omega_1, 1)$ is not respectful. This seems surprising as this forcing generally has very nice properties. As a consequence, there does not seem to be a reasonable iteration theorem for respectful forcings provable in ZFC. For example, the "iteration theorem for respectful semiproper forcing" is not provable.

Definition 11.60. An ω_1-preserving forcing $\mathbb{P}$ is weakly respectful if the following is true: Whenever

- θ is sufficiently large and regular,
- $X \prec H_\theta$ is countable with $\mathbb{P} \in X$,
- $\dot{S} \in X$ is a $\mathbb{P}$-name for a subset of ω_1 and
- $p \in \mathbb{P} \cap X$

then exactly one of the following holds:

(wRes.i) Either there is some $(X, \mathbb{P})$-semigeneric $q \leqslant p$ so that

$$q \Vdash \delta^{\check{X}} \notin \dot{S}$$

or

(wRes.ii) there is some $A \in X \cap \mathcal{P}(\omega_1)$ with $\delta^X \in A$ so that

$$p \Vdash \check{A} \subseteq \dot{S} \mod \mathrm{NS}_{\omega_1}.$$

It is not hard to see that respectful forcings are weakly respectful, just consider the name $\dot{I}$ for the ideal generated by NS_{ω_1} and $\dot{S}$ in the extension. All c.c.c. forcings are weakly respectful. However, in general not all σ-closed forcings are weakly respectful.

Lemma 11.61. *If $V = L$, then $\mathrm{Add}(\omega_1, 1)$ is not weakly respectful.*

Proof. Assume $V = L$. Consider the function $f\colon \omega_1 \to \omega_1$ defined via $f(\alpha)$ is the least β so that α is countable in $L_{\beta+1}$ (this is the well known example of a function in L not bounded by any canonical function). If G is $\mathrm{Add}(\omega_1, 1)$-generic then in $L[G]$ we can define

$$S := \{\alpha < \omega_1 \mid G \restriction \alpha \text{ is generic over } L_{f(\alpha)}\}.$$

Here, "$G \restriction \alpha$ is generic over $L_{f(\alpha)}$" means that

$$G \restriction \alpha := \{p \in G \mid \mathrm{dom}(p) < \alpha\}$$

is generic for $\mathrm{Add}(\alpha, 1)^{L_{f(\alpha)}}$ over $L_{f(\alpha)}$ (so in particular $G \restriction \alpha \subseteq L_{f(\alpha)}$). Let $\dot{S}$ be a name in L for this set S.

Claim 11.62. *It $T \in (\mathrm{NS}^+_{\omega_1})^L$ then both $T \cap S, T - S$ are stationary in $L[G]$.*

Proof. First, let us see that $T \cap S$ is stationary in $V[G]$. Let $\dot{C} \in L^{\mathrm{Add}(\omega_1,1)}$ be a name for a club and $p \in \mathrm{Add}(\omega_1, 1)$. Choose some sufficiently large and regular θ and find $X \prec H_\theta$ countable so that

$(X.i)$ $p, \dot{C} \in X$ and

$(X.ii)$ $\delta^X \in T$.

X collapses to L_β for some $\beta \leqslant f(\delta^X)$. We can now find a descending sequence $\langle p_n \mid n < \omega\rangle$ through $\mathrm{Add}(\delta^X, 1)^{L_\beta}$ so that for all dense $D \subseteq \mathrm{Add}(\delta^X, 1)^{L_\beta}$, $D \in L_{f(\delta^X)}$ there is n with $p_n \in D$. Clearly, $q := \bigcup_{n<\omega} p_n$ is $(X, \mathrm{Add}(\omega_1, 1))$-generic so that

$$q \Vdash \check{\delta^X} \in \dot{C}.$$

The final point is that by acceptability of the L-hierarchy[58],

$$\mathrm{Add}(\delta^X, 1)^{L_\beta} = \mathrm{Add}(\delta^X, 1)^{L_{f(\delta^X)}}$$

so that

$$q \Vdash \check{\delta^X} \in \dot{S}$$

as well.

To see that $T - S$ is stationary in $L[G]$, let $\dot{C}$, p, θ and X be as before. There is a descending sequence $\vec{p} := \langle p_\alpha \mid \alpha < \omega_1\rangle \in H_\theta$ of conditions in $\mathrm{Add}(\omega_1, 1)$ so that

$$\forall \alpha < \omega_1 \exists \alpha \leqslant \beta < \omega_1\ p_\alpha \Vdash \check{\beta} \in \dot{C}.$$

and $p_0 \leqslant p$. By elementarity, we may assume $\vec{p} \in X$. Let $q = \bigcup_{\alpha<\delta^X} p_\alpha$. It is clear that q is *not* $(X, \mathrm{Add}(\omega_1, 1))$-generic, but we have $q \Vdash \check{\delta}^X \in \dot{C}$. It follows that $q \Vdash \check{\delta^X} \in \dot{C} \cap (\check{T} - \dot{S})$. $\square$

[58] This means that if $\gamma < \delta$ are ordinals and $\mathcal{P}(\gamma) \cap L_\delta \subsetneq \mathcal{P}(\gamma) \cap L_{\delta+1}$, then there is a surjection $h\colon \gamma \to \delta$ in $L_{\delta+1}$.

It follows that (wRes.i) in the definition of weakly respectful fails for $\dot{S}$ (for any appropriate θ, X, p).
To get a failure of (wRes.ii) as well, it is enough to find a countable $X \prec H_\theta$ so that if $X \cong L_\gamma$ then $\mathcal{P}(\delta) \cap L_\gamma = \mathcal{P}(\delta) \cap L_{f(\delta)}$ for $\delta = \delta^X$: Any q that is $(X, \mathrm{Add}(\omega_1, 1))$-semigeneric is in fact $(X, \mathrm{Add}(\omega_1, 1))$-generic (as $\mathrm{Add}(\omega_1, 1)$ has size ω_1) and hence as L_γ and $L_{f(\delta)}$ have the same dense subsets of $\mathrm{Add}(\delta, 1)$, such q forces $G \upharpoonright \delta$ to be generic over $L_{f(\delta)}$.
Ralf Schindler explained the following example of such an X to the author: Let $X_0 = \mathrm{Hull}^{H_\theta}(\emptyset)$ and for $n < \omega$ let $X_{n+1} = \mathrm{Hull}^{H_\theta}(\{X_n\})$. Put $X = \bigcup_{n<\omega} X_n$. We will show that this X works as intended. Let $\delta := \delta^X$ and find γ with $X \cong L_\gamma$.

Claim 11.63. $f(\delta) = \gamma + 1$.

Proof. L_γ is a model of ZF^-, δ is uncountable in L_γ and so $f(\delta) > \gamma$. For $n < \omega$ let $X_n \cong L_{\gamma_n}$. Then $\langle \gamma_n \mid n < \omega \rangle$ is definable over $L_{\gamma+1}$: γ_0 is the least ξ with $L_\xi \prec L_\gamma$ and γ_{n+1} is the least $\xi > \gamma_n$ with $L_\xi \prec L_\gamma$. Thus $\langle \delta^{X_n} \mid n < \omega \rangle$ is definable over $L_{\gamma+1}$ and as $\delta = \sup_{n<\omega} \delta^{X_n}$, δ is countable in $L_{\gamma+2}$. □

As L_γ is a model of ZF^- we have

$$\mathcal{P}(\delta) \cap L_\gamma = \mathcal{P}(\delta) \cap L_{\gamma+1}$$

so that X has the desired property. □

Corollary 11.64. *Assume $V = L$. Then there is a nice (or* RCS*) iteration of semiproper respectful forcings which is not weakly respectful.*

Proof. $\mathrm{Add}(\omega_1, 1)$ is isomorphic to the iteration of length ω_1 of the respectful forcing consisting of two incompatible conditions together with a maximal element. □

We will finish by showing that (‡) can be phrased as a reflection principle.

Definition 11.65. Suppose I is a normal uniform ideal on ω_1 and A is uncountable with $\omega_1 \subseteq A$. A set $\mathcal{S} \subseteq [A]^\omega$ is *I-stationary* if for all sufficiently large regular λ and clubs $\mathcal{C} \subseteq [H_{\lambda^+}]^\omega$ there is some countable $X \prec H_{\lambda^+}$ which respects I so that $X \in \mathcal{C}$ and $X \cap A \in \mathcal{S}$.

Definition 11.66. Suppose I is a normal uniform ideal on ω_1.

(i) If A is uncountable with $\omega_1 \subseteq A$ then a stationary set $\mathcal{S} \subseteq [A]^\omega$ is *I-full* if whenever

- λ is large enough regular with $A \in H_\lambda$,
- $X \prec H_{\lambda^+}$ is countable with $A \in X$ and $X \cap A \in \mathcal{S}$,

- Y is countable with $X \sqsubseteq Y \prec H_{\lambda^+}$ and
- Y respects I

then $Y \cap A \in \mathcal{S}$.

(*ii*) I-SSR holds if for any uncountable A any I-full I-stationary $\mathcal{S} \subseteq [A]^\omega$ *I-reflects*, that is there is some $R \subseteq A$ of size ω_1 with $\omega_1 \subseteq R$ so that $\mathcal{S} \cap [R]^\omega \subseteq [R]^\omega$ is I-stationary.

Note that NS_{ω_1}-SSR is (equivalent to) the usual axiom SSR of Semistationary Reflection. Recall that the principle (†) holds if every stationary set preserving forcing is semiproper, see [FMS88].

Fact 11.67 (Shelah,[She98, XIII Claim 1.3])**.** *(†) is equivalent to* SSR.

(‡) is a natural strengthening of (†) and we will find an equivalent of (‡) in terms of the principles I-SSR.

Theorem 11.68. *The following are equivalent:*

(*i*) (‡).

(*ii*) *For all normal uniform ideals I on ω_1, I-*SSR *holds.*

Proof. (*i*)⇒(*ii*): Suppose I is a normal uniform ideal on ω_1, $\lambda \geqslant \omega_2$ is uncountable and $\mathcal{S} \subseteq [\lambda]^\omega$ is I-full I-stationary but does not I-reflect. Let $\mathbb{P}$ be the canonical forcing to shoot a continuous increasing chain of elementary substructures of H_{λ^+} of length ω_1 through

$$\mathcal{T} := [H_{\lambda^+}]^\omega - \{X \in [H_{\lambda^+}]^\omega \mid X \cap \lambda \in \mathcal{S} \wedge X \text{ respects } I\}.$$

Conditions in $\mathbb{P}$ are countable approximations

$$\langle X_i \mid i \leqslant \alpha\rangle$$

to such a sequence of successor length. Let $\dot{I}$ be a $\mathbb{P}$-name for the normal ideal generated by I in $V^{\mathbb{P}}$.

Claim 11.69. *$\dot{I}^p \subseteq I$ for all $p \in \mathbb{Q}$.*

Proof. Note that $\mathbb{P}$ forces $|I| = \omega_1$. Let $T \in I^+$ and $\theta > \lambda$ be sufficiently large and regular. Let $p \in \mathbb{P}$, $\dot{C}$ a $\mathbb{P}$-name for a club in ω_1. Let $\langle Y_i \mid i < \omega_1\rangle$ be a continuous increasing chain of countable elementary substructures of H_θ with $\lambda, \mathcal{S}, p, \dot{C} \in Y_0$. As $\mathcal{S}$ does not I-reflect, we have

$$\{\alpha < \omega_1 \mid Y_\alpha \cap \lambda \in \mathcal{S}\} \in I.$$

Thus we can find some $\alpha < \omega_1$ so that $\delta^{Y_\alpha} \in T$, Y_α respects I and $Y_\alpha \cap \lambda \notin \mathcal{S}$. It is straightforward to build a condition $q \leqslant p$ that is $(Y_\alpha, \mathbb{P})$-generic. If G is $\mathbb{P}$-generic with $q \in G$ then $Y_\alpha[G] \cap V = Y_\alpha$ and

$$\delta^{Y_\alpha} \in \dot{C}^G - \bigcup_{Z \in I \cap Y_\alpha[G]} Z.$$

This shows that $\neg(p \Vdash \dot{C} \cap T \in \dot{I})$. □

Let θ be sufficiently large and regular. As $\mathcal{S}$ is I-stationary, there is some countable $Y \prec H_\theta$ with

$(Y.i)$ $\mathbb{P} \in Y$,

$(Y.ii)$ $Y \cap \lambda \in \mathcal{S}$ and

$(Y.iii)$ Y respects I.

By (‡), there is some q which is $(Y, \mathbb{P})$-semigeneric so that

$$q \Vdash \text{“}\check{Y}[\dot{G}] \text{ respects } \dot{I}\text{”}.$$

Let G be $\mathbb{P}$-generic with $q \in G$. Then $Y \sqsubseteq Y[G]$ and let $\vec{X}$ be the generic sequence added by G. It is not difficult to see that

$$X_{\delta^Y} = Y[G] \cap H^V_{\lambda^+}.$$

Note that $Y[G] \cap H^V_{\lambda^+}$ respects I and hence $Y[G] \cap \lambda \in \mathcal{S}$ since $\mathcal{S}$ is I-full. But then $X_{\delta^Y} \notin \mathcal{T}$, contradiction.
$(ii) \Rightarrow (i)$: Suppose that $\mathbb{P}$ is an ω_1-preserving forcing and (‡) fails for $\mathbb{P}$. We can find a sufficiently large regular λ, some $p \in \mathbb{P}$ and some $\mathbb{P}$-name $\dot{I}$ for a normal uniform ideal on ω_1 so that the set $\mathcal{S}$ consisting of all countable $X \prec H_\lambda$ with

$(X.i)$ $p, \mathbb{P}, \dot{I} \in X$,

$(X.ii)$ X respects $\dot{I}^p$ and

$(X.iii)$ there is no $(X, \mathbb{P})$-semigeneric $q \leqslant p$ with $q \Vdash$ “$\check{X}[\dot{G}]$ respects $\dot{I}$”

is stationary in $[H_\lambda]^\omega$. Clearly $\mathcal{S}$ is $\dot{I}^p$-stationary and further note that $\mathcal{S}$ is $\dot{I}^p$-full by Proposition 3.56. By $\dot{I}^p$-SSR, there is some $R \subseteq H_\lambda$ of size ω_1 with $\omega_1 \subseteq R$ so that $\mathcal{S} \cap [R]^\omega \subseteq [R]^\omega$ is $\dot{I}^p$-stationary. Let

$$r\colon \omega_1 \to R$$

be a surjection and note that $T := \{\alpha < \omega_1 \mid r[\alpha] \in \mathcal{S}\} \notin \dot{I}^p$. Let

$$S := \{\alpha < \omega_1 \mid r[\alpha] \cap \omega_1 = \alpha \wedge r[\alpha] \in \mathcal{S}\}.$$

Claim 11.70. $S \in \dot{I}^p$.

Proof. Let G be generic with $p \in G$ and let $I := \dot{I}^G$. Suppose for a contradiction that $S \in I^+$. There is then some countable $Y \prec H_{\lambda^+}^{V[G]}$ so that $r \in Y$, Y respects I and $\delta^Y \in S$. Consider $X = r[\delta^Y]$. We have that $X \in \mathcal{S}$ and $\delta^X = \delta^Y$. Consequently $X[G] \sqsubseteq Y$ and $X[G]$ respects I, but this clearly contradicts $X \in \mathcal{S}$. □

But $T - S$ is nonstationary, contradiction. □

What is the exact relationship between (‡) and other reflection principles?

Question 11.71. Does (†) imply (‡)? If not, does it follow from WRP?

Part II

The Axiom of Choice in the κ-Mantle

12 Introduction

This second part belongs to the topic of Set Theoretic Geology.

12.1 Set-theoretic geology

The interest of this area is the study of the structure of grounds, that is inner models of ZFC that extend to V via forcing, and associated concepts. Motivated by the hope to uncover canonical structure hidden underneath generic sets, the mantle was born[59].

Definition 12.1. The mantle, denoted $\mathbb{M}$, is the intersection of all grounds.

This definition only makes sense due to the uniform definability of grounds.

Fact 12.2. *There is a first order* $\in$*-formula* $\varphi(x, y)$ *such that*

$$W_r = \{x|\varphi(x, r)\}$$

defines a ground for all $r \in V$ *and all grounds are of this form. Moreover, if* κ *is a cardinal and* W *extends to* V *via a forcing of size* $<\kappa$ *then there is* $r \in V_\kappa$ *with* $W = W_r$.

This was proven independently by Woodin [Woo11] [Woo04], Laver [Lav07] and was later strengthened by Hamkins, see [FHR15].

This allows us to quantify freely over grounds as we will frequently do. It was quickly realized that every model of ZFC is the mantle of another model of ZFC, see [FHR15], which eradicated any chance of finding nontrivial structure in the mantle. However, the converse question remained open for some while, namely whether the mantle is provably a model of ZFC. This tough nut was cracked by Toshimichi Usuba.

Fact 12.3 (Usuba,[Usu17])**.** *The mantle is always a model of* ZFC.

Thereby the mantle was established as a well behaved canonical object in the theory of forcing. Fuchs-Hamkins-Reitz [FHR15] suggested to study restricted forms of the mantle.

Definition 12.4. Let Γ be a class[60] Γ of forcings.

(*i*) A Γ-*ground* is a ground W that extends to V via a forcing $\mathbb{P} \in \Gamma^W$.

(*ii*) The Γ-*mantle* $\mathbb{M}_\Gamma$ is the intersection of all Γ-grounds.

[59] Rumor has it that Hamkins first thought of the mantle in the Mensa am Ring in Münster.

[60] In this case, we think of Γ as a definition, possibly with ordinal parameters, so that Γ can be evaluated grounds of V.

(*iii*) We say that the Γ-grounds are *downwards directed* if for any two Γ-grounds W_0, W_1 there is a Γ-ground $W_* \subseteq W_0, W_1$.

(*iv*) We say that the Γ-grounds are downwards set-directed if for any set-indexed collection of Γ-grounds $\langle W_r \mid r \in X\rangle$ there is a Γ-ground W_* contained in all W_r for $r \in X$.

(*v*) We say that Γ is *ground absolute* if the Γ-grounds of a Γ-ground W are exactly those common grounds of V and W that are Γ-grounds from the perspective of V, i.e. being a Γ-ground is absolute between V and all Γ-grounds.

Remark 12.5. Note that if Γ is provably (in ZFC) closed under quotients and two-step iterations then Γ is ground absolute.

Fuchs-Hamkins-Reitz [FHR15] have shown abstractly that if Γ is ground absolute and has directed grounds then $\mathbb{M}_\Gamma \models \mathrm{ZF}$. To prove $\mathbb{M} \models \mathrm{AC}$ they seemingly need the stronger assumption that the Γ-grounds are downwards set-directed, the argument is as follows: Suppose $X \in \mathbb{M}$ is not wellordered in $\mathbb{M}$. Then for every wellorder $\prec$ of X, we choose $W_\prec$ a Γ-ground from which $\prec$ is missing. By downwards set directedness, there is a Γ-ground W contained in all such grounds $W_\prec$, but then $X \in W$ is not wellordered in W either, contradiction. The main result of this part shows that indeed simple downwards directedness does not suffice to prove choice in $\mathbb{M}_\Gamma$ in general.

We will be interested in $\mathbb{M}_\Gamma$ for Γ the class of all forcings of size $<\kappa$, where κ is some given cardinal. In this case, we denote the Γ-mantle by $\mathbb{M}_\kappa$ and call it the *κ-mantle*. The associated grounds are the *κ-grounds*. The interest of the κ-mantle arose in different contexts.

The following is known:

Fact 12.6 (Usuba, [Usu18])**.** *If κ is a strong limit then* $\mathbb{M}_\kappa \models \mathrm{ZF}$.

Usuba proved this by showing that the κ-grounds are directed in this case. Usuba subsequently asked:

Question 12.7 (Usuba, [Usu18])**.** Is $\mathbb{M}_\kappa$ always a model of ZFC?

We will answer this question in the negative by providing counterexamples for three different types of cardinals κ.

We also mention that Fuchs-Hamkins-Reitz demonstrated that $\mathbb{M}_\Gamma$ can fail to be a model of choice for a different class of forcings, $\Gamma = \{\sigma\text{-closed forcings}\}$.

Fact 12.8 (Fuchs-Hamkins-Reitz, [FHR15])**.** *If Γ is the class of all σ-closed forcings it is consistent that* $\mathbb{M}_\Gamma \models \mathrm{ZF} \wedge \neg\mathrm{AC}$.

It turns out that there is an interesting tension between large cardinal properties of κ and the failure of choice in $\mathbb{M}_\kappa$. On the one side, Usuba has shown:

Fact 12.9 (Usuba, [Usu18]). *If κ is extendible then $\mathbb{M}_\kappa = \mathbb{M}$. In particular $\mathbb{M}_\kappa$ is a model of* ZFC.

Indeed, this result was the initial motivation of investigating the κ-mantle. Sargsyan-Schindler [SS18] showed that a similar situation arises in the least iterable inner model with a strong cardinal above a Woodin cardinal for κ the unique strong cardinal in this universe. See also [SSS21] and [Sch22b] for further results in this direction.
On another note, Schindler has proved the following.

Fact 12.10 (Schindler, [Sch18]). *If κ is measurable then $\mathbb{M}_\kappa \models$ ZFC.*

The big difference to Fact 12.9 is that the existence of a measurable is consistent with the failure of the Bedrock Axiom[61]. Particularly, we might have $\mathbb{M}_\kappa \neq \mathbb{M}$ for κ measurable.
If we go even lower in the large cardinal hierarchy then even less choice principles seem to be provable in the corresponding mantle. The relevant results here are due to Farmer Schlutzenberg.

Fact 12.11 (Schlutzenberg, [Sch22a]). *Suppose that κ is weakly compact. Then*

(i) $\mathbb{M}_\kappa \models \kappa$-DC *and*

(ii) for every $A \in H_{\kappa^+} \cap \mathbb{M}_\kappa$,

$$\mathbb{M}_\kappa \models \text{“}A \in H_{\kappa^+} \textit{ is wellorderable”}.$$

Definition 12.12. Suppose α is an ordinal and X is a set. $(<\alpha, X)$-choice holds if for any $\beta < \alpha$ and any sequence $\vec{x} := \langle x_\gamma \mid \gamma < \beta\rangle$ of nonempty elements of X there is a choice sequence for $\vec{x}$, that is a sequenece $\langle y_\gamma \mid \gamma < \beta\rangle$ with $y_\gamma \in x_\gamma$ for all $\gamma < \beta$.

Fact 12.13 (Schlutzenberg, [Sch22a]). *Suppose κ is inaccessible. Then we have*

(i) $V_\kappa \cap \mathbb{M}_\kappa \models$ ZFC *and*

(ii) $\mathbb{M}_\kappa \models$ “$(<\kappa, H_{\kappa^+})$-*choice*”.

[61] The Bedrock Axiom states that the universe has a minimal ground, which turns out to be equivalent to “$\mathbb{M}$ is a ground”.

12.2 Overview

In Section 13.1, we will argue that "κ is measurable" cannot be replaced by "κ is Mahlo" in Fact 12.10, as wells as that $(<\kappa, H_{\kappa^+})$-choice cannot be strengthened to $(<\kappa + 1, H_{\kappa^+})$-choice in Fact 12.13.

Theorem 12.14. *If* ZFC *is consistent with the existence of a Mahlo cardinal, then it is consistent with* ZFC *that there is a Mahlo cardinal κ so that $\mathbb{M}_\kappa$ fails to satisfy the axiom of choice. In fact we may have*

$$\mathbb{M}_\kappa \models \text{"}(<\kappa + 1, H_{\kappa^+})\text{-}\textit{choice fails}\text{"}.$$

In Section 13.2, we will investigate the κ-mantle for $\kappa = \omega_1$, as well as the Γ-mantle where $\Gamma = \{\text{Cohen forcing}\}$, denoted by $\mathbb{M}_{\mathsf{C}}$. We will first proof that these mantles are always models of ZF and will go on to provide a result analogous to Theorem 12.14.

Theorem 12.15. *It is consistent relative to a Mahlo cardinal that both $\mathbb{M}_{\omega_1}$ and $\mathbb{M}_{\mathsf{C}}$ fail to satisfy the axiom of choice.*

In Section 13.3, we will generalize this to any successor of a regular cardinal.

Theorem 12.16. *Suppose that*

(*i*) GCH *holds,*

(*ii*) *the Ground Axiom*[62] *holds and*

(*iii*) *κ is a regular uncountable cardinal.*

Then there is a cardinal preserving generic extension in which the κ^+-mantle fails to satisfy the axiom of choice.

In this case however, it is not known if the κ^+-mantle is a model of ZF in general. The proof of all these three theorems follows a similar pattern, though the details differ from case to case and it seems that we cannot employ a fully unified approach.

13 The Axiom of Choice May Fail in $\mathbb{M}_\kappa$

13.1 The case "κ is Mahlo"

Here, we will construct a model where the κ-mantle for a Mahlo cardinal κ does not satisfy the axiom of choice. We will start with L and assume that

[62]The Ground Axiom states that there is no nontrivial ground. See [Rei06] for more information on this axiom.

κ is the least Mahlo there. The final model will be a forcing extension of L by

$$\mathbb{P} = \prod_{\lambda \in I \cap \kappa}^{<\kappa\text{-support}} \operatorname{Add}(\lambda, 1)$$

where I is the class of all inaccessible cardinals. We define $\mathbb{P}$ to be a product forcing and not an iteration (in the usual sense), as we want to generate many κ-grounds. Let G be $\mathbb{P}$-generic over L. We will show that κ is still Mahlo in $L[G]$ and that $\mathbb{M}_\kappa^{L[G]}$ does not satisfy the axiom of choice. We remark that, would we start with a model in which κ is measurable, $\mathbb{P}$ would provably force κ to not be measurable.

First, let's fix notation. For $\lambda < \kappa$, we may factor $\mathbb{P}$ as $\mathbb{P}_{\leqslant\lambda} \times \mathbb{P}_{>\lambda}$ where in each case we only take a product over all $\gamma \in I \cap \kappa$ with $\gamma \leqslant \lambda$ and $\gamma > \lambda$ respectively. Observe that $\mathbb{P}_{>\lambda}$ is a $<\kappa$-support product while $\mathbb{P}_{\leqslant\lambda}$ is a full support product. We also factor G as $G_{\leqslant\lambda} \times G_{>\lambda}$ accordingly. For $\lambda \in I \cap \kappa$ we denote the generic for $\operatorname{Add}(\lambda, 1)^L$ induced by G as g_λ. In addition to this, for $\alpha \leqslant \kappa$ we denote the α-th inaccessible cardinal by I_α.
For $\alpha < \kappa$ let $E_\alpha\colon \kappa \to 2$ be the function induced by g_{I_α}. It will be convenient to think of G as a $\kappa \times \kappa$-matrix M which arises by stacking the maps $(E_\alpha)_{\alpha<\kappa}$ on top of each other, starting with E_{I_0} and proceeding downwards, and then filling up with 0's to produce rows of equal length κ. Let us write

$$e_{\alpha,\beta} = \begin{cases} E_\alpha(\beta) & \text{if } \beta < I_\alpha \\ 0 & \text{else.} \end{cases}$$

The $(e_{\alpha,\beta})_{\alpha,\beta<\kappa}$ are the entries of M:

$$M = \begin{array}{c} \\ \\ r_1 \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccc} & & c_2 & & & & & & & \end{array} \\ \left(\begin{array}{cccccccccc} e_{0,0} & e_{0,1} & e_{0,2} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ e_{1,0} & e_{1,1} & e_{1,2} & \cdots & e_{1,I_0} & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & 0 & \cdots & 0 & \cdots \\ e_{\alpha,0} & e_{\alpha,1} & e_{\alpha,2} & \cdots & e_{\alpha,I_0} & \cdots & e_{\alpha,I_1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 & \cdots \end{array}\right) \\ M_{\geqslant\alpha} \end{array}$$

We will give the α-th row of M the name r_α and we denote the β-th column of M by c_β. One trivial but key observation is that r_α carries the same information as g_{I_α}.

We will be frequently interested in the matrix M with its first α rows deleted for some $\alpha < \kappa$, so we will give this matrix the name $M_{\geqslant\alpha}$. Note that $M_{\geqslant\alpha}$ corresponds to the generic $G_{\geqslant I_\alpha}$. Finally observe that we may think of conditions in $\mathbb{P}$ as partial matrices that approximate such a matrix M in the sense that they already have the trivial 0's in the upper right corner, in any row $\alpha < \kappa$ they have information for $<I_\alpha$ many $\beta < I_\alpha$ on whether $e_{\alpha,\beta}$ is 0 or 1 and they contain non-trivial information in less than κ-many rows.

Lemma 13.1. *L and $L[G]$ have the same inaccessibles.*

Proof. First, we show that all limit cardinals of L are limit cardinals in $L[G]$. It is enough to prove that all double successors δ^{++} are preserved. This is obvious for $\delta \geqslant \kappa$ as $\mathbb{P}$ has size κ. For $\delta < \kappa$, $\mathbb{P}_{>\delta}$ is $\leqslant\delta^{++}$-closed so that all cardinals $\leqslant \delta^{++}$ are preserved in $L[G_{>\delta}]$. Furthermore, $\mathbb{P}_{<\delta}$ has size at most δ^+ in $L[G_{>\delta}]$ by GCH in L. Hence δ^{++} is still a cardinal in $L[G]$.

Now we have to argue that all $\lambda \in I$ remain regular. Again, this is clear if $\lambda > \kappa$. On the other hand, assume $\delta := \mathrm{cof}(\lambda)^{L[G]} < \lambda$. As $\mathbb{P}_{>\delta}$ is $\leqslant\delta$-closed, λ is still regular in $L[G_{>\delta}]$. Hence, a witness to $\mathrm{cof}(\lambda) = \delta$ must be added in the extension of $L[G_{>\delta}]$ by $\mathbb{P}_{\leqslant\delta}$. But this forcing has size $< \lambda$ in $L[G_{>\delta}]$ and thus could not have added such a sequence. □

In fact, $\mathbb{P}$ does not collapse any cardinals (if $V = L$), but some more work is required to prove this. This is, however, not important for our purposes. Next, we aim to show that κ remains Mahlo in $L[G]$.

To prove this, it is convenient to introduce a generalization of Axiom A.

Definition 13.2. For κ an ordinal, λ a cardinal we say that a forcing $\mathbb{Q}$ satisfies Axiom A(κ, λ), abbreviated by AA(κ, λ), if there is a sequence $\langle \leqslant_\alpha \mid \alpha < \kappa\rangle$ of partial orders on $\mathbb{Q}$ so that

(AA.*i*) $\forall \alpha \leqslant \beta < \kappa \;\; \leqslant_\beta \subseteq \leqslant_\alpha \subseteq \leqslant_{\mathbb{Q}}$,

(AA.*ii*) for all antichains A in $\mathbb{Q}$, $\alpha < \kappa$ and $p \in \mathbb{Q}$ there is $q \leqslant_\alpha p$ so that $|\{a \in A \mid a \| q\}| < \lambda$ and

(AA.*iii*) for all $\beta < \kappa$ if $\vec{p} = \langle p_\alpha \mid \alpha < \beta\rangle$ satisfies $p_\gamma \leqslant_\alpha p_\alpha$ for all $\alpha < \gamma < \beta$ then there is a fusion p_β of $\vec{p}$, that is $p_\beta \leqslant_\alpha p_\alpha$ for all $\alpha < \beta$.

Remark 13.3. The usual Axiom A is thus Axiom A$(\omega + 1, \omega_1)$.

Proposition 13.4. *Suppose λ is regular uncountable cardinal and $\mathbb{Q}$ satisfies* AA(λ, λ)*. Then $\mathbb{Q}$ preserves stationary subsets of λ.*

Proof. Suppose $S \subseteq \lambda$ is stationary, $\dot{C}$ is a $\mathbb{Q}$-name for a club in λ and $p \in \mathbb{P}$. We will imitate the standard proof that a $<\kappa$-closed forcing preserves stationary sets. Let $\langle \leqslant_\alpha \mid \alpha < \lambda\rangle$ witness that $\mathbb{Q}$ satisfies AA(λ, λ).

Claim 13.5. *For any $q \in \mathbb{Q}, \alpha < \lambda$ there is $r \leqslant_\alpha q$ and some $\alpha < \gamma < \lambda$ with $q \Vdash \check{\gamma} \in \dot{C}$.*

Proof. Construct a sequence $\langle q_\alpha \mid \alpha < \omega\rangle$ of conditions in $\mathbb{Q}$ and an ascending sequence $\langle \gamma_n \mid n < \omega\rangle$ of ordinals with (*i*) $q_0 = q$, $\gamma_0 = \alpha$,

(*ii*) $q_{n+1} \leqslant_{\alpha+n} q_n$ for all $n < \omega$ and

(*iii*) $q_{n+1} \Vdash$ "$\dot{C} \cap (\check{\gamma}_n, \check{\gamma}_{n+1}) \neq \varnothing$

for all $n < \omega$. The construction is immediate using that λ is regular uncountable and (AA.*iii*). Then by (AA.*ii*), there is $q_* \leqslant_\alpha q$ which is below all q_n, $n < \omega$. It follows that

$$q_* \Vdash \check{\gamma}_* \in \dot{C}$$

where $\gamma_* = \sup_{n<\omega} \gamma_n$. □

Suppose toward a contradiction that $p \Vdash \dot{C} \cap \check{S} = \varnothing$. By the claim above, we can build sequences $\langle p_\alpha \mid \alpha < \lambda\rangle$ of conditions in $\mathbb{Q}$ and an increasing sequence $\langle \gamma_\alpha \mid \alpha < \lambda\rangle$ of ordinals below λ so that

(*i*) $p_0 = p$,

(*ii*) $p_\beta \leqslant_\alpha p_\alpha$ for all $\alpha \leqslant \beta < \lambda$ and

(*iii*) $p_{\alpha+1} \Vdash \check{\gamma}_\alpha \in \dot{C}$ for all $\alpha < \lambda$.

Let D be the set of all limit points $<\lambda$ of $\{\gamma_\alpha \mid \alpha < \lambda\}$. For any $\alpha < \lambda$, we have

$$p_{\alpha+1} \Vdash \check{D} \cap \gamma_\alpha \subseteq \dot{C}$$

which shows that $D \cap S = \varnothing$, contradiction. □

Lemma 13.6. *$\mathbb{P}$ satisfies* AA(κ, κ).

Proof. For $\gamma < \kappa$ define $\leqslant_\gamma$ by $r \leqslant_\gamma q$ if $r \leqslant q$ and $r \upharpoonright \gamma = q \upharpoonright \gamma$ for $q, r \in \mathbb{P}$. We will only show that (AA.*ii*) holds. So let $p \in \mathbb{P}$, $\gamma < \kappa$ and $A \subseteq \mathbb{P}$ a maximal antichain. Let $\langle q_\alpha | \alpha < \delta\rangle$ be an enumeration of all conditions in $\mathbb{P}_{\leqslant gamma}$ below $p \upharpoonright \gamma + 1$ with $\delta = |\mathbb{P}_{\leqslant\gamma}|$. We construct a $\leqslant_\gamma$-descending sequence $\langle p_\alpha | \alpha \leqslant \delta\rangle$ of conditions in $\mathbb{P}$ starting with $p_0 = p$ as follows: If $\alpha \leqslant \delta$ then choose some $\leqslant_\gamma$-bound of $\langle p_\beta \mid \beta < \alpha\rangle$. This is possible as $\mathbb{P}_{>\gamma}$ is $\leqslant\delta$-closed, as the next forcing only appears at the next inaccessible. Moreover, if possible and $\alpha < \delta$ make sure that

$$q_\alpha^\frown p_\alpha \upharpoonright (\gamma, \kappa)$$

is below a condition in A. This completes the construction. Set $q := q_\kappa$, we will show that q is compatible with at most δ-many elements of A. Toward this goal, suppose $a \in A$ and q is compatible with a. We may find some $\alpha < \kappa$ so that $a \upharpoonright \gamma + 1 = q_\alpha$. It follows that we must have succeeded in the construction of p_α with the additional demand that

$$q_\alpha^\frown p_\alpha \upharpoonright (\gamma, \kappa)$$

is below a condition in A, but this can only be true for a. We have shown that for any $a \in A$ compatible with q there is $\alpha < \delta$ with $q_\alpha^\frown q \upharpoonright (\gamma, \kappa) \leqslant a$ and note that no single α can witness this for more than one element of A. □

Corollary 13.7. *κ is Mahlo in $L[G]$.*

Proof. This follows immediately from Lemma 13.1, Lemma 13.6 and Proposition 13.4. □

Next, we aim to find an easier description of $\mathbb{M}_\kappa^{L[G]}$. Recall the λ-approximation property introduced by Hamkins [Ham03]:

Definition 13.8. Let $W \subseteq V$ be an inner model, λ an infinite cardinal.

(i) For $x \in V$, a λ-approximation of x by W is of the form $x \cap y$ where $y \in W$ is of size $\leqslant\lambda$.

(ii) $W \subseteq V$ satisfies the λ-approximation property if whenever $x \in V$ and all λ-approximations of x by W are in W, then $x \in W$.

All κ-grounds satisfy the κ-approximation property (cf. [FHR15]).

Lemma 13.9. $\mathbb{M}_\kappa^{L[G]} = \bigcap_{\lambda \in I \cap \kappa} L[G_{>\lambda}]$.

Proof. Suppose W is a κ-ground of $L[G]$. It is enough to find $\lambda \in I \cap \kappa$ such that $L[G_{>\lambda}] \subseteq W$. Clearly, $\mathbb{P} \in L \subseteq W$. As κ is a limit of inaccessibles, we may take some $\lambda < \kappa$ inaccessible so that W is a λ-ground. Thus $W \subseteq L[G]$ satisfies the λ-approximation property. We will show $G_{>\lambda} \in W$ (even $G_{\geqslant\lambda} \in W$). Find α with $\lambda = I_\alpha$, it is thus enough to show $M_{\geqslant\alpha} \in W$. To any λ-approximation $M_{\geqslant\alpha} \cap a$ of $M_{\geqslant\alpha}$ by W corresponds some $a' \subseteq \kappa \backslash \alpha \times \kappa$, $a' \in W$ of size $<\lambda$ so that

$$M_{\geqslant\alpha} \cap a = M_{\geqslant\alpha} \upharpoonright a' := \langle e_{\gamma,\beta} \mid (\gamma, \beta) \in a' \rangle.$$

We will show that all such restrictions of $M_{\geqslant\alpha}$ are in W. So let $a \in W$, $a \subseteq \kappa \backslash \alpha \times \kappa$, $|a| < \lambda$. As $0^\#$ does not exist in W, there is $b \in L$, $b \subseteq \kappa \backslash \alpha \times \kappa$ of size $< \lambda$ with $a \subseteq b$. For all $\alpha \leqslant \gamma < \kappa$, the set of $\beta < I_\gamma$ with $(\gamma, \beta) \in b$ is bounded in I_γ. As described earlier, we may think of conditions in $\mathbb{P}$ as partial $\kappa \times \kappa$ matrices. With this in mind, the conditions $p \in \mathbb{P}$ that contain information on the entry $e_{\gamma,\beta}$ for all $(\gamma, \beta) \in b$ form a dense set of $\mathbb{P}$. Thus $M \upharpoonright b = \langle e_{\gamma,\beta} \mid (\gamma, \beta) \in b \rangle$ is essentially a condition $p \in \mathbb{P} \subseteq W$ and hence $M \upharpoonright a = (M \upharpoonright b) \upharpoonright a \in W$. As $W \subseteq L[G]$ satisfies the λ-approximation property, we have $M_{\geqslant\alpha} \in W$.

□

Remark 13.10. The above argument shows that for any $\lambda \in I \cap \kappa$

$$\mathbb{M}_\kappa^{L[G_{>\lambda}]} = \mathbb{M}_\kappa^{L[G]}.$$

In fact, whenever δ is a strong limit, the δ-mantle is always absolute to any δ-ground. The use of Jensen's covering lemma in the above argument is not essential, in fact a model in which the κ-mantle does not satisfy choice for κ Mahlo can be analogously constructed in the presence of $0^\sharp$. However, the absence of $0^\sharp$ simplifies the proof.

We will later show that $\mathcal{P}(\kappa)^{\mathbb{M}_\kappa^{L[G]}}$ does not admit a wellorder in $\mathbb{M}_\kappa^{L[G]}$. First, we analyze which subsets of κ $\mathbb{M}_\kappa^{L[G]}$ knows of. We call $a \subseteq \kappa$ *fresh* if $a \cap \lambda \in L$ for all $\lambda < \kappa$.

Proposition 13.11. *The subsets of κ in $\mathbb{M}_\kappa^{L[G]}$ are exactly the fresh subsets of κ in $L[G]$.*

Proof. First suppose $a \subseteq \kappa$, $a \in \mathbb{M}_\kappa^{L[G]}$. If $\lambda < \kappa$ then $a \in L[G_{>\lambda}]$. As $\mathbb{P}_{>\lambda}$ is $\leqslant\lambda$-closed in L, $a \cap \lambda \in L$.
For the other direction assume $a \in L[G]$ is a fresh subset of κ and assume W is a κ-ground of $L[G]$. There is $\lambda < \kappa$ so that $W \subseteq L[G]$ satisfies the λ-approximation property. As a is fresh, all the λ-approximations of a in W are in W. Thus $a \in W$. □

The columns c_β, $\beta < \kappa$, of M are the fresh subsets of κ relevant to our argument.

Proposition 13.12. *All c_β, $\beta < \kappa$, are* $\mathrm{Add}(\kappa, 1)$*-generic over L.*

Proof. The map $\pi\colon \mathbb{P} \to \mathrm{Add}(\kappa, 1)$ that maps $p \in \mathbb{P}$ to the information that p has on c_β is well-defined as $\mathbb{P}$ is a bounded support iteration of length κ. Clearly, π is a projection. □

This is exactly the reason we chose bounded support in the definition of $\mathbb{P}$.

We are now in good shape to complete the argument.

Theorem 13.13. $({<}\kappa + 1, H_{\kappa^+})$*-choice fails in $\mathbb{M}_\kappa^{L[G]}$.*

Proof. Note that any generic for $\mathrm{Add}(\kappa, 1)^L$ is the characteristic function of a fresh subset of κ so that $c_\beta \in \mathbb{M}_\kappa^{L[G]}$ for any $\beta < \kappa$. Of course, the sequence $\langle c_\beta | \beta < \kappa\rangle$ is not in $\mathbb{M}_\kappa^{L[G]}$, as one can compute the whole matrix M (and thus the whole generic G) from this sequence. However, we can make this sequence fuzzy to result in an element of $\mathbb{M}_\kappa^{L[G]}$. Let $\sim$ be the equivalence relation of eventual coincidence on $({}^\kappa 2)^{\mathbb{M}_\kappa^{L[G]}}$, i.e.

$$x \sim y \Leftrightarrow \exists \delta < \kappa\ x \upharpoonright [\delta, \kappa) = y \upharpoonright [\delta, \kappa).$$

We call $\langle [c_\beta]_\sim | \beta < \kappa\rangle$ the *fuzzy sequence.*

Claim 13.14. *The fuzzy sequence is an element of $\mathbb{M}_\kappa^{L[G]}$.*

Proof. By Lemma 13.9, it is enough to show that for every $\alpha < \kappa$, $L[G_{\geqslant I_\alpha}]$ knows of this sequence. But $L[G_{>I_\alpha}]$ contains the matrix $M_{\geqslant\alpha}$ and thus the sequence

$$\langle c_\beta \restriction (\kappa\backslash\alpha) | \beta < \kappa\rangle$$

so that $L[G_{\geqslant\alpha}]$ can compute the relevant sequence of equivalence classes from this parameter. $\square$

Finally, we argue that $\mathbb{M}_\kappa^{L[G]}$ does not contain a choice sequence for the fuzzy sequence[63]. Heading toward a contradiction, let us assume that

$$\langle x_\beta | \beta < \kappa\rangle \in \mathbb{M}_\kappa^{L[G]}$$

is such a sequence. $L[G]$ knows about the sequence

$$\langle \delta_\beta | \beta < \kappa\rangle$$

where δ_β is the least δ with $x_\beta \restriction (\kappa\backslash\delta) = c_\beta \restriction (\kappa\backslash\delta)$. The set of $\lambda < \kappa$ that are closed under the map $\beta \longmapsto \delta_\beta$ is club in κ. As κ is Mahlo in $L[G]$, there is an inaccessible $\alpha = I_\alpha < \kappa$ that is closed under $\beta \longmapsto \delta_\beta$. Now observe that

$$x_\beta(\alpha) = 1 \Leftrightarrow c_\beta(\alpha) = 1 \Leftrightarrow r_\alpha(\beta) = 1$$

holds for all $\beta < I_\alpha$, so that $r_\alpha \in \mathbb{M}_\kappa^{L[G]}$. But this is impossible as clearly r_α is not fresh.

$\square$

Theorem 12.14 follows.

Remark 13.15. The only critical property of L that we need to make sure that $\mathbb{M}_\kappa$ is not a model of choice in $L[G]$ is that L has no nontrivial grounds, i.e. L satisfies the ground axiom. GCH is convenient and implies that no cardinals are collapsed, but it is not necessary. The use of Jensen's covering lemma can also be avoided, as discussed earlier.

13.2 The ω_1-mantle

Up to now, we have focused on the κ-mantle for strong limit κ. We will get similar results for the ω_1-mantle. There is some ambiguity in the definition of the ω_1-mantle, depending on whether or not ω_1 is considered as a parameter or as a definition. In the former case, it is the intersections of all grounds W so that W extends to V via a forcing so that $W \models |\mathbb{P}| < \omega_1^V$, where in the latter case we would require $W \models |\mathbb{P}| < \omega_1^W$. These mantles are in general not equal. To make the distinction clear, we give the latter version

[63] That is, there is no sequence $\langle x_\beta \mid \beta < \kappa\rangle \in \mathbb{M}_\kappa^{L[G]}$ with $x_\beta \in [c_\beta]_\sim$ for all $\beta < \kappa$.

the name "Cohen mantle" and denote it by $\mathbb{M}_{\mathsf{C}}$. The reason for the name is, of course, that all non-trivial countable forcings are forcing-equivalent to Cohen forcing.

Lemma 13.16. $\mathbb{M}_{\omega_1} \models \mathrm{ZF}$ *and* $\mathbb{M}_{\mathsf{C}} \models \mathrm{ZF}$.

Proof. First let us do it for $\mathbb{M}_{\mathsf{C}}$. Clearly, $\mathbb{M}_{\mathsf{C}}$ is closed under the Gödel operations. It is thus enough to show that $\mathbb{M}_{\mathsf{C}} \cap V_\alpha \in \mathbb{M}_{\mathsf{C}}$ for all $\alpha \in \mathrm{Ord}$. Let W be any Cohen-ground. As Cohen-forcing is homogeneous, $\mathbb{M}_{\mathsf{C}}^V$ is a definable class in W. Hence, $\mathbb{M}_{\mathsf{C}} \cap V_\alpha = \mathbb{M}_{\mathsf{C}} \cap V_\alpha^W \in W$. As W was arbitrary, this proves the claim.

Now onto $\mathbb{M}_{\omega_1}$. The above argument shows that all we need to do is show that $\mathbb{M}_{\omega_1}$ is a definable class in all associated grounds. So let W be such a ground. There are two cases. First, assume that $\omega_1^W = \omega_1^V$. Then W extends to V via Cohen forcing, so $\mathbb{M}_{\omega_1}$ is definable in W. Next, suppose that $\omega_1^W < \omega_1^V$. This can only happen if ω_1^V is a successor cardinal in W, say $W \models \omega_1^V = \mu^+$. In this case, W extends to V via a forcing of W-size $\leqslant \mu$ and which collapses μ to be countable. It is well known that in this situation, W extends to V via $\mathrm{Col}(\omega, \mu)$, which is homogeneous as well, so once again, $\mathbb{M}_{\omega_1}$ is a definable class in W. $\square$

Once again, choice can fail.

Theorem 13.17. *Relative to the existence of a Mahlo cardinal, it is consistent that there is no wellorder of* $\mathcal{P}(\omega_1^V)^{\mathbb{M}_{\omega_1}}$ *in* $\mathbb{M}_{\omega_1}$.

We remark that the Mahlo cardinal is used in a totally different way than in the last section. In the model we will construct, ω_1 will be inaccessible in $\mathbb{M}_{\omega_1}$. Let us once again assume $V = L$ for the rest of the section and let κ be Mahlo. Let $\mathbb{P}$ be the "$<\kappa$-support version of $\mathrm{Col}(\omega, <\kappa)$", that is

$$\mathbb{P} = \prod_{\alpha<\kappa}^{<\kappa-\text{support}} \mathrm{Col}(\omega, \alpha).$$

Let us pick a $\mathbb{P}$-generic filter G over V. From now on, $\mathbb{M}_{\omega_1}$ will denote $\mathbb{M}_{\omega_1}^{V[G]}$ and $\mathbb{M}_{\mathsf{C}}$ will denote $\mathbb{M}_{\mathsf{C}}^{V[G]}$.

Proposition 13.18. *Suppose* $\mathbb{Q}$ *is a forcing,* $\gamma < \lambda$ *and* λ *is a cardinal. If* $\mathbb{Q}$ *is* $\mathrm{AA}(\gamma, \lambda)$ *then in* $V^{\mathbb{Q}}$ *there is no surjection from any* $\beta < \gamma$ *onto* λ.

Proof. This is a straightforward adaptation of the proof that Axiom A forcings preserve ω_1. $\square$

The following lemma is the only significant use of the Mahloness of κ.

Lemma 13.19. $\mathbb{P}$ *satisfies* $\mathrm{AA}(\kappa, \kappa)$.

Proof. We define $\leqslant_\alpha$ independent of $\alpha < \kappa$ as the order $\leqslant^*$: Let $p \leqslant^* q$ iff $p \leqslant q$ and $p \upharpoonright \operatorname{supp}(q) = q$. The only nontrivial part is showing that for any antichain A and any $p \in \mathbb{P}$ there is $q \leqslant^* p$ with

$$|\{a \in A \mid a \| q\}| < \kappa.$$

Let

$$\mathbb{P} \upharpoonright \alpha := \{p \in \mathbb{P} \mid \sup \operatorname{supp}(p) < \alpha\}$$

for all $\alpha < \kappa$. We will proceed to find some q with the desired property. For convenience, we may assume that A is a maximal antichain. As κ is Mahlo, there is a regular $\lambda < \kappa$ so that $p\mathbb{P} \upharpoonright \lambda$ and any $r \in \mathbb{P} \upharpoonright \lambda$ is compatible with some $a \in A \cap \mathbb{P} \upharpoonright \lambda$. As $V = L$, $\diamondsuit_\lambda$ holds. Thus there is a sequence $\vec{d} := \langle d_\alpha \mid \alpha < \lambda \rangle$ with

($\vec{d}.i$) $d_\alpha \in \mathbb{P}_{\leqslant \alpha}$ and

($\vec{d}.ii$) for all $r \in \mathbb{P}_{\leqslant \lambda}$ there are stationarily many $\alpha < \lambda$ with $d_\alpha = r \upharpoonright \alpha$.

Construct a sequence

$$\langle q_\alpha \mid \alpha < \lambda \rangle$$

of conditions in $\mathbb{P} \upharpoonright \lambda$ with $q_\alpha \leqslant^* q_\beta$ for all $\alpha < \beta < \lambda$ as follows: Set $q_0 = p$. If q_β is defined for all $\beta < \alpha$, let first $q'_\alpha = \bigcup_{\beta < \alpha} q_\beta$ and note that this is a condition. Let $\gamma_\alpha = \sup \operatorname{supp}(q'_\alpha)$. Now find $a \in A \cap \mathbb{P} \upharpoonright \lambda$ that is compatible with d_{γ_α} and let

$$q_\alpha := q'_\alpha {}^\frown a \upharpoonright [\gamma_\alpha, \lambda).$$

Finally, set $q = \bigcup_{\alpha < \lambda} q_\alpha$. We have to show that q is compatible with only a few elements of A, so suppose $b \in A$ is compatible with q. The properties of $\vec{d}$ guarantee that there is $\alpha < \lambda$ so that

($\alpha.i$) $\gamma_\alpha = \alpha$ and

($\alpha.ii$) $d_\alpha = b \upharpoonright \alpha$.

Hence in the construction of $q_{\alpha+1}$ we found some $a \in A \cap \mathbb{P} \upharpoonright \lambda$ compatible with $b \upharpoonright \alpha$ and have $q_{\alpha+1} \upharpoonright [\alpha, \lambda) \leqslant a \upharpoonright [\alpha, \lambda)$. If $a \neq b$, then $a \perp b$ and the incompatibility must lie in the interval $[\alpha, \lambda)$. But then $q_{\alpha+1}$ and b are incompatible as well, contradiction. Thus $b = a$ and it follows that q is compatible with at most λ-many elements of A. □

Corollary 13.20. *We have*

($G.i$) $\omega_1^{L[G]} = \kappa$ *and*

($G.ii$) *if* $g \colon \omega \to \mathrm{Ord} \in L[G]$ *then there is some* $\alpha < \kappa$ *so that* $g \in V[G_{\leqslant \alpha}]$.

Proof. To see $(G.i)$, note that $\mathbb{P}$ collapses all cardinals $<\kappa$ to ω, so $\omega_1^{L[G]} \geqslant \kappa$. As $\mathbb{P}$ satisfies $\mathrm{AA}(\kappa,\kappa)$, there is no surjection from ω onto κ in $L[G]$.
Next, let us prove $(G.ii)$. Let $\dot{g} \in L$ be a name for g. In $L[G]$, find a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$ in G so that p_n decides the value of $\dot{g}(\check{n})$ (from the perspective of L). Let $\alpha = \sup_{n<\omega} \sup \mathrm{supp}(p_n)$. By $(G.i)$, $\alpha < \kappa$. But then $L[G_{\leqslant\alpha}]$ can compute the whole of g. □

From now on, $\mathbb{M}_{\omega_1}$ denotes $\mathbb{M}_{\omega_1}^{L[G]}$ and $\mathbb{M}_{\mathsf{C}}$ is $\mathbb{M}_{\mathsf{C}}^{L[G]}$. Let us define an auxiliary model $N = \bigcap_{\alpha<\kappa} L[G_{>\alpha}]$. It is clear that $\mathbb{M}_{\omega_1} \subseteq N$.

Recall the following fact due to Solovay.

Fact 13.21 (Solovay, [Sol70])**.** *If G, H are mutually generic filters over V (for any forcings) then $V[G] \cap V[H] = V$.*

Proposition 13.22. *We have that*

$(N.i)$ $N \models \mathrm{ZF}$ *and*

$(N.ii)$ $N \cap \mathcal{P}(\kappa) = \mathbb{M}_{\omega_1} \cap \mathcal{P}(\kappa) = \mathbb{M}_{\mathsf{C}} \cap \mathcal{P}(\kappa) = \{a \subseteq \kappa \mid \forall\beta < \kappa\ a \cap \beta \in V\}$.

Proof. First, we will prove $(N.i)$. Once again it is enough to show that N is definable in all models of the form $L[G_{>\alpha}]$ for $\alpha < \kappa$. But this is clear from the definition of N.
Next, we show $(N.ii)$. $\mathbb{M}_{\omega_1} \cap \mathcal{P}(\kappa) \subseteq \mathbb{M}_{\mathsf{C}} \cap \mathcal{P}(\kappa) \subseteq N \cap \mathcal{P}(\kappa)$ is trivial. If $a \in N \cap \mathcal{P}(\kappa)$ and $\beta < \kappa$ then $a \cap \beta \in L[G_{\leqslant\alpha}]$ for some α by clause $(G.ii)$ of Corollary 13.20. As $a \in N$, $a \cap \beta \in L[G_{>\alpha}]$, too. Thus by Fact 13.21

$$a \in L[G_{\leqslant\alpha}] \cap L[G_{>\alpha}] = L.$$

The final inclusion $N \cap \mathcal{P}(\kappa) \subseteq \mathbb{M}_{\omega_1} \cap \mathcal{P}(\kappa)$ holds since if W is a ground of $L[G]$ which extends to $L[G]$ via $\mathbb{Q}$ of size $< \kappa$ then $\mathbb{Q}$ cannot add a fresh subset of κ. □

Proof of Theorem 13.17. We will show that in $L[G]$, neither $\mathbb{M}_{\omega_1}$ nor $\mathbb{M}_{\mathsf{C}}$ possess a wellorder of its version of $\mathcal{P}(\kappa)$. In fact, we will show that N does not have such a wellorder, which is enough by $(N.ii)$ of the above proposition. Once again, let $\sim$ be the equivalence relation on functions $f\colon \kappa \to \kappa \in N$ of eventual coincidence. For $n < \omega$, let

$$d_n\colon \kappa \to \kappa,\ d_n(\alpha) = g_\alpha(n)$$

where g_α is the map $\omega \to \alpha$ induced by the slice of G generic for $\mathrm{Col}(\omega,\alpha)$. As before, we get that the fuzzy sequence $\langle [d_n]_\sim \mid n < \omega\rangle \in N$. If N had a wellorder of $\mathcal{P}(\kappa)$, then there would be a choice sequence $\langle x_n \mid n < \omega\rangle \in N$ for the fuzzy sequence. In $L[G]$, one can define the sequence $\langle \delta_n \mid n < \omega\rangle$ where δ_n is the least point after which x_n and d_n coincide. As $\kappa = \omega_1$ in $L[G]$, the δ_n are bounded uniformly by some $\delta < \kappa$. But this means that $G_{>\delta} \in N$, a contradiction. □

It is natural to conjecture that $N = \mathbb{M}_{\mathsf{C}} = \mathbb{M}_{\omega_1}$, though we do not have a proof of any of these equalities. The problem is that we cannot follow the strategy from Section 13.1: $L[G]$ has Cohen-grounds which do not contain any g_α for $\alpha < \kappa$, let alone a tail of the sequence $(g_\alpha)_{\alpha<\kappa}$.

Question 13.23. Is $N = \mathbb{M}_{\mathsf{C}} = \mathbb{M}_{\omega_1}$?

13.3 The successor of a regular uncountable cardinal case

We show that, again under $V = L$, for every regular uncountable κ there is a forcing extension in which $\mathbb{M}_{\kappa^+}$ is not a model of ZFC. The upside here is that we do not need any large cardinals at all in our construction, however we pay a price: We do not know whether $\mathbb{M}_{\kappa^+}$ is a model of ZF in general.

Theorem 13.24. *Assume $V = L$ and suppose κ is regular uncountable. Then after forcing with*

$$\mathbb{P} := \prod_{\alpha<\kappa^+}^{<\kappa^+-support} \mathrm{Add}(\kappa, 1)$$

$\mathbb{M}_{\kappa^+}$ *is not a model of* ZFC.

First, lets do a warm-up with an initial segment of $\mathbb{P}$. We thank Elliot Glazer for explaining (the nontrivial part of) the following argument to the author.

Lemma 13.25 (Elliot Glazer)**.** *If κ is regular and $\diamondsuit_\kappa$ holds then*

$$\mathbb{P}_{\leqslant\kappa} = \prod_{\alpha<\kappa}^{\textit{full support}} \mathrm{Add}(\kappa, 1)$$

satisfies $\mathrm{AA}(\kappa + 1, \kappa^+)$.

An additional assumption beyond "κ is regular" is necessary here: It is well known that

$$\prod_{n<\omega}^{\text{full support}} \mathrm{Add}(\omega, 1)$$

collapses 2^ω to ω.

Proof. We let $p \leqslant_\alpha q$ if $p \leqslant q$ and $p \upharpoonright \alpha = q \upharpoonright \alpha$. It is easy to see that (AA.*i*) and (AA.*iii*) of Definition 13.2 hold, so let us show (AA.*ii*). Therefore, let $\alpha < \kappa$, $p \in \mathbb{P}_{\leqslant\kappa}$ and an antichain A in $\mathbb{P}_{\leqslant\kappa}$ be given. As $\diamondsuit_\kappa$ holds, there is a sequence $\langle d_\beta \mid \beta < \kappa\rangle$ with $d_\beta \in \mathbb{P}_{\leqslant\beta}$ so that for any $q \in \mathbb{P}_{\leqslant\kappa}$ there is some β with $q \upharpoonright \beta = d_\beta$. We will define a sequence $(p_\beta)_{\alpha\leqslant\beta\leqslant\kappa}$ inductively so that $p_\gamma \leqslant_\beta p_\beta$ for all $\beta \leqslant \gamma \leqslant \kappa$. We put $p_\alpha = p$. At limit stages β we let p_β

be the canonical fusion of $\langle p_\gamma \mid \alpha \leqslant \gamma < \beta\rangle$. So assume p_β is defined. We choose $p_{\beta+1} \leqslant_\beta p_\beta$ so that, if possible,

$$d_\beta^\frown p_{\beta+1} \leqslant a$$

for some $a \in A$. Otherwise, we are lazy and set $p_{\beta+1} = p_\beta$.
Now clearly $q := p_\kappa \leqslant_\alpha p$ and we will show that q is compatible with at most κ-many conditions in A. To see this, suppose $a \in A$ is compatible with q. We may find $\beta < \kappa$ so that $d_\beta = a \upharpoonright \beta$. In the construction of $p_{\beta+1}$ from p_β, we tried to achieve that

$$d_\beta^\frown p_{\beta+1} \upharpoonright [\beta, \kappa)$$

is below some condition in A, which is possible and only possible for a. This shows that for any $a \in A$ that is compatible with q, there is $\beta < \kappa$ so that $q \upharpoonright [\beta, \kappa) \leqslant a \upharpoonright [\beta, \kappa)$. As $\mathbb{P}_{\leqslant\beta}$ has size $\leqslant\kappa$, it follows that there are at most κ-many such $a \in A$. □

Corollary 13.26. *Under the same assumptions as before, $\mathbb{P}_{\leqslant\kappa}$ preserves all cardinals $\leqslant \kappa^+$.*

Proof. $\mathbb{P}_{\leqslant\kappa}$ is $<\kappa$-closed and satisfies $\mathrm{AA}(\kappa + 1, \kappa^+)$. □

We aim to prove a similar result for $\mathbb{P}$.

Lemma 13.27. *If κ is regular and $\diamondsuit_\kappa$ holds then $\mathbb{P}$ preserves all cardinals $\leqslant \kappa^+$. Moreover, if G is $\mathbb{P}$-generic and $g\colon \kappa \to \mathrm{Ord}$ is in $V[G]$ then there is $\alpha < \kappa^+$ with $g \in V[G_{\leqslant\alpha}]$.*

The argument is similar, but somewhat more complicated. To do so, we introduce a further abstraction of $\mathrm{AA}(\kappa, \lambda)$.

Definition 13.28. Suppose that $\mathcal{P} = (P, \preceq)$ is a partial order, $\mathbb{Q}$ is a forcing, $\kappa < \lambda$ are ordinals. $\mathbb{Q}$ satisfies *Strategic Axiom A$(\kappa, \lambda, \mathcal{P})$* ($\mathrm{SAA}(\kappa, \lambda, \mathcal{P})$) if there is a family $\langle \leqslant_x \mid x \in P\rangle$ of partial orders on $\mathbb{Q}$ so that

(SAA.*i*) $\leqslant_y \subseteq \leqslant_x \subseteq \leqslant_{\mathbb{Q}}$ whenever $x \preceq y$ for $x, y \in P$,

(SAA.*ii*) for any antichain $A \subseteq \mathbb{Q}$, any $x \in P$ and $p \in \mathbb{Q}$, there is $q \leqslant_x p$ with

$$|\{a \in A \mid a \| p\}| < \lambda$$

and

(SAA.*iii*) player II has a winning strategy in the following game we call $\mathcal{G}(\mathbb{Q}, \kappa, \mathcal{P})$:

I	p_0		p_1		$\dots$	p_ω		$\dots$
II		x_0		x_1	$\dots$		x_ω	$\dots$

The game has length κ. In an even round $\alpha \cdot 2$, Player I plays some condition $p_\alpha \in \mathbb{Q}$ so that $p_\alpha \leqslant_{x_\beta} p_\beta$ for all $\beta < \alpha$ played so far. In an odd round $\alpha \cdot 2 + 1$, player II plays some $x_\alpha \in P$ with $x_\beta \preceq x_\alpha$ for all $\beta < \alpha$.
Player I wins the game iff some player has no legal moves in some round $<\kappa$. If the game last all κ rounds instead, II wins.

It is straightforward to generalize Proposition 13.18.

Proposition 13.29. *Suppose $\mathbb{Q}$ satisfies* $\mathrm{SAA}(\kappa, \lambda, \mathcal{P})$*. Then in $V^{\mathbb{Q}}$, there is no surjection $f \colon \beta \to \lambda$ for any $\beta < \kappa$.*

□

Lemma 13.30. *If κ is regular and $\diamondsuit_\kappa$ holds then $\mathbb{P}$ satisfies*

$$\mathrm{SAA}(\kappa + 1, \kappa^+, \mathcal{P}_\kappa(\kappa^+))$$

where $\mathcal{P}_\kappa(\kappa^+)$ is ordered by inclusion.

Proof. For $x \in \mathcal{P}_\kappa(\kappa^+)$ we will write $p \leqslant_x q$ if $p \leqslant q$ and $p \upharpoonright x = q \upharpoonright x$. It is clear that (SAA.$i$) holds.
Next, we aim to establish (SAA.iii). We describe a strategy for player II in the relevant game. We will need to do some additional bookkeeping. Let

$$h \colon \kappa \to \kappa \times \kappa$$

be a surjection such that if $h(\beta) = (\alpha, \gamma)$ then $\alpha \leqslant \beta$. Suppose that p_α is the last condition played by player I and $(x_\beta)_{\beta<\alpha}$ have been played already. In the background, we already have chosen some surjections $s_\beta \colon \kappa \to \mathrm{supp}(p_\beta)$ for $\beta < \alpha$ and we will adjoin a surjection $s_\alpha : \kappa \to \mathrm{supp}(p_\alpha)$ to that list. We set

$$x_\alpha = s_{\gamma_0}(\gamma_1) \cup \bigcup_{\beta<\alpha} x_\beta$$

where $(\gamma_0, \gamma_1) = h(\alpha)$. As κ is regular, $x_\alpha \in \mathcal{P}_\kappa(\kappa^+)$.

Claim 13.31. *Player I does not run out of moves before the game ends.*

Proof. Suppose we reached round $2 \cdot \alpha \leqslant \kappa$ and let $x = \bigcup_{\beta<\alpha} x_\beta$. We will find a legal play p_* for player I. For $\gamma \in \kappa^+ \setminus \bigcup_{\beta<\alpha} \mathrm{supp}(p_\beta)$, let $p_*(\gamma)$ be trivial. The point is that for $\gamma \in x$, $\langle p_\beta(\gamma) \mid \beta < \alpha \rangle$ stabilizes eventually to some $p_*(\gamma)$. If $\alpha = \kappa$, then our bookkeeping made sure that we have

$$x = \bigcup_{\beta<\kappa} \mathrm{supp}(p_\beta)$$

so that p_* is already fully defined and a legal play. If $\alpha < \kappa$ instead, then there are possibly $\gamma \in \bigcup_{\beta<\alpha} \mathrm{supp}(p_\beta) - x$, but then $\langle p_\beta(\gamma) \mid \beta < \alpha \rangle$ is a sequence of length $<\kappa$, so we may pick a lower bound $p_*(\gamma) \in \mathrm{Add}(\kappa, 1)$ for it. □

It remains to show (SAA.*ii*) and here we will use that $\diamondsuit_\kappa$ holds. Let $\langle d_\beta \mid \beta < \kappa\rangle$ be the "$\diamondsuit_\kappa$-sequence for $\mathbb{P}_{\leqslant\kappa}$" that appeared in the proof of Lemma 13.25 and let A be a maximal antichain in $\mathbb{P}$. Choose τ to be a winning strategy for player II in $\mathcal{G}(\mathbb{P}, \kappa+1, \mathcal{P}_\kappa(\kappa^+))$ and we will describe a strategy σ for player I: Suppose $\alpha \leqslant \kappa$ and p_β, x_β have already been played for $\beta < \alpha$. This time, we will have picked some surjections $s_\beta\colon \kappa \to x_\beta$ for $\beta < \alpha$ in the background. Let $x_{<\alpha} := \bigcup_{\beta<\alpha} x_\beta$. Then, assuming there is a legal move, pick some p_α so that

($p_\alpha.i$) $p_\alpha \leqslant_{x_\beta} p_\beta$ for all $\beta < \alpha$ and

($p_\alpha.ii$) if possible, $p_\alpha \restriction (\kappa^+ \backslash x_{<\alpha}) \cup e_\alpha \restriction x_{<\alpha}$ is below a condition in A

where e_α is defined by

$$e_\alpha(s_{\gamma_0}(\gamma_1)) = d_\alpha(\gamma)$$

whenever $\gamma < \alpha$ and $h(\gamma) = (\gamma_0, \gamma_1)$ (and e_α is trivial where we did not specify a value)[64].

Let $\langle p_\alpha \mid \alpha \leqslant \kappa\rangle$, $\langle x_\alpha \mid \alpha < \kappa\rangle$ be the sequences of moves played by player I and II in a game where player I follows σ and player II follows τ. As τ is a winning strategy, the sequence must be of length $\kappa+1$. We will show that $q := p_\kappa$ is compatible with at most κ-many elements of A. So let $a \in A$ and assume that q is compatible with a.

Claim 13.32. *There is $\alpha < \kappa$ so that $e_\alpha \in \mathbb{P}$ and $e_\alpha \restriction x_{<\alpha} = a \restriction x_{<\alpha}$.*

Proof. We define $b \in \mathbb{P}_{\leqslant\kappa}$ by $b(\gamma) = a(s_{\gamma_0}(\gamma_1))$ whenever $h(\gamma) = (\gamma_0, \gamma_1)$. Then there is $\alpha < \kappa$ with

($\alpha.i$) $b \restriction \alpha = d_\alpha$ and

($\alpha.ii$) $x_{<\alpha} = \{s_{\gamma_0}(\gamma_1) \mid \exists\gamma < \alpha\ h(\gamma) = (\gamma_0, \gamma_1)\}$.

It is easy to see now that α is as desired. □

Thus in round $\alpha \cdot 2$ in the game, player I tried to make sure that

$$a \restriction x_{<\alpha} \cup p_\alpha \restriction (\kappa^+ \backslash x_{<\alpha})$$

is below some condition in A. This is possible for a, and only for a as q and a are compatible.

We have shown that for any $a \in A$ that is compatible with q, there is $\alpha < \kappa$ such that $q \restriction (\kappa^+ \backslash x_{<\alpha}) \leqslant a \restriction (\kappa^+ \backslash x_{<\alpha})$. As there are only $\leqslant\kappa$-many $r \in \mathbb{P}$ with support contained in $x_{<\alpha}$, this implies that there are at most κ-many such $a \in A$. □

[64] e_α may fail to be a function, in which case ($p_\alpha.ii$) is void.

Lemma 13.27 follows from Lemma 13.30 and Proposition 13.29 similarly to how we proved Corollary 13.20.

Remark 13.33. If additionally GCH holds at κ^+ then $\mathbb{P}$ does not collapse any cardinals at all by a standard Δ-system argument.

Proof of Theorem 13.24. Let G be $\mathbb{P}$-generic over L. By Lemma 13.27, all L-cardinals $\leqslant \kappa^+$ are still cardinals in $L[G]$ (in fact, all cardinals are preserved). Let $N = \bigcap_{\alpha<\kappa^+} L[G_{>\alpha}]$. Using that N is definable in every model of the form $L[G_{>\alpha}]$, it is easy to check that N is a model of ZF. Once again, we call $A \subseteq \kappa^+$ fresh if $A \cap \alpha \in L$ for all $\alpha < \kappa^+$.

Claim 13.34. $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}} = \mathcal{P}(\kappa^+)^N = \{A \subseteq \kappa^+ \mid A \textit{ is fresh}\}^{L[G]}$.

Proof. $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}} \subseteq \mathcal{P}(\kappa^+)^N$ is trivial. Suppose $A \subseteq \kappa^+, A \in N$. Given $\alpha < \kappa^+$, by Lemma 13.27, there is $\beta < \kappa^+$ so that $A \cap \alpha \in L[G_{\leqslant\beta}]$ so that

$$A \cap \alpha \in L[G_{\leqslant\beta}] \cap L[G_{>\beta}] = L$$

by Fact 13.21. For the last inclusion assume $A \in L[G]$ is a fresh subset of κ^+ and W is any κ^+-ground of $L[G]$. It follows that $W \subseteq L[G]$ satisfies the κ^+-approximation property so that $A \in W$ as any bounded subset of A is in $L \subseteq W$. □

We will show that there is no wellorder of $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}}$ in $\mathbb{M}_{\kappa^+}$. So assume otherwise. Let $\sim$ be the equivalence relation of eventual coincidence on ${}^{\kappa^+}2$ in N. We can realise G as a matrix where the α-th row is $\mathrm{Add}(\kappa,1)$-generic over L. Now the columns are in fact $\mathrm{Add}(\kappa^+,1)$-generic over L. Let us write c_α for the α-th column ($\alpha < \kappa^+$) and d_β for the β-th row ($\beta < \kappa$). For any $\alpha < \kappa^+$ we have that $\langle d_\beta \upharpoonright [\alpha, \kappa^+) \mid \beta < \kappa\rangle \in L[G_{>\alpha}]$. Thus

$$\langle [d_\beta]_\sim \mid \beta < \kappa\rangle \in N$$

and by our assumption there must be a choice function, say $\langle x_\beta \mid \beta < \kappa\rangle$, in N. In $L[G]$, we can define the sequence $\langle \delta_\beta \mid \beta < \kappa\rangle$, where δ_β is the least point after which x_β and d_β coincide. As κ^+ is not collapsed by $\mathbb{P}$, we can strictly bound all δ_β by some $\delta_* < \kappa^+$. But then

$$\langle x_\beta(\delta_*) \mid \beta < \kappa\rangle \in N$$

is $\mathrm{Add}(\kappa,1)$-generic over L, which contradicts that N and L have the same subsets of κ. □

Note that Fact 12.6 does not apply in the situation here, so we may ask:

Question 13.35. Is $\mathbb{M}_{\kappa^+}$ a model of ZF? Is $\mathbb{M}_{\kappa^+} = N$?

13.4 Conclusion

There are a number of open questions regarding the interplay between large cardinal properties of κ and the κ-mantle. The following table summarizes what is known as presented in the introduction.

Large cardinal property of κ	Theory of $\mathbb{M}_\kappa$ extends...
extendible	ZFC + GA
measurable	ZFC
weakly compact	ZF + κ-DC
inaccessible	ZF + $(<\kappa, H_{\kappa^+})$-choice

There is certainly much more to discover here. How optimal are these results? Optimality has only been proven for one of them, namely the first. This is due to Gabriel Goldberg.

Fact 13.36 (Goldberg, [Gol21])**.** *Suppose κ is an extendible cardinal. Then there is a class forcing extension in which κ remains extendible and $\mathbb{M}_\kappa$ is not a κ-ground. In particular, if $\lambda < \kappa$ and $\mathbb{M}_\lambda \models$ ZFC then $\mathbb{M}_\lambda$ has a nontrivial ground.*

The most interesting question seems to be up to when exactly the axiom of choice can fail to hold in $\mathbb{M}_\kappa$. Since this can happen at a Mahlo cardinal, the natural next test question is whether this is possible at a weakly compact cardinal.

Question 13.37. Suppose that κ is weakly compact. Must $\mathbb{M}_\kappa \models$ ZFC?

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